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HOMOLOGY OF REAL ALGEBRAIC VARIETIES AND MORPHISMS TO SPHERES

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ABSTRACT. Let X and Y be affine nonsingular real algebraic varieties. One of the classical problems in real algebraic geometry is whether a given C^∞ mapping $f : X \to Y$ can be approximated by entire rational mappings in the space of C^∞ mappings. In this work, we obtain some sufficient conditions in the case when Y is the standard sphere S^n.

1. Introduction and the results

Given two nonsingular affine real algebraic varieties X and Y, we regard the set $R(X, Y)$ of all entire rational maps from X into Y as a subset of the space $C^\infty(X, Y)$ of all C^∞ maps from X into Y endowed with the C^∞-topology.

In this study we focus on the question of when C^∞ maps between nonsingular affine real algebraic varieties can be approximated by entire rational maps. If X is compact and nonsingular, as indicated by the classical Stone-Weierstrass approximation theorem, every C^∞ mapping $f : X \to \mathbb{R}^n$ can be approximated by the polynomial maps in $C^\infty(X, \mathbb{R}^n)$. In particular, every C^∞ mapping from X into Euclidean space can be approximated by entire rational maps in the C^∞-topology. The general idea is to try to extend this result to different target spaces. The next natural case is to take the standard n-dimensional unit sphere

$$S^n = \{x_0, ..., x_n \in \mathbb{R}^{n+1} \mid x_0^2 + ... + x_n^2 = 1\}$$

as a target space. In this case, the approximation problem becomes very difficult. There are some positive results in this direction. First, Ivanov proved that the smooth map $f : X \to S^1$ can be approximated by entire rational maps from X to S^1 if and only if $f^*(u)$ belongs to $H^1_{alg}(X, \mathbb{Z}_2)$, where u is a generator of $H^1_{alg}(S^1, \mathbb{Z}_2)$ [9]. After that Bochnak and Kucharz extended this result to S^2 and obtained some partial results for S^4 [3, 5]. There are also some negative results. Loday showed that any polynomial map from T^n to S^n is null homotopic [10]. Bochnak and Kucharz proved that any entire rational map from $X \times S^{2n-k}$ to S^{2n} is null homotopic, where k is the dimension of X and $k < 2n$ [3] (see also [11, 12]).

We examine this approximation problem for maps to spheres that factor through the real or the complex projective spaces. Our main results follow.

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Theorem 1.1. *Let X^{2n} be a nonsingular compact orientable real algebraic variety and $f : X^{2n} \to S^{2n}$ be a continuous map. If there is a cohomology class $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$ such that $u^n = f^*(\alpha)$, where $\alpha \in H^{2n}(S^{2n}, \mathbb{Z})$ is a generator, then f is homotopic to an entire rational map.*

The next theorem gives a partial answer to the converse of the above theorem.

Theorem 1.2. *Let $f : X^{2n} \to S^{2n}$ be a continuous map where X^{2n} is a nonsingular compact orientable real algebraic variety. If there is an entire rational map $\tilde{f} : X \to \mathbb{CP}^n$ such that $\pi \circ \tilde{f}$ is homotopic to f, then there is a cohomology class $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$ such that $u^n = f^*(\alpha)$, where α is a generator of $H^{2n}(S^{2n}, \mathbb{Z})$.*

For a nonorientable real algebraic variety we have the following result.

Theorem 1.3. *Let X^n be a nonorientable, closed, nonsingular variety and $f : X \to S^n$ be a continuous map. If there is some $v \in H^1_{alg}(X, \mathbb{Z}_2)$ such that $v^n = f^*(\alpha)$ and $\alpha \in H^n(S^n, \mathbb{Z}_2)$ is a generator, then f is homotopic to an entire rational map.*

Remark 1.4. Clearly in Theorems 1.1 and 1.3, the assumption of the existence of certain algebraic cohomology classes on X is not necessary since the identity map $id : S^k \to S^k$ is entire rational for any k.

