PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 2, February 2009, Pages 401–408
S 0002-9939(08)09587-7
Article electronically published on October 3, 2008

CYCLOTOMIC UNITS IN FUNCTION FIELDS

SUNGHAN BAE AND LINSHENG YIN

(Communicated by Wen-Ching Winnie Li)

ABSTRACT. Let k be a global function field over the finite field $\mathbb{F}_q$ with a fixed place ∞ of degree 1. Let K be a cyclic extension of degree dividing $q-1$, in which ∞ is totally ramified. For a certain abelian extension L of k containing K, there are two notions of the group of cyclotomic units arising from sign normalized rank 1 Drinfeld modules on k and on K. In this article we compare these two groups of cyclotomic units.

0. INTRODUCTION

Let K be an imaginary quadratic number field and L an abelian extension of $\mathbb{Q}$ containing K. There exist two subgroups of the group of units of L. One is the group of *cyclotomic* units of the extension $L/\mathbb{Q}$ and the other the group of *elliptic* units of the extension L/K. Both have finite indices in the full group of units of L, which are closely related to the class number of L. The relation between these two groups was studied by Gillard [Gi] and Kersey [Ke]. In fact, it is shown that some power of an elliptic unit is a cyclotomic unit.

In this article we consider the analogous problem in the function field setting. Let k be a global function field over the finite field $\mathbb{F}_q$ with a fixed place ∞ of degree 1. Let ℓ be an integer dividing $q-1$. Let K be a cyclic extension of k of degree ℓ in which ∞ is totally ramified, and let L be an abelian extension of both k and K, such that ∞ splits completely in L/K. In L there exist two notions of the group of cyclotomic units. One group is over k and the other over K. The latter can be viewed as an analogue of the group of elliptic units ([ABJ], [Yi1], [Ou]). We will compare these two groups, adopting the method of [Gi].

We note that in [Gi] there are some misprints. In the statement of Theorem 3, $12fe(\mathfrak{f})$ should be changed to $12eh$. The reason for this is that the wrong formula was used (5 bis) on p. 187; it should be (see [GR], Proposition 7.19, or [Ou], (3.3))

$$L'(0,\chi,K/k) = -\frac{1}{6eh}\sum_{c\in Cl((1))}\chi(c)\log|\delta(c)|.$$

Received by the editors February 16, 2007.
2000 *Mathematics Subject Classification.* Primary 11R58.
The first author was supported by KOSEF research grants R01-2006-000-10320-0, F01-2006-000-10040-0 and SRC program (ASARC R11-2007-035-01001-0).
The second author was supported by NSFC (No. 10571097).

Notation. k: a global function field over the finite field $\mathbb{F}_q$ of q elements

∞: a fixed place of degree 1 of k

$\mathbb{A}$: the ring of functions in k, which are regular away from ∞

ℓ: an integer dividing $q-1$

$K := k(m^{1/\ell})$, where $m \in \mathbb{A}$ has degree prime to ℓ

$\chi_K :=$ a fixed generator of the character group of $\mathrm{Gal}(K/k)$

It is clear that ∞ is totally ramified in K/k, so we use the same ∞ to denote the unique place of K lying over ∞.

$\mathbb{B}$: the integral closure of $\mathbb{A}$ in K, which is the same as the ring of functions in K regular away from ∞

$\mathfrak{a}, \mathfrak{b}, \mathfrak{m}, \mathfrak{n}, \mathfrak{f}, \cdots$: ideals of $\mathbb{A}$

$\mathfrak{A}, \mathfrak{B}, \mathfrak{M}, \mathfrak{N}, \mathfrak{F}, \cdots$: ideals of $\mathbb{B}$

h_k (resp. h_K): the class number of k (resp. K), which is the same as the ideal class number of $\mathbb{A}$ (resp. $\mathbb{B}$) since ∞ has degree 1

Fix a sign function $sgn : k_\infty = K_\infty \longrightarrow \mathbb{F}_q$ with $sgn(0) = 0$.

