PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 2, February 2009, Pages 641–651
S 0002-9939(08)09609-3
Article electronically published on September 4, 2008

THE ESCAPING SET OF A QUASIREGULAR MAPPING

WALTER BERGWEILER, ALASTAIR FLETCHER, JIM LANGLEY, AND JANIS MEYER

(Communicated by Mario Bonk)

ABSTRACT. We show that if the maximum modulus of a quasiregular mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ grows sufficiently rapidly, then there exists a nonempty escaping set $I(f)$ consisting of points whose forward orbits under iteration of f tend to infinity. We also construct a quasiregular mapping for which the closure of $I(f)$ has a bounded component. This stands in contrast to the situation for entire functions in the complex plane, for which all components of the closure of $I(f)$ are unbounded and where it is in fact conjectured that all components of $I(f)$ are unbounded.

1. INTRODUCTION

In the study [1] of the dynamics of nonlinear entire functions $f : \mathbb{C} \to \mathbb{C}$ considerable recent attention has focussed on the escaping set

$$I(f) = \{z \in \mathbb{C} : \lim_{n\to\infty} f^n(z) = \infty\},$$

where $f^1 = f, f^{n+1} = f \circ f^n$ denote the iterates of f. Eremenko [5] proved that if f is transcendental, then $I(f) \neq \emptyset$ and indeed that, in keeping with the nonlinear polynomial case [18], the boundary of $I(f)$ is the Julia set $J(f)$. The proof in [5] that $I(f)$ is nonempty is based on the Wiman-Valiron theory [6].

For transcendental entire functions f, Eremenko went on to prove in [5] that all components of the closure of $I(f)$ are unbounded, and to conjecture that the same is true of $I(f)$ itself. For entire functions with bounded postcritical set this conjecture was proved by Rempe [12], and for the general case it was shown by Rippon and Stallard [16] that $I(f)$ has at least one unbounded component.

In the meromorphic case the set $I(f)$ was first studied by Dominguez [4], who proved that again $I(f) \neq \emptyset$ and $\partial I(f) = J(f)$. For meromorphic f it is possible that all components of $I(f)$ are bounded [4], and the closure of $I(f)$ may have bounded components even if f has only one pole [4, p. 229]. On the other hand $I(f)$ always has at least one unbounded component if the inverse function f^{-1} has a direct transcendental singularity over infinity: this was proved by Bergweiler, Rippon and Stallard [3] by developing an analogue of the Wiman-Valiron theory in the presence of a direct singularity.

Received by the editors February 7, 2008.
2000 *Mathematics Subject Classification.* Primary 30C65, 30C62; Secondary 37F10.
This research was supported by the G.I.F, the German-Israeli Foundation for Scientific Research and Development, Grant G-809-234.6/2003, and the EU Research Training Network CODY (first author); EPSRC grant RA22AP (second and third authors); the ESF Research Networking Programme HCAA (first and third authors); and DFG grant ME 3198/1-1 (fourth author).

The present paper is concerned with the escaping set for quasiregular mappings $f : \mathbb{R}^N \to \mathbb{R}^N$ [15], which represent a natural counterpart in higher real dimensions of analytic functions in the plane, and exhibit many analogous properties, a highlight among these being Rickman's Picard theorem for entire quasiregular maps [13, 15]. For the precise definition and further properties of quasiregular mappings we refer the reader to Rickman's text [15]. Now the iterates of an entire quasiregular map are again quasiregular, and properties such as the existence of periodic points were investigated in [2, 17]. Further, there is increasing interest in the dynamics of quasiregular mappings on the compactification $\overline{\mathbb{R}^N}$ of $\mathbb{R}^N$, although attention has been restricted to mappings which are uniformly quasiregular in the sense that all iterates have a common bound on their dilatation; see [8, Section 21] and [7]. In the absence of this uniform quasiregularity there are evidently some difficulties in extending some concepts of complex dynamics to quasiregular mappings in general, but the escaping set $I(f)$ makes sense nevertheless, and we shall prove the following theorem.

Theorem 1.1. *Let $N \geq 2$ and $K > 1$. Then there exists $J > 1$, depending only on N and K, with the following property.*

Let $R > 0$ and let $f : D_R \to \mathbb{R}^N$ be a K-quasiregular mapping, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set

$$B_R = \{x \in \mathbb{R}^N : R \leq |x| < \infty\}. \tag{1}$$

Assume that f satisfies

$$\liminf_{r\to\infty} \frac{M(r,f)}{r} \geq J, \quad \text{where} \quad M(r,f) = \max\{|f(x)| : |x| = r\}, \tag{2}$$

and define the escaping set by

$$I(f) = \{x \in \mathbb{R}^N : \lim_{n\to\infty} f^n(x) = \infty\}, \quad f^1 = f, \quad f^{n+1} = f \circ f^n. \tag{3}$$

Then $I(f)$ is nonempty. If, in addition, f is K-quasiregular on $\mathbb{R}^N$, then $I(f)$ has an unbounded component.