Example 1.5. Let M be a smooth closed orientable manifold of dimension $2n$ and $u \in H^2(M \sharp \mathbb{CP}^n, \mathbb{Z})$ be such that $u^n \in H^{2n}(M \sharp \mathbb{CP}^n, \mathbb{Z})$ is a generator. Then, by Theorem 1.2 of [6], $M \sharp \mathbb{CP}^n$ has an algebraic model X such that $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$. Hence, there are plenty of examples of algebraic varieties satisfying the hypothesis of Theorem 1.1.

Example 1.6. Let M be a smooth closed orientable manifold of dimension n and $w \in H^1(M, \mathbb{Z}_2)$ such that w^n is a generator of $H^n(M, \mathbb{Z}_2)$. Let G be a subgroup of $H^1(M, \mathbb{Z}_2)$ generated by w and $w_1(M)$. Then, by Theorem 4.1 of [7], there exist an algebraic model X of M and a diffeomorphism $h : X \to M$ such that $h^*(G) = H^1_{alg}(X, \mathbb{Z}_2)$.

In general, let N be any smooth manifold of dimension n. Then $M = N \sharp \mathbb{RP}^n$ has a class $w \in H^1(M, \mathbb{Z}_2)$ such that w^n is a generator of $H^n(M, \mathbb{Z}_2)$ and hence by the above paragraph there exists an algebraic model X of M such that $w \in H^1_{alg}(X, \mathbb{Z}_2)$ with w^n a generator of $H^n(X, \mathbb{Z}_2)$.

2. Proofs

All real algebraic varieties under consideration in this report are nonsingular. It is well known that real projective varieties are affine (cf. Proposition 2.4.1 [1] or Theorem 3.4.4 [2]). Moreover, compact affine real algebraic varieties are projective (cf. Corollary 2.5.14 [1]), and therefore we do not distinguish between real compact affine varieties and real projective varieties.

For real algebraic varieties $X \subseteq \mathbb{R}^r$ and $Y \subseteq \mathbb{R}^s$, a map $F : X \to Y$ is said to be entire rational if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots, x_r]$, $i = 1, \ldots, s$, such that each g_i vanishes nowhere on X and $F = (f_1/g_1, \ldots, f_s/g_s)$. We say X and Y are isomorphic if there are entire rational maps $F : X \to Y$ and $G : Y \to X$ such that $F \circ G = id_Y$ and $G \circ F = id_X$. Isomorphic algebraic varieties will be regarded as the same.

An algebraic homology group $H_k^{alg}(X,R)$ ($R=\mathbb{Z}$ or $\mathbb{Z}_2$) is defined as the subgroup of $H_k(X,R)$ generated by the compact real algebraic subsets of X. Define $H^*_{alg}(X,R)$ to be the Poincaré dual of the groups $H_*^{alg}(X,R)$ where it is defined.

For a compact nonsingular affine real algebraic variety X, $H^{2k}_{\mathbb{C}-alg}(X,\mathbb{Z})$, consisting of the elements which are the restriction of the classes in $H^{2k}(X_{\mathbb{C}},\mathbb{Z})$ via the projective nonsingular complexification map $j: X \to X_{\mathbb{C}}$ whose Poincaré dual is represented by complex algebraic cycles is defined to be the subgroup of $H^{2k}(X,\mathbb{Z})$ [4]. We refer the reader for the basic definitions and facts about real algebraic geometry to [1, 2].

First we have a purely topological result.

Lemma 2.1. *Let M be a smooth closed orientable manifold of dimension $2n$ and $f: M \to S^{2n}$ be any smooth map. Then there is a smooth map $\tilde{f}: M \to \mathbb{CP}^n$ such that the diagram*

$$\begin{array}{ccc} & & \mathbb{CP}^n \\ & {\scriptstyle\tilde{f}}\nearrow & \downarrow \pi \\ M & \xrightarrow{f} & S^{2n} \end{array}$$

commutes up to homotopy if and only if there is a cohomology class $u \in H^2(M,\mathbb{Z})$ such that $u^n = f^*(\alpha)$, where $\alpha \in H^{2n}(S^{2n},\mathbb{Z})$ is a generator.