$k_\mathfrak{n}$: the cylotomic function field over k of conductor $\mathfrak{n}$ with respect to sgn

$K_\mathfrak{N}$: the cylotomic function field over K of conductor $\mathfrak{N}$ with respect to sgn

$G_\mathfrak{n} := \mathrm{Gal}(k_\mathfrak{n}/k)$ and $\Gamma_\mathfrak{N} := \mathrm{Gal}(K_\mathfrak{N}/K)$

$k_{(1)}$ (resp. $K_{(1)}$) : the Hilbert class field of k (resp. K)

We choose the sign of m so that K is contained in $k_{(m)}$.

$\xi(\mathfrak{n})$ (resp. $\xi(\mathfrak{N})$): ξ-invariant associated to $\mathfrak{n}$ (resp. $\mathfrak{N}$)

$e_\mathfrak{n}$ (resp. $e_\mathfrak{N}$): the lattice function associated to the ideal $\mathfrak{n}$ (resp. $\mathfrak{N}$)

For $\mathfrak{n} \neq (1)$ and $\mathfrak{N} \neq (1)$, $\lambda_\mathfrak{n} := \xi(\mathfrak{n})e_\mathfrak{n}(1)$, $\Lambda_\mathfrak{N} := \xi(\mathfrak{N})e_\mathfrak{N}(1)$.

For details of this notation we refer to [Ha1], [Yi1].

1. Preparation and statement of main theorem

Let L be an abelian extension of k which is contained in some cyclotomic function field over k and suppose ∞ splits completely in L/K. Let O_L be the integral closure of $\mathbb{A}$ in L. For each ideal class c (resp. C) of $\mathbb{A}$ (resp. $\mathbb{B}$) containing an ideal $\mathfrak{a}$ (resp. $\mathfrak{A}$), let

$$\delta(c) := a\xi(\mathfrak{a})^{h_k} \quad \text{and} \quad \Delta(C) := A\xi(\mathfrak{A})^{h_K},$$

where $(a) = \mathfrak{a}^{h_k}$ and $(A) = \mathfrak{A}^{h_K}$ with $sgn(a) = sgn(A) = 1$.

For $\sigma \in G_\mathfrak{n}$, we define the partial zeta function by

$$Z_\mathfrak{n}(s,\sigma) := \sum_{\sigma_\mathfrak{b} = \sigma,\, (\mathfrak{b},\mathfrak{n})=(1)} N(\mathfrak{b})^{-s}.$$

Note that $Z_\mathfrak{n}(0,\sigma)$ is a rational number. We return to $Z_\mathfrak{n}(s,\sigma)$ in the last section.

Let $\mathfrak{n}$ be the conductor of L over k and $\mathfrak{N}$ the conductor of L over K, that is, $\mathfrak{n}$ (resp. $\mathfrak{N}$) is the smallest ideal $\mathfrak{n}$ (resp. $\mathfrak{N}$) such that L is contained in $k_\mathfrak{n}$ (resp. $K_\mathfrak{N}$). Let $\mathfrak{n}_1$ be the ideal $k \cap \mathfrak{N}$. Let $\Gamma := \mathrm{Gal}(L/K)$ and $G := \mathrm{Gal}(L/k)$. For $\mathfrak{n} \neq (1)$ and $g \in G$ (resp. $\mathfrak{N} \neq (1)$ and $\gamma \in \Gamma$), let

$$\varphi_L(g) := \prod_{\tau \in G_\mathfrak{n},\, \tau|_L = g} \lambda_\mathfrak{n}^\tau, \qquad \Phi_L(\gamma) = \prod_{\tau \in \Gamma_\mathfrak{N},\, \tau|_L = \gamma} \Lambda_\mathfrak{N}^\tau.$$

For $\mathfrak{n} = (1)$ (resp. $\mathfrak{N} = (1)$), we let

$$\delta_L(g) := \prod_{\sigma_c|_L = g} \delta(c), \quad \Delta_L(\gamma) := \prod_{\sigma_C|_L = \gamma} \Delta(C),$$

where σ_c and σ_C are the Artin automorphisms associated to c and C, respectively.

