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ON AMALGAMATIONS OF HEEGAARD SPLITTINGS WITH HIGH DISTANCE

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ABSTRACT. Let M be a compact, orientable 3-manifold and F an essential closed surface which cuts M into M_1 and M_2. Suppose that M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with distance $D(S_i) \geqslant 2g(M_i)+1$, $i=1,2$. Then $g(M) = g(M_1)+g(M_2)-g(F)$, and the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is the unique minimal Heegaard splitting of M up to isotopy.

1. INTRODUCTION

Let M_i be a connected, compact, orientable 3-manifold, F_i an essential boundary component of M_i with $g(F_i) \geqslant 1$, $i=1,2$, and $F_1 \cong F_2$. Let $\varphi : F_1 \to F_2$ be a homeomorphism, and $M = M_1 \cup_\varphi M_2$. Suppose $V_i \cup_{S_i} W_i$ is a Heegaard splitting of M_i $(i=1,2)$. Then $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ induce a natural Heegaard splitting $V \cup_S W$ of M with $g(S) = g(S_1)+g(S_2)-g(F)$, which is called the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ along F_1 and F_2. Clearly, $g(M) \leqslant g(M_1)+g(M_2)-g(F)$.

There exist examples which show that an amalgamation of two minimal genus Heegaard splittings of M_1 and M_2 is stabilized (refer to [1], [8], etc.). On the other hand, it has been shown that under some conditions on the manifolds and the gluing maps, the equality $g(M) = g(M_1)+g(M_2)-g(F)$ holds; see [10], [11], [17], etc.

The concept of Hempel's Heegaard distance of a Heegaard splitting ([5]) is a natural generalization of the concept of Casson-Gordon's weakly reducible Heegaard splitting ([3]); its relations to the genus of the Heegaard splitting have been discussed in [4], [6], [14], etc. For a Heegaard splitting $V \cup_S W$, we use $D(S)$ to denote the Heegaard distance of $V \cup_S W$.

Recently, Kobayashi and Qiu ([9]) proved the following theorem:

Theorem 1.0. *Let M be a connected, compact, orientable 3-manifold, and F an essential closed surface which cuts M into two 3-manifolds M_1 and M_2. Suppose that M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $D(S_i) \geqslant 2(g(M_1)+g(M_2)-g(F))$, $i=1,2$. Then M has a unique minimal Heegaard splitting up to isotopy, i.e. the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$.*

The main result of this paper is as follows:

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Theorem 1.1. *Let M_i be a connected, compact, orientable 3-manifold, F_i an essential boundary component of M_i with $g(F_i) \geqslant 1$, $i = 1, 2$, and $F_1 \cong F_2$. Let $\varphi : F_1 \to F_2$ be a homeomorphism, and $M = M_1 \cup_\varphi M_2$, $F = F_2 = \varphi(F_1)$. Suppose M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $D(S_i) \geqslant 2g(M_i) + 1$, $i = 1, 2$. Then the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ is the unique minimal Heegaard splitting of M up to isotopy. In particular, it is unstabilized.*

As a direct consequence of Theorem 1.1, we have

Corollary 1.2. *Under the conditions as in Theorem* 1.1*, the minimal Heegaard splitting of M is weakly reducible.*

The paper is organized as follows. In section 2, we introduce some preliminaries, lemmas and propositions. The main part of section 2 is to prove Proposition 2.5, which is a stronger version of Lemma 3.3 in [2]. In section 3, we first prove some results that will be used in the proof of Theorem 1.1, and then give a proof of Theorem 1.1, where Proposition 2.5 plays a key role in our proofs.

The concepts and terminologies which are not defined in the paper are standard; see, for example, [5], [7].

2. Preliminaries

In this section, we will review some fundamental definitions and facts on surfaces in 3-manifolds.

Let F be either a properly embedded connected surface in a 3-manifold M or a subsurface of ∂M. If there is an essential curve in F which bounds a disk in M or F is a 2-sphere which bounds a 3-ball in M, then we say F is *compressible* in M. Otherwise, F is *incompressible* in M. If F is an incompressible surface in M and not parallel to a subsurface of ∂M, then F is an *essential* surface in M. When F is not connected, then F is said to be incompressible if each component of F is incompressible. F is said to be essential if F is incompressible and at least one component of F is essential in M.