The proof of Theorem 1.1 is based on the approach of Dominguez [4], as well as that of Rippon and Stallard [16]. A key role is played also by the analogue of Zalcman's lemma [19, 20] developed for quasiregular mappings by Miniowitz [10] (see §2). It seems worth observing that in Theorem 1.1 the hypothesis (2) cannot be replaced by

$$\liminf_{r\to\infty} \frac{M(r,f)}{r} > 1,$$

as the following example shows. Take cylindrical polar coordinates $r\cos\theta, r\sin\theta, x_3$ in $\mathbb{R}^3$, let $\lambda > 0$ and let f be the mapping defined by

$$0 \to 0, \quad (re^{i\theta}, x_3) \to (re^{\lambda\cos\theta + i(\theta+\pi)}, x_3).$$

Then f^2 is given by

$$(re^{i\theta}, x_3) \to (re^{\lambda\cos\theta + \lambda\cos(\theta+\pi) + i(\theta+2\pi)}, x_3)$$

and so is the identity, while since f is C^1 on $\mathbb{R}^3 \setminus \{0\}$ and satisfies $f(2x) = 2f(x)$ it is easy to see that f is quasiconformal on $\mathbb{R}^3$. On the other hand if $f : \mathbb{R}^N \to \mathbb{R}^N$ is quasiregular with an essential singularity at infinity, then $M(r,f)/r \to \infty$ as $r \to \infty$ (see, for example, [2, Lemma 3.4]) so that (2) holds with any $J > 1$.

Next, we show in §6 that there exists a quasiregular mapping f on $\mathbb{R}^2$ with an essential singularity at infinity, such that the closure of $I(f)$ has a bounded component. Thus while the result of [16] that $I(f)$ has at least one unbounded component extends to quasiregular mappings by Theorem 1.1, Eremenko's theorem [5] that all components of the closure of $I(f)$ are unbounded does not.

We remark finally that it is easy to show that if f is quasimeromorphic with infinitely many poles in $\mathbb{R}^N$, then $I(f)$ is nonempty, and for completeness we outline how this is proved in §7, using the "jumping from pole to pole" method [3, 4].

2. Theorems of Rickman and Miniowitz

Let G be a domain in $\mathbb{R}^N$. A continuous mapping $f : G \to \mathbb{R}^N$ is called quasiregular [15] if f belongs to the Sobolev space $W^1_{N,loc}(G)$ and there exists $K \in [1, \infty)$ such that $|f'(x)|^N \leq K J_f$ a.e. in G. Moreover, f is called K-quasiregular if its inner and outer dilatations do not exceed K: for the details and equivalent definitions we refer the reader to [15]. Furthermore, if $f : G \to \overline{\mathbb{R}^N} = \mathbb{R}^N \cup \{\infty\}$ is continuous, then f is called quasimeromorphic [9] if each $x \in G$ has a neighbourhood U_x such that either f or $g \circ f$ is a quasiregular map of U_x into $\mathbb{R}^N$, where g is a sense-preserving Möbius map of $\overline{\mathbb{R}^N}$ with $g(\infty) \in \mathbb{R}^N$.

Rickman proved [13, 15] that given $N \geq 2$ and $K \geq 1$ there exists an integer $C(N, K)$ such that if f is K-quasiregular on $\mathbb{R}^N$ and omits $C(N, K)$ distinct values $a_j \in \mathbb{R}^N$, then f is constant. Here $C(2, K) = 2$ because a quasiregular mapping in $\mathbb{R}^2$ may be written as the composition of a quasiconformal mapping with an entire function, but for $N \geq 3$ this integer $C(N, K)$ in general exceeds 2 [14, 15].

Miniowitz [10] established for quasiregular mappings the following direct analogue of Zalcman's lemma [19, 20]. A family F of K-quasiregular mappings on the unit ball B^N of $\mathbb{R}^N$ is not normal if and only if there exist

$$f_n \in F, \quad x_n \in B^N, \quad x_n \to \hat{x} \in B^N, \quad \rho_n \to 0+$$

and a nonconstant K-quasiregular mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ with the property that $f_n(x_n + \rho_n x) \to f(x)$ locally uniformly in $\mathbb{R}^N$, with respect to the the spherical distance $\chi(x, y)$ on $\mathbb{R}^N$. Using this she established the following analogue of Montel's theorem, in which $C(N, K)$ is the integer from Rickman's theorem [13].

Theorem 2.1 ([10]). *Let $N \geq 2, K > 1, \varepsilon > 0$ and let D be a domain in $\mathbb{R}^N$. Let F be a family of functions K-quasiregular on D with the following property. Each $f \in F$ omits $q = C(N, K)$ values $a_1(f), \dots, a_q(f)$ on D, which may depend on f but satisfy*

$$\chi(a_j(f), a_k(f)) \geq \varepsilon \quad \text{for} \quad j \neq k.$$

Then F is normal on D.