Proof of Lemma 2.1. By the Hopf classification theorem there is a continuous degree one map $\pi: \mathbb{CP}^n \to S^{2n}$ (cf. Theorem 11.6, p. 300 [8]). Next, assume that such an $\tilde{f}$ exists. Then,

$$\begin{aligned} f^*(\alpha) &= (\pi\circ\tilde{f})^*(\alpha) \\ &= (\tilde{f}^*\circ\pi^*)(\alpha) \\ &= \tilde{f}^*(a^n) \qquad (\pi \text{ is a degree one map}) \\ &= (\tilde{f}^*(a))^n \\ &= u^n; \end{aligned}$$

here $a \in H^2(\mathbb{CP}^n,\mathbb{Z})$ is a generator and $u = \tilde{f}^*(a)$. So, one side has been proved.

Conversely, assume that there is a cohomology class $u \in H^2(M,\mathbb{Z})$ in the form of $u^n = f^*(\alpha)$. Let $\tilde{f}: M \to \mathbb{CP}^\infty$, which is the Eilenberg-Mac Lane space $K(\mathbb{Z},2)$, be a map such that $\tilde{f}^*(a) = u$, where $a \in H^2(\mathbb{CP}^\infty,\mathbb{Z})$ is a generator. Since M is $2n$-dimensional, we can change $\tilde{f}$ by a homotopy so that $\tilde{f}(M) \subseteq \mathbb{CP}^n \subseteq \mathbb{CP}^\infty$, where $\mathbb{CP}^n$ is the $2n$-th skeleton of $\mathbb{CP}^\infty$. Now we can assume that $\tilde{f}: M \to \mathbb{CP}^n$ is a map such that $\tilde{f}^*(a) = u$. Then, $(\tilde{f}^*(a))^n = u^n$, where $a^n \in H^{2n}(\mathbb{CP}^n,\mathbb{Z})$. Since $a^n = \pi^*(\alpha)$, we get

$$\begin{aligned} (\pi\circ\tilde{f})^*(\alpha) &= \tilde{f}^*(\pi^*(\alpha)) \\ &= u^n \\ &= f^*(\alpha). \end{aligned}$$

Thus, $\pi\circ\tilde{f}$ and f have the same degree and hence $\pi\circ\tilde{f}$ and f are homotopic. $\square$

Proof of Theorem 1.1. By Lemma 2.1, there is a map $\tilde{f}: X \to \mathbb{CP}^n$ such that $\pi\circ\tilde{f}$ is homotopic to f. The pull-back complex line bundle $\tilde{f}^*(\gamma_{n,1})$, where $(\gamma_{n,1}) \to \mathbb{CP}^n$ is the canonical line bundle over $\mathbb{CP}^n$, is strongly algebraic because its Chern class, $c_1(\tilde{f}^*(\gamma_{n,1})) = u$, is in $H^2_{\mathbb{C}-alg}(X,\mathbb{Z})$ (cf. Remark 5.4 [4]). Now by Theorem 13.3.1

of [2] the map $\tilde{f}$ classifying the pull-back bundle can be homotoped to an entire rational map. $\square$

Proof of Theorem 1.2. Since $\pi : \mathbb{CP}^n \to S^{2n}$ has degree one we have $\pi^*(\alpha) = a^n$, where $a \in H^2(\mathbb{CP}^n, \mathbb{Z})$ is a generator. It is well known that $H^2(\mathbb{CP}^n, \mathbb{Z}) = H^2_{\mathbb{C}-alg}(\mathbb{CP}^n, \mathbb{Z})$. Now, let $u = \tilde{f}^*(a)$. Then, $u \in H^2_{\mathbb{C}-alg}(X, \mathbb{Z})$ because $\tilde{f}$ is an entire rational map. By assumption, $\pi \circ \tilde{f}$ is homotopic to f and hence we get

$$f^*(\alpha) = (\tilde{f})^*\pi^*(\alpha) = \tilde{f}^*(a^n) = (\tilde{f}^*(a))^* = u^n. \qquad \square$$