Let F be a properly embedded connected surface in a 3-manifold M. If there is an essential arc α in F and an arc β in ∂M such that $\alpha \cap \beta = \partial\alpha = \partial\beta$ and $\alpha \cup \beta$ bounds a disk Δ in M, then F is said to be *∂-compressible* in M.

A *compression body* is a 3-manifold V obtained from a connected closed orientable surface S by attaching some 2-handles to $S \times \{0\} \subset S \times I$ and capping off any resulting 2-sphere boundary components. We denote $S \times \{1\}$ by $\partial_+ V$ and $\partial V - \partial_+ V$ by $\partial_- V$. An essential disk for V is a compressing disk of $\partial_+ V$ in V.

A *Heegaard splitting* of a 3-manifold M is a decomposition $M = V \cup_S W$ of M in which V and W are compression bodies such that $V \cap W = \partial_+ V = \partial_+ W = S$ and $M = V \cup W$. S is called a *Heegaard surface* of M. The genus $g(S)$ of S is called the *genus* of the splitting $V \cup_S W$. We use $g(M)$ to denote the *Heegaard genus* of M, which is equal to the minimal genus of all Heegaard splittings of M. A Heegaard splitting $V \cup_S W$ for M is *minimal* if $g(S) = g(M)$.

Let $V \cup_S W$ be a Heegaard splitting. $V \cup_S W$ is *reducible (weakly reducible, or stabilized*, respectively) if there are essential disks $D_1 \subset V$ and $D_2 \subset W$ such that $\partial D_1 = \partial D_2$ ($\partial D_1 \cap \partial D_2 = \emptyset$, or $|\partial D_1 \cap \partial D_2| = 1$, respectively). Otherwise, $V \cup_S W$ is *irreducible (strongly irreducible, unstabilized*, respectively).

A *generalized* Heegaard splitting for a 3-manifold M is a structure $M = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m)$, where each $V_i \cup_{S_i} W_i$ is a Heegaard splitting, and $\{M_i = V_i \cup_{S_i} W_i, 1 \leqslant i \leqslant m\}$ is a union of submanifolds of M.

It was shown by Scharlemann and Thompson [12] that any irreducible Heegaard splitting $M = V \cup_S W$ can be broken up into a series of strongly irreducible Heegaard splittings by rearranging the order of adding the 1-handles and 2-handles as

$$M = V \cup_S W = (V_1 \cup_{S_1} W_1) \cup_{F_1} (V_2 \cup_{S_2} W_2) \cup_{F_2} \cdots \cup_{F_{m-1}} (V_m \cup_{S_m} W_m),$$

such that each $V_i \cup_{S_i} W_i$ is a strongly irreducible Heegaard splitting with $\partial_- W_i \cap \partial_- V_{i+1} = F_i$, $1 \leqslant i \leqslant m-1$, $\partial_- V_1 = \partial_- V$, $\partial_- W_m = \partial_- W$, and for each i, each component of F_i is a closed incompressible surface of positive genus, and only one component of $M_i = V_i \cup_{S_i} W_i$ is not a product, and none of the compression bodies V_i, W_{i-1}, $2 \leqslant i \leqslant m$, is trivial. Such a rearrangement of handles is called an *untelescoping* of the Heegaard splitting $V \cup_S W$. Then it is easy to see $g(S) \geqslant g(S_i), g(F_i)$ for each i, and when $m \geqslant 2$, $g(S) > g(S_i), g(F_i)$ for each i. In fact, $\chi(S) = \sum_{i=1}^{m} \chi(S_i) - \sum_{i=1}^{m-1} \chi(F_i)$.

Let $M = V \cup_S W$ be a Heegaard splitting, α and β be two essential simple closed curves in S. The *distance* $d(\alpha, \beta)$ of α and β is the smallest integer $n \geqslant 0$ such that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \cdots, \alpha_n = \beta$ in S with $\alpha_{i-1} \cap \alpha_i = \emptyset$, for $1 \leqslant i \leqslant n$. The *distance* of the Heegaard splitting $V \cup_S W$ is defined to be $D(S) = \min\{d(\alpha, \beta)\}$, where α bounds an essential disk in V and β bounds an essential disk in W.