Proposition 1 ([Ou, Chap. 3, Chap. 4]). *Let $g_1, g_2, g \in G$. Then we have*

1) $\delta_L(g_1)/\delta_L(g_2)$ *and* $\varphi_L(g_1)/\varphi_L(g_2)$ *are units in* O_L.
2) $\left(\frac{\delta_L(g_1)}{\delta_L(g_2)}\right)^g = \frac{\delta_L(gg_1)}{\delta_L(gg_2)}$ *and* $\varphi_L(g_1)^g = \varphi_L(gg_1)$.
3) *If* $\mathfrak{n}$ *is not a prime power, then* $\varphi_L(g)$ *is a unit in* O_L.

The same holds for Δ, Φ *and* Γ.

Let $P_{L/k}$ (resp. $P_{L/K}$) be the subgroup of L^* generated by $\mathbb{F}_q^*$, $\varphi_L(g)$ and $\delta_L(g)/\delta_L(id)$ for $g \in G$ (resp. $\Phi_L(\gamma)$ and $\Delta_L(\gamma)/\Delta_L(id)$ for $\gamma \in \Gamma$), which we call the *group of cyclotomic numbers* over k (resp. over K) in L. Let

$$C_{L/k} := P_{L/k} \cap O_L^*, \quad \text{and} \quad C_{L/K} := P_{L/K} \cap O_L^*,$$

which we call the *group of cyclotomic units* of L over k and K, respectively. These are slightly different from the group of cyclotomic units defined in [ABJ] or [Yi1].

Let S be the set of all prime ideals of $\mathbb{A}$, which are ramified in L/k but unramified in L/K. For $\mathfrak{p} \in S$ denote by $T_\mathfrak{p}$ the inertia group in L/k at $\mathfrak{p}$. Decompose S into a disjoint union $S = \bigcup_{i \in I} S_i$, where two ideals in S lie in the same S_i if and only if they have the same inertia groups. Let J be a subset of I and J_1 its complementary subset. Let L_J^0 (resp. L_J^1) be the subfield of L fixed by the subgroup $T_J \subset G$ (resp. T_{J_1}) generated by $T_\mathfrak{p}$ for any $\mathfrak{p} \in S_i$ with $i \in J$ (resp. $i \in J_1$). Let $\mathfrak{f}_J^0$ (resp. $\mathfrak{f}_J$) be the conductor of L_J^0 (resp. L_J^1) over k. Let

$$\theta_J := \begin{cases} N_{k_{\mathfrak{f}_J^0}/L_J^0}(\lambda_{\mathfrak{f}_J^0}) & \text{if } \mathfrak{f}_J^0 \neq (1), \\ N_{k_{(1)}/L_J^0}\delta(1) & \text{if } \mathfrak{f}_J^0 = (1), \end{cases}$$

and for $i = 1, ..., \ell - 1$,

$$\beta_J^{(i)}(\sigma) := \chi_K^i(\sigma) \sum_{\tau \in G_{\mathfrak{f}_J^1},\, \tau|_{L_J^1} = \sigma} Z_{\mathfrak{f}_J^1}(0, \tau),$$

where $\sigma \in \mathrm{Gal}(L/k)$ and $\mathfrak{f}_J^1$ is the least common multiple of $\mathfrak{n}_1$ and $\mathfrak{f}_J$.

Let

$$m_J := \frac{[L : L_I^0]^2}{[L : L_J^0][L : L_J^1]},$$

$$\beta_J^{(i)} := \sum_{\sigma \in G} \beta_J^{(i)}(\sigma)\sigma^{-1}, \quad \beta_J = \prod_{i=1}^{\ell-1} \beta_J^{(i)}$$

and

$$\gamma_J := \prod_{\mathfrak{p}|\mathfrak{n}_1,\, \mathfrak{p}\nmid\mathfrak{f}_J^0} (1 - \sigma_\mathfrak{p}^{-1}),$$

where $\sigma_\mathfrak{p}$ is the Frobenius automorphism of $\mathfrak{p}$ in L_J^0/k. Let $\alpha_J := \gamma_J \beta_J$. But $\beta_J \in \mathbb{Q}(\zeta)[G]$, where ζ is a primitive ℓ-th root of 1. To proceed we need the following lemma.