Theorem 2.1 leads at once to the following standard lemma of Schottky type.

Lemma 2.1. *Let $N \geq 2$ and $K > 1$. Then there exists $Q > 2$ with the following property. Let f be K-quasiregular on the set $\{x \in \mathbb{R}^N : 1 < |x| < 4\}$ such that f omits $q = C(N, K)$ values $y_1, \dots, y_q$, with*

$$|y_j| = 4^{j-1}, \quad j = 1, \dots, q.$$

If $\min\{|f(x)| : |x| = 2\} \leq 2$, then $\max\{|f(x)| : |x| = 2\} \leq Q$.

3. Two lemmas needed for Theorem 1.1

We need the following two facts, the first of which is from Newman's book [11, Exercise, p. 84]:

Lemma 3.1. *Let G be a continuum in $\overline{\mathbb{R}^N} = \mathbb{R}^N \cup \{\infty\}$ such that $\infty \in G$, and let H be a component of $\mathbb{R}^N \cap G$. Then H is unbounded.*

This leads to the second fact we need:

Lemma 3.2. *Let E be a continuum in $\overline{\mathbb{R}^N}$ such that $\infty \in E$, and let $g : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous open mapping. Then the preimage*

$$g^{-1}(E) = \{x \in \mathbb{R}^N : g(x) \in E\}$$

cannot have a bounded component.

For completeness we give a proof of Lemma 3.2 in §8.

4. An analogue of Bohr's theorem

Let $f : D_R \to \mathbb{R}^N$ be K-quasiregular, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set B_R in (1), and assume that f satisfies (2) for some $J > 1$. For $0 \le r < s \le \infty$ set

$$A(r, s) = \{x \in \mathbb{R}^N : r < |x| < s\}.$$

Using (2) choose $s_0 > R$ such that

$$M(r, f) > M(R, f) \quad \text{for all} \quad r \ge s_0.$$

Then $M(r, f)$ is strictly increasing on $[s_0, \infty)$ because if $s_0 \le r_1 < r_2 < \infty$ and $M(r_2, f) \le M(r_1, f)$, then $|f(x)|$ has a local maximum at some $\hat{x} \in A(R, r_2)$, which contradicts the fact that nonconstant quasiregular mappings send open sets to open sets [15, Theorem 4.1, p. 16]. Following Dominguez [4] we establish a lemma analogous to Bohr's theorem.

Lemma 4.1. *Let $c = 1/(2Q)$, where Q is the constant of Lemma 2.1. Then for all sufficiently large ρ there exists $L \ge cM(\rho/2, f)$ such that*

$$S(0, L) = \{x \in \mathbb{R}^N : |x| = L\} \subseteq f(A(R, \rho)).$$

Proof. Using (2) let ρ be so large that

$$\rho > 4R \quad \text{and} \quad S = cM(\rho/2, f) > 2T = 4M(R, f), \tag{4}$$

and assume that the assertion of the lemma is false for ρ. Then for $j = 1, \dots, q$, where $q = C(N, K)$ is the integer from Rickman's Picard theorem [13] (see §2), there exists $a_j \in \mathbb{R}^N$ with

$$|a_j| = 4^{j-1} S \quad \text{and} \quad a_j \notin f(A(R, \rho)). \tag{5}$$

Furthermore, there exists $x_1 \in A(R, \rho/2)$ such that $|f(x_1)| = S$. To see this join a point x_0 on $S(0, \rho/2)$ such that $|f(x_0)| = M(\rho/2, f)$ to $S(0, R)$ by a radial segment and use (4) and the fact that $c < 1$. Let G be the component of the set

$$\{x \in \mathbb{R}^N : T < |f(x)| < 2S\}$$

which contains x_1. Then $G \subseteq A(R, \infty)$ by (4). Suppose first that $G \subseteq A(R, \rho/2)$. Then the closure $\overline{G}$ of G lies in $A(R, \rho)$ by (4) again. Choose a geodesic $\sigma \subseteq S(0, S)$ joining $f(x_1)$ to a_1. Let

$$\mu = \inf\{|f(x) - a_1| : x \in \overline{G}, f(x) \in \sigma\}$$

and take $\zeta_n \in \overline{G}$ with $f(\zeta_n) \in \sigma$ and $|f(\zeta_n) - a_1| \to \mu$. Then we may assume that $\zeta_n \to \hat{\zeta} \in \overline{G}$, and we have $f(\hat{\zeta}) \in \sigma$ and so $\hat{\zeta} \in G$. But then the open mapping theorem forces $\mu = |f(\hat{\zeta}) - a_1| = 0$, which contradicts (5).