$D(S)$ was first defined by Hempel [6]. It is clear that $V \cup_S W$ is reducible if and only if $D(S) = 0$, and $V \cup_S W$ is weakly reducible if and only if $D(S) \leqslant 1$.

Next we introduce some basic results on Heegaard splittings and the distance of a Heegaard splitting.

Lemma 2.1. *Let V be a compression body and F be a properly embedded incompressible surface in V with $\partial F \subset \partial_+ V$. Then each component of $V \backslash F$ is a compression body.*

The proof of Lemma 2.1 can be found in [15].

Lemma 2.2. *Let $M = V \cup_S W$ be a strongly irreducible Heegaard splitting. If α is an essential simple loop in S which bounds a disk D in M such that D is transverse to S, then α bounds an essential disk in V or W.*

The proof of Lemma 2.2 can be found in [13].

Lemma 2.3. *Let $V \cup_S W$ be a Heegaard splitting of M and F be a properly embedded incompressible surface (maybe not connected) in M. Then any component of F is parallel to ∂M or $D(S) \leqslant 2 - \chi(F)$.*

The proof of Lemma 2.3 can be found in [4].

Lemma 2.4. *Let $M = V \cup_S W$ and $M = V' \cup_{S'} W'$ be two different Heegaard splittings. Then $V' \cup_{S'} W'$ is a stabilization of $V \cup_S W$ or $D(S) \leqslant 2g(S')$.*

The proof of Lemma 2.4 can be found in [14].

The following proposition is a stronger version of Lemma 3.3 in [2].

Proposition 2.5. *Let $M = V \cup_S W$ be a non-trivial strongly irreducible Heegaard splitting and F be a 2-sided essential surface (not a disk or 2-sphere) in M. Then F can be isotoped such that*

(1) *each component of $S \cap F$ is an essential loop in both F and S;*
(2) *at most one component of $S \backslash F$ is compressible in $M \backslash F$.*

Proof. (1) is due to Schultens [16].

If (2) is not true, then at least two components of $S \backslash F$ are compressible in $M \backslash F$ and by Lemma 2.2, at least two components of $S \backslash F$ are compressible in V or W. Since $V \cup_S W$ is strongly irreducible, we may assume that at least two components of $S \backslash F$ are compressible in V and any component of $S \backslash F$ is incompressible in W. Choose an essential disk D of W and isotope F if necessary so that $|D \cap (F \cap W)|$ is minimal subject to the conditions that any component of $S \cap F$ is essential in both F and S, and at least two components of $S \backslash F$ are compressible in V.

Since $V \cup_S W$ is strongly irreducible, $D \cap (F \cap W) \neq \emptyset$. By the standard innermost circle argument, we know that $D \cap (F \cap W)$ has no circle component. Let α be an outermost arc of $D \cap (F \cap W)$ in D and $\triangle$ be the corresponding outermost disk. We denote $\overline{\partial \triangle - \alpha}$ by β. α is an essential arc in $F \cap W$ by the minimality of $|D \cap (F \cap W)|$. β is an essential arc in $S \backslash F$, too. Otherwise, there is an arc γ in $S \backslash F$ with $\gamma \cap \beta = \partial \gamma = \partial \beta$ and $\beta \cup \gamma$ bounds a disk $\triangle'$. Then either $\triangle \cup \triangle'$ is a compressing disk of F, a contradiction, or $\alpha \cup \gamma$ is trivial, contradicting the minimality of $|D \cap (F \cap W)|$.

If the component P of $F \cap W$ containing α is not an annulus, then ∂-compress P along $\triangle$ to get F^*, which is, isotopic to F. Any component of $F^* \cap S$ is essential in both S and F^*. At least one component of $S \backslash F^*$ is compressible in V since at least two components of $S \backslash F$ are compressible in V. If only one component of $S \backslash F^*$ is compressible, then Proposition 2.5 (2) is true. If at least two components of $S \backslash F^*$ are compressible in V, we have $|D \cap (F^* \cap W)| < |D \cap (F \cap W)|$, again a contradiction to the minimality of $|D \cap (F \cap W)|$.