Lemma 2. $\beta_J \in \mathbb{Q}[G]$.

Proof. Let $\eta \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$. We extend the action of η to $\mathbb{Q}(\zeta)[G]$ in an obvious way. Then it is not hard to see that η fixes β_J, which implies that $\beta_J \in \mathbb{Q}[G]$. $\square$

Theorem 3. *If L/K is ramified, then*

$$\Phi_L^{[L:L_I^0]^2} = \epsilon \prod_{J \subset I} \theta_J^{-\ell^2 \alpha_J m_J}$$

with $\epsilon \in \mathbb{F}_q^*$. If L/K is unramified and $\sum_{\sigma \in G} n(\sigma)\sigma \in \mathbb{Z}[G]$ satisfies $\sum n(\sigma) = 0$, then

$$\prod_{\sigma \in G} \Delta_L(\sigma)^{n(\sigma)[L:L_I^0]^2} = \epsilon \left(\prod_{J \subset I} \theta_J^{-\ell^2 r_J \alpha_J m_J} \right)^{\sum n(\sigma)\sigma},$$

with $\epsilon \in \mathbb{F}_q^*$, where

$$r_J = \begin{cases} \frac{h_K}{h_k} & \text{if } \mathfrak{f}_J^0 = (1) \\ h_K & \text{if } \mathfrak{f}_J^0 \neq (1). \end{cases}$$

Note that h_K is divisible by h_k.

2. Proof of the main theorem

Let v_k be the normalized valuation of k at ∞, and v_K be the normalized valuation of K at ∞. Then $v_K(a) = \ell v_k(a)$ for $a \in k$. We choose an extension v_1 (resp. v_2) of v_k (resp. v_K) to $\bar{k} = \bar{K}$ so that $v_2(\alpha) = \ell v_1(\alpha)$, which we also denote by v_k (resp. v_K). Let χ be a character of Γ. Let $\chi_0, ..., \chi_{\ell-1}$ be the characters of G extending χ. Assume that χ_0 is real, that is, χ_0 is trivial on the inertia group at ∞, and $\chi_i = \chi_K^i \chi_0$.

Let

$$\Sigma_J^0(\chi) := \begin{cases} \frac{1}{[L:L_J^0]} \sum_{\sigma \in G} \chi_0(\sigma) v_k(\sigma(\theta_J^{\gamma_J})) \text{ if } \mathfrak{f}_J^0 \neq (1) \\ \frac{1}{h_k [L:L_J^0]} \sum_{\sigma \in G} \chi_0(\sigma) v_k(\sigma(\theta_J^{\gamma_J})) \text{ if } \mathfrak{f}_J^0 = (1) \end{cases}$$

$$\Sigma_J^i(\chi) := \frac{1}{[L:L_J^1]} \sum_{\sigma \in G} \chi_0(\sigma) \beta_J^{(i)}(\sigma)$$

and

$$\Sigma(\chi) := \sum_{\sigma \in \Gamma} \chi(\sigma) v_K(\Phi_L(\sigma)) \quad \text{if } \mathfrak{N} \neq (1)$$

$$\Sigma(\chi) := \frac{1}{h_K} \sum_{\sigma \in \Gamma} \chi(\sigma) v_K(\Delta_L(\sigma)) \quad \text{if } \mathfrak{N} = (1).$$

To prove Theorem 3, it suffices to show that

$$[L:L_I^0]^2 \Sigma(\chi) = \ell^2 \sum_{J \subset I} \sum_{\sigma \in \Gamma} \chi(\sigma) v_K(\sigma(\theta_J^{\alpha_J})) = \sum_{J \subset I} \sum_{\sigma \in G} \chi_0(\sigma) v_k(\sigma(\theta_J^{\alpha_J}));$$

that is, it suffices to show that

$$\Sigma(\chi) = \sum_{J \subset I} \left(\prod_{i=0}^{\ell-1} \Sigma_J^i(\chi) \right).$$