Thus $G \not\subseteq A(R, \rho/2)$, and this implies, using (4) again, that there must exist x_2 on $S(0, \rho/2)$ such that $|f(x_2)| \le 2S$. By (4) and (5) the function $g(x) = f(x\rho/4)/S$ is K-quasiregular on $A(1,4)$ and omits the q values $y_j = a_j/S$, which satisfy $|y_j| = 4^{j-1}$. Since $|g(4x_2/\rho)| \le 2$, Lemma 2.1 implies that $|g(x)| \le Q$ for $|x| = 2$, which gives

$$M(\rho/2, f) \le QS = QcM(\rho/2, f) = \frac{M(\rho/2, f)}{2},$$

a contradiction. □

5. Proof of Theorem 1.1

Again let $f : D_R \to \mathbb{R}^N$ be K-quasiregular, where $D_R \subseteq \mathbb{R}^N$ is a domain containing the set B_R in (1), but this time assume that f satisfies (2) for some large positive J. Retain the notation of §4. Following Dominguez' method [4] let $\rho_0 > R$ be so large that every $\rho \ge \rho_0$ satisfies the conclusion of Lemma 4.1 and further that with the same constant c as in Lemma 4.1,

$$cM(\rho/2, f) > 4\rho > \rho > M(R, f) \quad \text{for all} \quad \rho \ge \rho_0, \tag{6}$$

which is possible by (2) and the assumption that J is large. Fix $\rho \ge \rho_0$.

Lemma 5.1. *There exist bounded open sets $G_0, G_1, \ldots$ with the following properties.*

(i) *The set $\overline{\mathbb{R}^N} \setminus G_n$ has two components, namely*

$$\widetilde{G}_n = \overline{B(0,R)} = \{x \in \mathbb{R}^N : |x| \le R\}$$

and $G_n^* = A_n$, which satisfies $\infty \in A_n$.

(ii) *We have*

$$\{x \in \mathbb{R}^N : R < |x| \le 2^n \rho\} \subseteq G_n. \tag{7}$$

(iii) *The sets G_n, A_n and $\gamma_n = \partial A_n$ satisfy*

$$\gamma_{n+1} \subseteq f(\gamma_n) \quad \textit{and} \quad f(G_n) \cap A_{n+1} = \emptyset. \tag{8}$$

Proof. The open sets G_n will be constructed inductively. We begin by setting $G_0 = A(R, \rho')$ for some $\rho' > \rho$, so that (7) obviously is satisfied for $n = 0$. It remains to show how to construct G_{n+1} given the existence of $G_0, \ldots, G_n$ for some $n \ge 0$. The fact that f maps open sets to open sets gives

$$\partial f(G_n) \subseteq f(\partial G_n) = f(S(0,R)) \cup f(\gamma_n), \tag{9}$$

using (i) and the definition $\gamma_n = \partial A_n$. By Lemma 4.1, (6) and (7) there exists

$$T_n \ge cM(2^{n-1}\rho, f) > 2^{n+2}\rho \quad \text{with} \quad S(0, T_n) \subseteq f(A(R, 2^n\rho)) \subseteq f(G_n). \tag{10}$$

Now $f(G_n)$ is a bounded open set, so let A_{n+1} be the component of $\overline{\mathbb{R}^N} \setminus f(G_n)$ which contains ∞ and set

$$\gamma_{n+1} = \partial A_{n+1}. \tag{11}$$

Then by (10) we have

$$\gamma_{n+1} \subseteq A_{n+1} \subseteq A(2^{n+2}\rho, \infty), \tag{12}$$

and (6), (9) and (11) imply the first assertion of (8). Let

$$G_{n+1} = \mathbb{R}^N \setminus (\overline{B(0,R)} \cup A_{n+1}).$$

Then (i) is satisfied with n replaced by $n+1$, and the second assertion of (8) follows from the definition of A_{n+1}. Finally (12) shows that (7) is satisfied with n replaced by $n+1$, and so the induction is complete. □

Lemma 5.2. *Let $w \in \gamma_n$. Then there exists $z_n \in \gamma_0$ with $f^n(z_n) = w$ and*

$$f^m(z_n) \in \gamma_m \quad \text{for} \quad m = 0, \ldots, n. \tag{13}$$

Proof. This is easily proved using induction and (8). □

Now take a sequence of points $z_n \in \gamma_0$ satisfying (13). We may assume that (z_n) converges to $\hat{z} \in \gamma_0$, and we have, by (13),

$$f^m(\hat{z}) = \lim_{n\to\infty} f^m(z_n) \in \gamma_m \quad \text{for each } m \geq 0. \tag{14}$$

Using (12) we get $\hat{z} \in I(f)$, and hence $I(f)$ is nonempty. This proves the first assertion of Theorem 1.1.

The second assertion will be established by modifying the method of Rippon and Stallard [16], so assume that f is K-quasiregular in $\mathbb{R}^N$ and take $\hat{z}$ satisfying (14). As before let $A_n = G_n^*$ be the component of $\overline{\mathbb{R}^N} \setminus G_n$ containing ∞, and let L_n be the component of $f^{-n}(A_n)$ containing $\hat{z}$, which is well-defined since $f^n(\hat{z}) \in \gamma_n$ and $\gamma_n = \partial A_n$ by definition.