Now assume P is an annulus. P is not parallel to any component of $S \backslash F$. Otherwise, pushing P from W into V, this corresponds to an isotopy of F, denoted by F^*, too. Then any component of $F^* \cap S$ is essential in both S and F^*. At least one component of $S \backslash F^*$ is compressible in V. Then by the same argument as above, either Proposition 2.5 (2) is true or we get a contradiction.

So P is an essential annulus in W. We ∂-compress P along $\triangle$ to get an essential disk E with $E \cap F = \emptyset$ in W. At least two components of $S \backslash F$ are compressible in V. This is a contradiction to the assumption that $V \cup_S W$ is strongly irreducible.

This completes the proof. □

3. The main results and proofs

First, we have

Theorem 3.1. *Let M be a compact, orientable 3-manifold and F be an essential closed surface which cuts M into M_1 and M_2. If M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $D(S_i) \geqslant 2g(M_i) + 1$, $i = 1, 2$, and $V \cup_S W$ is a Heegaard splitting of M with $g(S) \leqslant g(M_1) + g(M_2) - g(F)$, then $V \cup_S W$ is weakly reducible.*

Proof. Suppose $V \cup_S W$ is a strongly irreducible Heegaard splitting. F is essential in M, so $F \cap S \neq \emptyset$. Then by Proposition 2.5, we may assume that $F \cap S$ consists of loops which are essential in both F and S and at most one component of $S \backslash F$ is compressible in W or V, so in M_1 or M_2. With no loss of generality, we assume any component of $S \cap M_1$ is incompressible. Thus any component of $S \cap M_1$ is essential in M_1. By Lemma 2.3, $2 - \chi(S \cap M_1) \geqslant D(S_1) \geqslant 2g(M_1) + 1$.

By assumption, $g(S) \leqslant g(M_1) + g(M_2) - g(F)$. So

$$\begin{aligned}\chi(S \cap M_1) + \chi(S \cap M_2) &= \chi(S) = 2 - 2g(S) \\ &\geqslant 2 - 2(g(M_1) + g(M_2) - g(F)).\end{aligned}$$

Therefore,

$$\begin{aligned}-\chi(S \cap M_2) &\leqslant \chi(S \cap M_1) + 2(g(M_1) + g(M_2) - g(F)) - 2 \\ &\leqslant 2g(M_2) - 2g(F) - 1.\end{aligned}$$

By Proposition 2.5, $S \cap M_2$ has at most one component which is compressible in M_2, and since $2 - \chi(S \cap M_2) \leqslant 2g(M_2) - 2g(F) + 1 < 2g(M_2) < D(S_2)$, any incompressible component of $S \cap M_2$ is parallel to a subsurface of F in M_2.

If $S \cap M_2$ is incompressible, then we can isotope F such that $F \cap S = \emptyset$, a contradiction. So $S \cap M_2$ has only one component Q which is compressible in V or W, say V. We compress Q as much as possible in V, and the resulting surface is denoted by Q^*. Then Q^* is incompressible in M_2 since $V \cup_S W$ is strongly irreducible. Since $2 - \chi(Q^*) \leqslant 2 - \chi(S \cap M_2) < D(S_2)$, Q^* is parallel to the subsurfaces of F; see Figure 1.