Considering χ as a character of $G(L_J^0/k)$ or $G(L_J^1/k)$ if possible, we have

$$\Sigma_J^0(\chi) = \begin{cases} \prod_{\mathfrak{p} | \mathfrak{n}_1 \, \mathfrak{p} \nmid \mathfrak{f}_J^0} (1 - \chi_0(\mathfrak{p})) \sum_{\sigma \in G_{\mathfrak{f}_J^0}} \chi_0(\sigma) v_k(\sigma(\lambda_{\mathfrak{f}_J^0})) & \text{if } \mathfrak{f}_J^0 \neq (1) \\ \prod_{\mathfrak{p} | \mathfrak{n}_1} (1 - \chi_0(\mathfrak{p})) \sum_{\sigma \in G_{(1)}} \chi_0(\sigma) v_k(\sigma(\delta(1))) & \text{if } \mathfrak{f}_J^0 = (1) \end{cases}$$

if χ_0 is trivial on $G(L/L_J^0)$ and 0 otherwise, and

$$\Sigma_J^i(\chi) = \sum_{\sigma \in G_{\mathfrak{f}_J^1}} \chi_i(\sigma) Z_{\mathfrak{f}_J^1}(0, \sigma)$$

if χ_i is trivial on $G(L/L_J^1)$ and 0 otherwise.

Remark. For $\mathfrak{p} \in S$ let $t_\mathfrak{p}$ be a generator of $T_\mathfrak{p}$. Then

$$\chi_i(t_\mathfrak{p}) = \chi_0(t_\mathfrak{p}) \chi_K^i(t_\mathfrak{p}) = \zeta^i \chi_0(t_\mathfrak{p}),$$

since $\mathfrak{p}$ is ramified in K/k, where $\zeta \neq 1$ is an ℓth root of 1. Thus χ_i is trivial on $T_\mathfrak{p}$ for some $i > 0$ if and only if χ_0 is not. Thus $\prod_{i=0}^{\ell-1} \Sigma_J^i(\chi) \neq 0$ if and only if the union of S_i, for $i \in J$ is exactly the set of $\mathfrak{p} \in S$ such that $\chi_0(t_\mathfrak{p}) = 1$. For each χ this can happen for a unique J.

Thus Theorem 3 is equivalent to

Proposition 4. *Let χ be a character of Γ, nontrivial if $\mathfrak{N} = (1)$. For the subset J of I as above, we have*

$$\Sigma(\chi) = \prod_{i=0}^{\ell-1} \Sigma_J^i(\chi).$$

Let $\mathfrak{f}_i$ be the conductor of χ_i for $i = 0, 1, ..., \ell - 1$, and let $\mathfrak{F}$ be the conductor of χ as a character over K. We have the following properties of L-series (cf. [Yi1], [Yi2], [Ou]):

$$L_K(s, \chi) = \prod_{i=0}^{\ell-1} L_k(s, \chi_i). \tag{1}$$

For a nontrivial character χ, we have

$$L_k(0, \chi_0) = \frac{1}{q-1} \sum_{\sigma \in G(k_{\mathfrak{f}_0}/k)} \chi_0(\sigma) v_k(\lambda_{\mathfrak{f}_0}^\sigma) \quad \text{if } \mathfrak{f}_0 \neq (1), \tag{2}$$

$$L_k(0, \chi_0) = \frac{1}{h_k(q-1)} \sum_{\sigma \in G(k_{(1)}/k)} \chi_0(\sigma) v_k(\delta(\sigma)) \quad \text{if } \mathfrak{f}_0 = (1), \tag{2'}$$

$$L_K(0, \chi) = \frac{1}{q-1} \sum_{\sigma \in G(K_\mathfrak{F}/K)} \chi(\sigma) v_K(\lambda_\mathfrak{F}^\sigma) \quad \text{if } \mathfrak{F} \neq (1), \tag{3}$$

$$L_K(0, \chi) = \frac{1}{h_K(q-1)} \sum_{\sigma \in G(K_{(1)}/K)} \chi(\sigma) v_K(\Delta(\sigma)) \quad \text{if } \mathfrak{F} = (1), \tag{3'}$$

$$L_k(0, \chi_i) = B_{\chi_i} := \sum_{\sigma \in G(k_{\mathfrak{f}_i}/k)} \bar{\chi}_i(\sigma) Z_{\mathfrak{f}_i}(0, \sigma), \quad i = 1, ..., \ell - 1. \tag{4}$$