Lemma 5.3. *L_n is closed and unbounded.*

Proof. L_n is closed since A_n is closed, and L_n is unbounded by Lemma 3.2. □

Lemma 5.4. *We have $L_{n+1} \subseteq L_n$ for $n = 0, 1, \ldots$.*

Proof. Suppose that $f^{n+1}(z') \in A_{n+1}$ but $f^n(z') \notin A_n$. Thus either $|f^n(z')| \leq R$ or $f^n(z') \in G_n$, from which we obtain $f^{n+1}(z') \notin A_{n+1}$, in the first case from (6) and (7) and in the second case from (8), and this is a contradiction. Hence if $z' \in L_{n+1}$, then z' lies in a component of $f^{-n-1}(A_{n+1})$ which contains $\hat{z}$, and this component in turn lies in a component of $f^{-n}(A_n)$. Hence we get $z' \in L_n$. □

We may now write

$$K_n = L_n \cup \{\infty\}, \quad \{\hat{z}, \infty\} \subseteq K_{n+1} \subseteq K_n, \quad \{\hat{z}, \infty\} \subseteq K = \bigcap_{n=0}^{\infty} K_n.$$

Since K_n is compact and connected so is K [11, Theorem 5.3, p. 81]. Let Γ be the component of $K \setminus \{\infty\}$ which contains $\hat{z}$. Then Γ is unbounded by Lemma 3.1. Now for $w \in \Gamma$ we have $w \in L_n$ and so $f^n(w) \in A_n = G_n^*$, so that $w \in I(f)$ by (7). This completes the proof of Theorem 1.1.

We do not know whether the second conclusion of Theorem 1.1 holds if f is only quasiregular on the set B_R in (1), but this seems unlikely. The difficulty is that for large n we cannot control the behaviour of f^n near $S(0, R)$ and so the component L_n in Lemma 5.3 may in principle be bounded.

6. A quasiregular mapping f for which $\overline{I(f)}$ has a bounded component

To show that there exists a quasiregular mapping $f : \mathbb{C} \to \mathbb{C}$ such that the closure of the escaping set $I(f)$ has a bounded component, we begin by constructing a quasiconformal map g with the following properties. For each z in the punctured disc $A := \{z \in \mathbb{C} : 0 < |z| < 1\}$ the iterates g^n satisfy $\lim_{n\to\infty} |g^n(z)| = 1$, and we have $\lim_{n\to\infty} g^n(1/2) = 1$. On the other hand there exist annuli $A_n \subseteq A$ such that g maps A_n onto A_{n+1}, but with sufficient rotation that for each $z \in A_n$ infinitely many of the forward images $g^k(z)$ lie away from 1. A map h is then obtained from g by conjugation with a Möbius map L which sends 1 to ∞, and finally h is interpolated on a sector to ensure that the resulting function has an essential singularity at infinity.

We will use the fact that if p is quasiregular on a domain $D \subseteq \mathbb{C}$ and

$$p_z = \frac{\partial p}{\partial z} = \frac{1}{2}\left(\frac{\partial p}{\partial x} - i\frac{\partial p}{\partial y}\right)$$

is bounded below in modulus on D, and if q is continuous and such that the partial derivatives q_x, q_y are sufficiently small on D, then $p + q$ is quasiregular on D. If $0 \notin D \cup p(D)$ the same property may be applied locally to $\log p$ as a function of $\log z$.

Turning to the detailed construction, we define $a : [1, 2] \to [0, \pi/4]$ by

$$a(r) = \frac{\pi}{4} - \arcsin\left(\frac{\sqrt{2}}{2r}\right).$$

Then an application of the sine rule shows that the line segment

$$\operatorname{Re} z = 1 + \operatorname{Im} z, \quad 1 \le |z| \le 2,$$

is parametrized by $z = re^{ia(r)}$.

For $c > 0$ we define $g : \mathbb{C} \to \mathbb{C}$ as follows. Let $g(0) = 0$ and for $z = re^{it}$ with $r > 0$ and $-\pi \le t \le \pi$ set:

$$g(z) = \begin{cases} \frac{4}{3} r \exp\left(i\left(t + c|\sin t|\right)\right), & 0 < r < \frac{1}{2}; \\ \frac{1}{2-r}\exp\left(i\left(t + c|\sin t| + c(1-r)^2\left|\sin\left(\frac{\pi}{1-r}\right)\right|\right)\right), & \frac{1}{2} \le r < 1; \\ r\exp\left(i\left(t + c(2-r)\sin\left(\frac{|t|-a(r)}{\pi - a(r)}\pi\right)\right)\right), & 1 \le r \le 2,\ a(r) < |t|; \\ r\exp(it), & 1 \le r \le 2,\ |t| \le a(r); \\ r\exp(it), & r > 2. \end{cases}$$

Then g is continuous on $\mathbb{C}$. Moreover, if c is sufficiently small, then g is quasiconformal, and in particular we choose $c < \pi/4$. Note that, by the choice of $a(r)$,

$$g(z) = z \quad \text{if} \quad \operatorname{Re} z \ge |\operatorname{Im} z| + 1. \tag{15}$$

For $n \in \mathbb{N}$ we have

$$g\left(1 - \frac{1}{n+1}\right) = 1 - \frac{1}{n+2}. \tag{16}$$

For $n \in \mathbb{N}$, $n \geq 2$, we consider the annulus

$$A_n := \left\{ z \in \mathbb{C} : 1 - \frac{1}{n+1/4} < |z| < 1 - \frac{1}{n+3/4} \right\}.$$

Then $g(A_n) = A_{n+1}$.