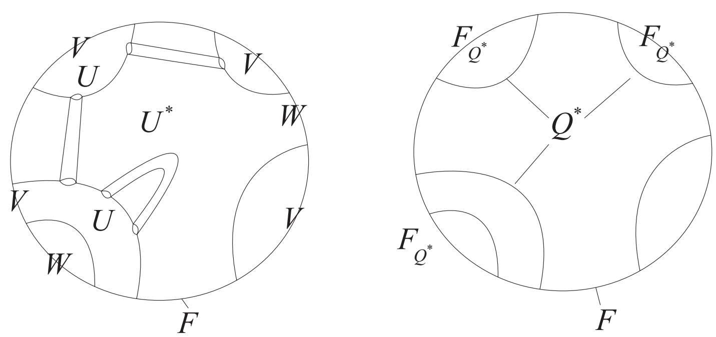

FIGURE 1

Obviously, any component of $F \cap V$ is incompressible in V and any component of $F \cap W$ is incompressible in W. Then by Lemma 2.1, any component of $V \cap M_2$ and $W \cap M_2$ is a compression body. We denote the component of $V \backslash F$ which contains the component Q by U and the component of $W \backslash F$ which contains the component Q by U^*. Then $U \cup_Q U^*$ is homeomorphic to M_2 since any incompressible component of $S \cap M_2$ is parallel to F in M_2. For any component A of Q^*, let F_A be the subsurface of F which is parallel to A with $\partial A = \partial F_A$ and $F_{Q^*} = \{F_A : A \in Q^*\}$. If there are two components A and B of Q^* such that $F_A \subseteq F_B$, then set $\mathcal{A}_1 = \{A' : A' \in Q^*, F_{A'} \subset F_B, F_{A'} \neq F_B\}$ and $\mathcal{A}_2 = \{A' : A' \in Q^*, F_{A'} \cap F_B = \emptyset\}$, and we may assume that Q is compressed into Q^* in V by cutting Q open along a collection $\mathcal{D} = \{D_1, \cdots, D_n\}$ of pairwise disjoint compressing disks in V. We claim that $\mathcal{A}_2 = \emptyset$. Otherwise, since Q is connected, there must exist $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$, and $D_{p_1}, D_{p_2} \in \mathcal{D}$ such that the two cutting sections of D_{p_i} lie in A_i and B respectively, $i = 1, 2$. But this contradicts the assumption that Q is separating. So $\mathcal{A}_2 = \emptyset$. Then $M_2 \cong R$ is a compression body, where R, A and B are shown as in Figure 2 and $V_2 \cup_{S_2} W_2$ is weakly reducible, a contradiction to the fact that $D(S_2) \geqslant 2g(M_2) + 1$.

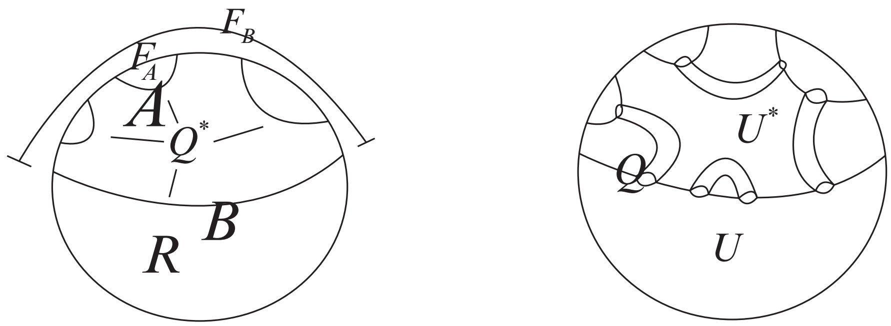

FIGURE 2

Let $C = U \cup N(F \cap U^*, U^*)$ and $C^* = U^* \backslash N(F \cap U^*, U^*)$. Then C is a compression body and C^* is a compression body with $\partial_+ C = \partial_+ C^* = S^*$ and $C \cup_{S^*} C^*$ is a Heegaard splitting of $U \cup_Q U^* = M_2$; see Figure 3.

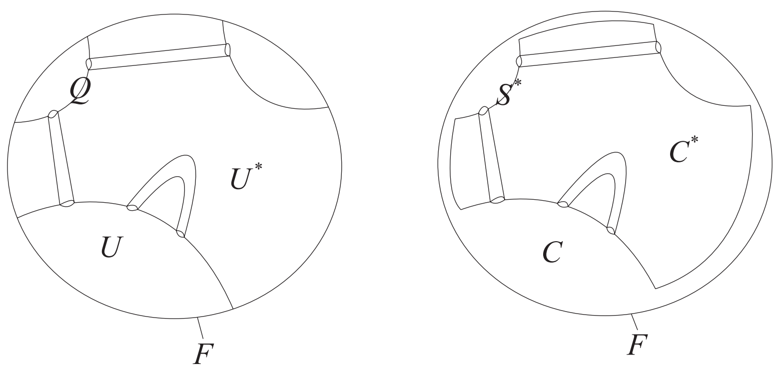

FIGURE 3

Clearly, $2g(S^*) = 2 - \chi(S^*) \leqslant 2 - \chi(Q) - \chi(F) \leqslant 2 - \chi(S \cap M_2) - \chi(F)$.