Suppose that χ is nontrivial. Then

$$\Sigma(\chi) = (q-1) \prod_{\mathfrak{P} | \mathfrak{N}} (1 - (\chi(\mathfrak{P})) L_K(0, \chi). \tag{5}$$

Here $\chi(\mathfrak{P})$ is 0 if $\mathfrak{p}$ divides the conductor of χ and $\chi(\sigma_\mathfrak{P})$ otherwise.

Note that, for $\mathfrak{g} \neq (1)$ and $\mathfrak{f}_0 = (1)$,

$$\sum_{\sigma \in G_\mathfrak{g}} \chi_0(\sigma) v_\infty(\lambda_\mathfrak{g}^\sigma) = \prod_{\mathfrak{p} | \mathfrak{g}} (1 - \chi_0(\mathfrak{p})) L_k(0, \chi_0).$$

Then

$$\Sigma_J^0(\chi) = (q-1)\prod_{\mathfrak{p}|\mathfrak{n}_1}(1-\chi_0(\mathfrak{p}))L_k(0,\chi) \tag{6}$$

and

$$\Sigma_J^i = \prod_{\mathfrak{p}|\mathfrak{n}_1}(1-\chi_i(\mathfrak{p}))L_k(0,\chi_i), \quad i = 1, ..., \ell-1. \tag{7}$$

Using the fact that

$$\prod_{\mathfrak{P}|\mathfrak{N}}(1-\chi(\mathfrak{P})) = \prod_{\mathfrak{p}|\mathfrak{n}_1}\prod_{i=0}^{\ell-1}(1-\chi_i(\mathfrak{p})),$$

we get

$$\Sigma(\chi) = \prod_{i=0}^{\ell-1}\Sigma_J^i(\chi).$$

Now assume that χ is trivial and $\mathfrak{N} \neq (1)$. Then $J = I$ in this case.

Case 1: $\mathfrak{n}_1$ contains at least two prime divisors.

Then so does $\mathfrak{N}$. Hence Φ_K and θ_I are units. Thus $\Sigma(\chi) = 0 = \Sigma_I^0(\chi)$.

Suppose that $\mathfrak{n}_1$ is a power of a prime $\mathfrak{p}$. Let e, f, r be the ramification index, inertia degree and the number of primes over $\mathfrak{p}$ in K, respectively.

Case 2. $\mathfrak{n}_1$ is a power of a prime $\mathfrak{p}$ and $r > 1$.

Then $\mathfrak{N}$ is not a prime power. So $\Sigma(\chi) = 0$. On the other hand, $\Sigma_I^i(\chi)$ contains the factor $(1-\chi_K^i(\mathfrak{p}))$, which is 0 if $f \mid i$ and $e = 1$.

Case 3. $\mathfrak{n}_1$ is a power of $\mathfrak{p}$ and $r = 1$.

Let $\mathfrak{P}$ be the prime ideal of K lying over $\mathfrak{p}$. Then one can show that (cf. [Ha2], (2.3))

$$\Sigma(\chi) = h_K \deg\mathfrak{P} = f h_K \deg\mathfrak{p},$$
$$\Sigma_I^0(\chi) = h_k \deg\mathfrak{p}$$

and

$$\prod_{i=1}^{\ell-1}\Sigma_I^i(\chi) = \left(\prod_{i=1}^{\ell-1}(1-\chi_K^i(\mathfrak{p}))\right)\frac{h_K}{h_k} = f\frac{h_K}{h_k},$$

since

$$\chi_K^i(\mathfrak{p}) = \begin{cases} 0 & \text{if } e > 1 \\ \zeta_f^i & \text{if } e = 1, \end{cases}$$

where ζ_f is a primitive f-th root of 1. Hence we get the result in this case too.