Lemma 6.1. *For each $z \in A_n$ with $\operatorname{Re} z > 0$ there exists $k \in \mathbb{N}$ with $\operatorname{Re} g^k(z) \leq 0$.*

Proof. Let $z \in A_m$ and suppose first that $0 < t := \arg z < \pi/2$. Then

$$\pi > t + \frac{\pi}{2} > t + 2c \geq \arg g(z) \geq t + c \sin t \geq t + \frac{2c}{\pi} t = \left(1 + \frac{2c}{\pi}\right) t. \tag{17}$$

On the other hand, if $-\pi/2 < t = \arg z \leq 0$, then

$$\frac{\pi}{2} > \arg g(z) \geq t + c|\sin t| + \frac{c\sqrt{2}}{2(m+3/4)^2} \geq \left(1 - \frac{2c}{\pi}\right) t + \frac{c'}{(m+1)^2} > -\frac{\pi}{2}, \tag{18}$$

where $c' := \frac{1}{2} c\sqrt{2}$. In particular, (17) and (18) both hold with $\arg g(z)$ the principal argument.

Suppose then that there exists $z \in A_n$ with $\operatorname{Re} g^k(z) > 0$ for all integers $k \geq 0$, and set $t_k = \arg g^k(z) \in (-\pi/2, \pi/2)$. Then $g^k(z) \in A_{n+k}$. If there exists $k \geq 0$ with $0 < t_k < \pi/2$, then by repeated application of (17) we obtain $k' > k$ with $t_{k'} \in (\pi/2, \pi)$, a contradiction. Hence we must have $-\pi/2 < t_k \leq 0$ for all $k \geq 0$. But then repeated application of (18) gives, for large k,

$$t_{k-1} \geq \left(1 - \frac{2c}{\pi}\right)^{k-1} t_0, \quad t_k \geq \left(1 - \frac{2c}{\pi}\right) t_{k-1} + \frac{c'}{(n+k)^2} > 0,$$

again a contradiction. $\square$

With the Möbius transformation

$$L(z) = \frac{1}{1-z}$$

we now consider the map $h := L \circ g \circ L^{-1}$. Then h is a quasiconformal self-map of the plane. Moreover, (15) gives $h(z) = z$ if $\operatorname{Re} L^{-1}(z) \geq |\operatorname{Im} L^{-1}(z)| + 1$, which is equivalent to $\operatorname{Re} z \leq -|\operatorname{Im} z|$, and we have

$$L(A_n) \subseteq \{z \in \mathbb{C} : \operatorname{Re} z > 0\} \quad \text{and} \quad h(L(A_n)) = L(A_{n+1}), \tag{19}$$

using the fact that $g(A_n) = A_{n+1}$.

It follows from (16) that

$$h(n+1) = n+2 \quad \text{for} \quad n \in \mathbb{N}, \tag{20}$$

and we deduce at once that $2 \in I(h)$. Next we show that $L(A_n) \cap I(h) = \emptyset$ for every integer $n \geq 2$. In fact, suppose that $n \geq 2$ and $u \in L(A_n) \cap I(h)$. Then there exists $j_0 \in \mathbb{N}$ such that $|h^j(u)| > 1$ for $j \geq j_0$. Put $w := h^{j_0}(u)$ and $m := n + j_0$. Then $L^{-1}(w) \in A_m$ by (19), and Lemma 6.1 gives $k \geq 0$ with $\operatorname{Re} g^k\left(L^{-1}(w)\right) \leq 0$. Since $|L(z)| \leq 1$ for $\operatorname{Re} z \leq 0$ we deduce that

$$\left|h^{k+j_0}(u)\right| = \left|h^k(w)\right| = \left|L\left(g^k\left(L^{-1}(w)\right)\right)\right| \leq 1,$$

contradicting the choice of j_0. Thus $L(A_n) \cap I(h) = \emptyset$.

Since A_2 separates $\frac{1}{2}$ from 1 it follows that 2 lies in the bounded component of the complement of $L(A_2)$, and we deduce that the component of $\overline{I(h)}$ containing 2 is bounded.