Note that we have proved that $-\chi(S \cap M_2) \leqslant 2g(M_2) - 2g(F) - 1$. Thus $2g(S^*) \leqslant 2 + 2g(M_2) - 2g(F) - 1 + 2g(F) - 2 = 2g(M_2) - 1$. So $g(S^*) < g(M_2)$, a contradiction.

□

Proposition 3.2. *Let M be a compact, orientable 3-manifold and F be an essential closed surface which cuts M into M_1 and M_2. If M_i has a Heegaard splitting $V_i \cup_{S_i} W_i$ with $D(S_i) \geqslant 2g(M_i) + 1$, $i = 1, 2$, then for any closed incompressible surface F^* in M with $g(F^*) < g(M_1) + g(M_2)$, we can isotope F in M such that $F \cap F^* = \emptyset$.*

Proof. Since F and F^* are incompressible, we can isotope F such that any component of $F \cap F^*$ is essential in both F and F^*. Suppose $|F \cap F^*|$ is minimal. If $|F \cap F^*| > 0$, then any component of $F^* \cap M_i$ is essential in M_i since $|F \cap F^*|$ is minimal.

So by Lemma 2.3, we have

$$2 - \chi(F^* \cap M_1) \geqslant D(S_1) \geqslant 2g(M_1) + 1$$

and

$$2 - \chi(F^* \cap M_2) \geqslant D(S_2) \geqslant 2g(M_2) + 1.$$

Then

$$4 - \chi(F^* \cap M_1) - \chi(F^* \cap M_2) \geqslant 2g(M_1) + 2g(M_2) + 2;$$

i.e., $4 - \chi(F^*) \geqslant 2g(M_1) + 2g(M_2) + 2$, so $g(F^*) \geqslant g(M_1) + g(M_2)$, a contradiction to the assumption. □

Now we come to the proof of Theorem 1.1.

Proof. By assumption, $M = M_1 \cup_F M_2$ and $V_i \cup_{S_i} W_i$ is a Heegaard splitting with $D(S_i) \geqslant 2g(M_i) + 1$, $i = 1, 2$. Then by Lemma 2.4, $V_i \cup_{S_i} W_i$ is the unique minimal genus Heegaard splitting of M_i. Obviously, M_i is irreducible, so M is irreducible. We may assume that $F \subset \partial_- W_1, \partial_- V_2$. Let $V^{'} \cup_{S^{'}} W^{'}$ be an unstabilized Heegaard splitting of M with

$$g(S^{'}) \leqslant g(M_1) + g(M_2) - g(F).$$

Then by Theorem 3.1, $V^{'} \cup_{S^{'}} W^{'}$ is a weakly reducible and irreducible Heegaard splitting. By the result of [12], $V^{'} \cup_{S^{'}} W^{'}$ is an amalgamation of n strongly irreducible Heegaard splittings $V^{'} \cup_{S^{'}} W^{'} = (V_1^{'} \cup_{S_1^{'}} W_1^{'}) \cup_{F_1^{'}} (V_2^{'} \cup_{S_2^{'}} W_2^{'}) \cup_{F_2^{'}} \cdots \cup_{F_{n-1}^{'}} (V_n^{'} \cup_{S_n^{'}} W_n^{'})$. Since $g(F_i^{'}) < g(S^{'}) \leqslant g(M_1) + g(M_2) - g(F) < g(M_1) + g(M_2)$, by Proposition 3.2, we can isotope F so that $(\bigcup F_i^{'}) \cap F = \emptyset$. So F lies in the non-trivial component $V_j^* \cup_{S_j^*} W_j^*$ of $V_j^{'} \cup_{S_j^{'}} W_j^{'}$, for some $1 \leqslant j \leqslant n$.