3. Integrality of exponents

Now the question is to know whether $\ell^2 m_J\alpha_J$ is an element of $\mathbb{Z}[G]$. We want to determine $\beta_J^{(i)}(\sigma)$. For this we need more information about $Z_\mathfrak{m}(0,\sigma)$ for an ideal $\mathfrak{m}$ and $\sigma \in G$. It is well-known that $(\mathbb{A}/\mathfrak{m})^* \simeq \mathrm{Gal}(k_\mathfrak{m}/k_\mathfrak{e}) \subset \mathrm{Gal}(k_\mathfrak{m}/k)$. Let X be the image of $\mathbb{F}_q^*$ under this isomorphism. X is called the *sign group* of $G(k_\mathfrak{m}/k)$. Shu [Sh] constructed a set $G'_\mathfrak{m}$ of coset representatives of $G_\mathfrak{m}/X$ and called the elements of $G'_\mathfrak{m}$ *monic*. The following is due to Shu [Sh].

Proposition 5. *Let $\mathfrak{m}$ be an ideal of $\mathbb{A}$ and $\sigma \in G_\mathfrak{m}$.*

a) *The partial zeta function* $Z_{\mathfrak{m}}(s,\sigma)$ *is a rational function in* q^{-s} *and* $(1-q^{1-s})Z_{\mathfrak{m}}(s,\sigma)$ *is a polynomial in* q^{-s} *with integer coefficients.*

b) *For any* $a \neq id \in X$ *and any* $\sigma \in G'_{\mathfrak{m}}$, *we have*

$$Z_{\mathfrak{m}}(s,\sigma) - Z_{\mathfrak{m}}(s,a\sigma) = q^{n(\sigma)}q^{-sj(\sigma)},$$

for some appropriate nonnegative integers $n(\sigma)$ *and* $j(\sigma)$.

Let

$$Y = \{a \in X : \chi_K(a) = 1\}.$$

Corollary 6. *For any* $\sigma \in G_{\mathfrak{m}}$, $\sum_{a\in Y} Z_{\mathfrak{m}}(0,a\sigma)$ *is either* $q^{n(\sigma)} + \frac{D}{\ell}$ *or* $\frac{D}{\ell}$ *for some integer* D.

Proof. There exist integers C and D such that $Z_{\mathfrak{m}}(0,\sigma)$ equals $\frac{C}{q-1}$ if $\sigma \in G'_{\mathfrak{m}}$ and $\frac{D}{q-1}$ otherwise. From Proposition 5 b), $\frac{C}{q-1} - \frac{D}{q-1} = q^{n(\sigma)}$. Then the sum will be either

$$\frac{C}{q-1} + (\frac{q-1}{\ell} - 1)\frac{D}{q-1} = q^{n(\sigma)} + \frac{D}{\ell}$$

or

$$\frac{D}{q-1}\frac{q-1}{\ell} = \frac{D}{\ell}. \qquad \square$$

Write a set of representatives of the quotient group $G_{\mathfrak{f}_J^1}/Y$ by W. Then

$$\sum_{\tau\in G_{\mathfrak{f}_J^1},\, \tau|_{L_J^1}=\sigma} Z_{\mathfrak{f}_J^1}(0,\tau) = \sum_{\tau\in W,\, \tau|_{L_J^1}=\sigma} \left(\sum_{a\in Y} Z_{\mathfrak{f}_J^1}(0,a\tau)\right).$$

Now letting $\mathfrak{m} = \mathfrak{f}_J^1$ in Corollary 6, we see that the sum in the parentheses is either $q^{n(\tau)} + \frac{D}{\ell}$ or $\frac{D}{\ell}$ for some integer D. Thus $\ell\beta_J^{(i)} \in \mathbb{Z}[\zeta][G]$ and $\ell^{\ell-1}\beta_J \in \mathbb{Z}[G]$. Therefore $\Phi_L^{\ell^{\ell-3}[L:L_I]^2}$ is a cyclotomic number over k.

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Department of Mathematics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Republic of Korea

E-mail address: `shbae@math.kaist.ac.kr`

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, People's Republic of China

E-mail address: `lsyin@math.tsinghua.edu.cn`