To construct a quasiregular map $f : \mathbb{C} \to \mathbb{C}$ with an essential singularity at ∞ for which the closure of $I(f)$ has a bounded component, we put $f(z) = h(z)$ for $\operatorname{Re} z \geq -|\operatorname{Im} z|$ and $f(z) = z + d \exp\left(z^4\right)$ for $\operatorname{Re} z \leq -|\operatorname{Im} z| - 1$, where d is a small positive constant. In the remaining region Ω we define f by interpolation, using

$$f(z) = z - d\phi(z), \quad \phi(z) = (\operatorname{Re} z + |\operatorname{Im} z|) \exp\left(z^4\right) \quad \text{for} \quad -1 < \operatorname{Re} z + |\operatorname{Im} z| < 0.$$

Since $\exp\left(z^4\right)$ tends to 0 rapidly as z tends to infinity in Ω, it is then clear that the partial derivatives of ϕ are bounded on Ω, so that f is quasiregular on Ω because d is small.

In particular we have $f(z) = h(z)$ for $\operatorname{Re} z > 0$, and so it follows from (20) that $2 \in I(f)$, whereas $L(A_n) \cap I(f)$ is again empty using (19). Thus the component of $\overline{I(f)}$ containing 2 is bounded.

7. The quasimeromorphic case

Let f be nonconstant and K-quasimeromorphic on the set B_R defined in (1), with a sequence of poles tending to ∞, and set $R_{-1} = R$. Choose x_j, D_j, R_j for $j = 0, 1, 2, \ldots$ as follows. Each x_j is a pole of f, and D_j is a bounded component of the set $\{x \in B_R : R_j < |f(x)| \leq \infty\}$ which contains x_j but no other pole of f, such that D_j is mapped by f onto $\{y \in \overline{\mathbb{R}^N} : R_j < |y| \leq \infty\}$. Moreover, by choosing R_{j+1} and x_{j+1} sufficiently large, we may ensure that

$$|x_{j+1}| > 4R_j \quad \text{and} \quad D_{j+1} \subseteq \{x \in \mathbb{R}^N : 2R_j < |x| < \infty\} \quad \text{for} \quad j \geq -1. \tag{21}$$

Since $|f(x)| = R_j$ for all $x \in \partial D_j$ we may write, for $j \geq 0$, using (21),

$$C_j = \{x \in D_j : f(x) \in D_{j+1}\} \subseteq \overline{C_j} \subseteq D_j. \tag{22}$$

Now set

$$X_0 = \overline{C_0}, \quad X_{j+1} = \{x \in X_j : f^{j+1}(x) \in \overline{C_{j+1}}\}. \tag{23}$$

Evidently X_0 is compact. Assuming that X_j is compact, it then follows that X_{j+1} is the intersection of a compact set with the closed set $f^{-j-1}(\overline{C_{j+1}})$ and so is compact. Hence the X_j form a nested sequence of compact sets. We assert that

$$f^j(X_j) = \overline{C_j}. \tag{24}$$

We clearly have $f^j(X_j) \subseteq \overline{C_j}$ by (23), and (24) is obviously true for $j = 0$, so assume the assertion for some $j \geq 0$ and take $w \in \overline{C_{j+1}}$. Since f maps D_j onto $\{y \in \overline{\mathbb{R}^N} : R_j < |y| \leq \infty\}$, it follows from (21) and (22) that there exists $v \in C_j$ with $f(v) = w$. Hence there exists $x \in X_j$ with $f^j(x) = v$ and $f^{j+1}(x) = w$, completing the induction.

Again since f maps D_j onto $\{y \in \overline{\mathbb{R}^N} : R_j < |y| \leq \infty\}$, we evidently have $C_j \neq \emptyset$ and so X_j is nonempty by (24). Hence there exists x lying in the intersection of the X_j, so that $f^j(x) \in \overline{C_j}$ and $x \in I(f)$ by (21) and (22).

8. Proof of Lemma 3.2

To establish Lemma 3.2 let E and g be as in the statement and assume that $g^{-1}(E)$ is nonempty since otherwise there is nothing to prove. Note first that $g^{-1}(E)$ is a closed subset of $\mathbb{R}^N$ by continuity. Thus

$$F = g^{-1}(E) \cup \{\infty\}$$

is a compact subset of $\overline{\mathbb{R}^N}$. In order to prove Lemma 3.2 it therefore suffices in view of Lemma 3.1 to show that F is connected. Suppose that this is not the case. Then there is a partition of F into nonempty disjoint relatively closed (and so closed) sets H_1, H_2 such that $\infty \in H_2$. Let $W = \mathbb{R}^N \setminus H_2$. Then W is an open subset of $\mathbb{R}^N$, and $g(W \setminus H_1) \cap E = \emptyset$. Moreover, H_1 is a closed subset of $\overline{\mathbb{R}^N}$ and thus compact, and hence a compact subset of $\mathbb{R}^N$ since $\infty \in H_2$. Thus $g(H_1)$ is compact and thus a nonempty closed subset of E.