Next we give a similar proof for Theorem 1.3 using the real projective space instead of the complex projective space. Let $\pi : \mathbb{RP}^n \to S^n$ be an entire rational map defined by

$$\pi([x_0 : ... : x_n]) = \|x\|^{-2}(2x_0x_n, ..., 2x_{n-1}x_n, (\textstyle\sum_{i=0}^{n-1} x_i^2) - x_n^2).$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{RP}^n & \xrightarrow{\pi} & S^n \\ \varphi \uparrow & & \uparrow i \\ \mathbb{R}^n & \xrightarrow{\phi^{-1}} & S^n - (N), \end{array}$$

where $N = (0, 0, ..., 1)$ is the north pole of S^n, ϕ is the stereographic projection, φ is the embedding defined by $\varphi(x_1, ..., x_n) = [x_1 : ... : x_n : 1]$, and i is the inclusion map. We may consider π as an extension of ϕ^{-1} so that $\deg(\pi) = 1$, where we consider the $\mathbb{Z}_2$ degree when n is even.

Lemma 2.2. *Let M^n be a nonorientable manifold and $f : M^n \to S^n$ be a continuous map. Then there is a continuous map $\tilde{f} : M^n \to \mathbb{RP}^n$ such that the diagram*

$$\begin{array}{ccc} & & \mathbb{RP}^n \\ & \tilde{f} \nearrow & \downarrow \pi \\ M^n & \xrightarrow{f} & S^n \end{array}$$

commutes up to homotopy if and only if there is a cohomology class $v \in H^1(M, \mathbb{Z}_2)$ such that $v^n = f^*(\alpha)$, where $\alpha \in H^n(S^n, \mathbb{Z}_2)$ is a generator.

Proof of Lemma 2.2. Assume that there exists an $\tilde{f}$. Then,

$$\begin{aligned} f^*(\alpha) &= (\pi \circ \tilde{f})^*(\alpha) \\ &= (\tilde{f}^* \circ \pi^*)(\alpha) \\ &= \tilde{f}^*(a^n) \qquad (\pi \text{ is a degree one map}) \\ &= (\tilde{f}^*(a))^n \\ &= v^n; \end{aligned}$$

here $a \in H^1(\mathbb{RP}^n, \mathbb{Z}_2)$ is a generator and $v = \tilde{f}^*(a)$.

Conversely, assume that $v \in H^1(M^n, \mathbb{Z}_2)$ such that $v^n = f^*(\alpha)$. Let $\tilde{f} : M^n \to \mathbb{RP}^\infty = K(\mathbb{Z}_2, 1)$ be a map such that $\tilde{f}^*(a) = v$, where $a \in H^1(\mathbb{RP}^\infty, \mathbb{Z}_2)$ is a generator. Since M^n is n-dimensional, we can change $\tilde{f}$ by a homotopy so that $\tilde{f}(M^n) \subseteq \mathbb{RP}^n \subseteq \mathbb{RP}^\infty$, where $\mathbb{RP}^n$ is the n-th skeleton of $\mathbb{RP}^\infty$. Now we assume that $\tilde{f} : M^n \to \mathbb{RP}^n$ is a map such that $\tilde{f}^*(a) = v$, where we can assume $a \in$

$H^1(\mathbb{RP}^n, \mathbb{Z}_2)$. Then, $(\tilde{f}^*(a))^n = v^n$. Since $a^n = \pi^*(\alpha)$, we get

$$\begin{aligned}(\pi \circ \tilde{f})^*(\alpha) &= \tilde{f}^*(\pi^*(\alpha)) \\ &= v^n \\ &= f^*(\alpha).\end{aligned}$$

Thus, we get that $\pi \circ \tilde{f}$ and f have the same $\mathbb{Z}_2$ degree and thus they are homotopic. □

Proof of Theorem 1.3. By Lemma 2.2, there is an $\tilde{f} : X \to \mathbb{RP}^n$ such that $\pi \circ \tilde{f}$ is homotopic to f. The pull-back real line bundle $\tilde{f}^*(\gamma_{n,1})$, where $(\gamma_{n,1}) \to \mathbb{RP}^n$ is the canonical line bundle over $\mathbb{RP}^n$, is strongly algebraic because its Stiefel-Whitney class $w_1(\tilde{f}^*(\gamma_{n,1})) = v$ is in $H^1_{alg}(X, \mathbb{Z}_2)$ (cf. Theorem 12.4.5 [2]). Now by Theorem 13.3.1 of [2], the map $\tilde{f}$ classifying the pull-back bundle can be homotoped to an entire rational map. □

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