Now suppose that there exist $y_n \in E \setminus g(H_1)$ with $y_n \to \widetilde{y} \notin E \setminus g(H_1)$. Since E is compact we have $\widetilde{y} \in E$ and so $\widetilde{y} \in g(H_1)$. Hence there exists $\widetilde{x} \in H_1$ with $g(\widetilde{x}) = \widetilde{y}$ and for large enough n there exists x_n close to $\widetilde{x}$ with $g(x_n) = y_n \in E \setminus g(H_1)$. But then we must have $x_n \in H_1$, since $g(W \setminus H_1) \cap E = \emptyset$, and this is a contradiction. So $E \setminus g(H_1)$ is also closed, but evidently nonempty since $g(\mathbb{R}^N) \subseteq \mathbb{R}^N$ and $\infty \in E$, which contradicts the hypothesis that E is connected.

References

[1] W. Bergweiler, Iteration of meromorphic functions, Bull. Amer. Math. Soc. 29 (1993), 151-188. MR1216719 (94c:30033)

[2] W. Bergweiler, Fixed points of composite entire and quasiregular maps, Ann. Acad. Sci. Fenn. Math. 31 (2006), 523-540. MR2248829 (2007g:30043)

[3] W. Bergweiler, P.J. Rippon and G.M. Stallard, Dynamics of meromorphic functions with direct or logarithmic singularities, Proc. London Math. Soc. 2008; doi: 10.1112/plms/pdn007.

[4] P. Dominguez, Dynamics of transcendental meromorphic functions, Ann. Acad. Sci. Fenn. Math. 23 (1998), 225-250. MR1601879 (99b:30031)

[5] A. Eremenko, On the iteration of entire functions, Dynamical Systems and Ergodic Theory, Banach Center Publ. 23, Polish Scientific Publishers, Warsaw, 1989, 339-345. MR1102727 (92c:30027)

[6] W.K. Hayman, The local growth of power series: A survey of the Wiman-Valiron method, Canad. Math. Bull. 17 (1974), 317-358. MR0385095 (52:5965)

[7] A. Hinkkanen, G.J. Martin and V. Mayer, Local dynamics of uniformly quasiregular mappings, Math. Scand. 95 (2004), 80-100. MR2091483 (2005f:37094)

[8] T. Iwaniec and G. Martin, Geometric function theory and non-linear analysis, Clarendon Press, Oxford University Press, New York, 2001. MR1859913 (2003c:30001)

[9] O. Martio, S. Rickman and J. Väisälä, Distortion and singularities of quasiregular mappings, Ann. Acad. Sci. Fenn. Ser. A I 465 (1970), 1-13. MR0267093 (42:1995)

[10] R. Miniowitz, Normal families of quasimeromorphic mappings, Proc. Amer. Math. Soc. 84 (1982), 35-43. MR633273 (83c:30026)

[11] M.H.A. Newman, Elements of the topology of plane sets of points, Cambridge, At the University Press, 1951. MR0044820 (13:483a)

[12] L. Rempe, On a question of Eremenko concerning escaping components of entire functions, Bull. London Math. Soc. 39 (2007), 661-666. MR2346947

[13] S. Rickman, On the number of omitted values of entire quasiregular mappings, J. Analyse Math. 37 (1980), 100-117. MR583633 (81m:30030)

[14] S. Rickman, The analogue of Picard's theorem for quasiregular mappings in dimension three, Acta Math. 154 (1985), 195-242. MR781587 (86h:30039)

[15] S. Rickman, Quasiregular mappings, Ergebnisse der Mathematik und ihrer Grenzgebiete 26, Springer-Verlag, Berlin, 1993. MR1238941 (95g:30026)

[16] P.J. Rippon and G.M. Stallard, On questions of Fatou and Eremenko, Proc. Amer. Math. Soc. 133 (2005), 1119-1126. MR2117213 (2005j:37069)

[17] H. Siebert, Fixed points and normal families of quasiregular mappings, J. Analyse Math. 98 (2006), 145-168. MR2254483 (2007e:30023)

[18] N. Steinmetz, Rational iteration, de Gruyter Studies in Mathematics 16, Walter de Gruyter, Berlin-New York, 1993. MR1224235 (94h:30035)

[19] L. Zalcman, A heuristic principle in complex function theory, Amer. Math. Monthly 82 (1975), 813-817. MR0379852 (52:757)

[20] L. Zalcman, Normal families: New perspectives, Bull. Amer. Math. Soc. (N.S.) 35 (1998), 215-230. MR1624862 (99g:30048)

Mathematisches Seminar, Christian–Albrechts–Universität zu Kiel, Ludewig–Meyn–Str. 4, D–24098 Kiel, Germany

E-mail address: bergweiler@math.uni-kiel.de

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom

E-mail address: alastair.fletcher@nottingham.ac.uk

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom

E-mail address: jkl@maths.nott.ac.uk

School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom

E-mail address: janis.meyer@maths.nottingham.ac.uk