If F is parallel to some component, say F^*, of $\bigcup F_i^{'}$, we amalgamate the Heegaard splitting sequence $V_1^{'} \cup_{S_1^{'}} W_1^{'}, V_2^{'} \cup_{S_2^{'}} W_2^{'}, \cdots, V_n^{'} \cup_{S_n^{'}} W_n^{'}$ along $\bigcup F_i^{'} - F^*$, and we obtain an unstabilized Heegaard splitting $V_1^{'*} \cup_{S_1^{'*}} W_1^{'*}$ of M_1 and an unstabilized Heegaard splitting $V_2^{'*} \cup_{S_2^{'*}} W_2^{'*}$ of M_2, such that $\partial_- W_1^{'*} = \partial_- V_2^{'*} = F^*$ and $g(S_1^{'*}) + g(S_2^{'*}) - g(F) = g(S^{'}) \leqslant g(M_1) + g(M_2) - g(F)$. Then by Lemma 2.4, we have $g(S_1) = g(M_1) \leqslant g(S_1^{'*})$ and $g(S_2) = g(M_2) \leqslant g(S_2^{'*})$, so $g(S_1) = g(S_1^{'*})$ and $g(S_2) = g(S_2^{'*})$. By Lemma 2.4, $V^{'} \cup_{S^{'}} W^{'}$ is the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$.

So we may assume that F is not parallel to any component of $\bigcup F_i^{'}$. Then by Proposition 2.5, we may assume that any component of $F \cap S_j^*$ is essential in both S_j^* and F, and at most one component of $S_j^* \backslash F$ is compressible in $M \backslash F$. Since F is essential, $F \cap S_j^* \neq \emptyset$. We may assume that any component of $S_j^* \backslash F$ is incompressible in M_1. Then $S_j^* \cap M_1$ is essential in M_1. So $2 - \chi(S_j^* \cap M_1) \geqslant D(S_1) \geqslant 2g(M_1) + 1$.

If any component of $S_j^* \cap M_2$ is incompressible in M_2, then any component of $S_j^* \cap M_2$ is parallel to a subsurface of F. Since

$$\begin{aligned} 2 - \chi(S_j^* \cap M_2) &= 2 - \chi(S_j^*) + \chi(S_j^* \cap M_1) \\ &\leqslant 2g(S^{'}) - 2g(M_1) + 1 \\ &\leqslant 2g(M_2) - 2g(F) + 1 \\ &< D(S_2), \end{aligned}$$

we can isotope S_j^* and F such that $F \cap S_j^* = \emptyset$, a contradiction.

Then we denote the compressible component of $S_j^* \cap M_2$ by $Q^{'}$ and assume that $Q^{'}$ is compressible in V_j^*. We compress $Q^{'}$ as much as possible in V_j^* to obtain a surface $Q^{'*}$. Then any component of $Q^{'*}$ is incompressible in $V_j^* \cup_{S_j^*} W_j^*$ since $V_j^* \cup_{S_j^*} W_j^*$ is strongly irreducible. Furthermore, $Q^{'*}$ is incompressible in M_2 since $\bigcup F_i^{'}$ is incompressible in M. $Q^{'*}$ is parallel to the subsurfaces $F_{Q^{'*}}$ of F since $2-\chi(Q^{'*}) \leqslant 2-\chi(Q^{'}) \leqslant 2-\chi(S_j^* \cap M_2) < D(S_2)$. If one component of $F_{Q^{'*}}$ contains another component of $F_{Q^{'*}}$, then by the similar arguments of Theorem 3.1, $V_j^{'} \cup_{S_j^{'}} W_j^{'}$ is a non-trivial compression body. The Heegaard splitting $V_j^{'} \cup_{S_j^{'}} W_j^{'}$ is not strongly irreducible, a contradiction.

Any component of $F \cap V_j^*$ is incompressible in V_j^*. Then by Lemma 2.1 any component of $V_j^* \backslash F$ is a compression body. By the same reason as above, any component of $W_j^* \backslash F$ is a compression body. Let $U_1^{'}$ be the component of $V_j^* \backslash F$ containing $Q^{'}$ and $U_2^{'}$ be the component of $W_j^* \backslash F$ containing $Q^{'}$. We amalgamate the Heegaard splitting $V_j^* \cup_{S_j^*} W_j^*$ and the Heegaard splittings contained in M_2 of the Heegaard sequence $V_1^{'} \cup_{S_1^{'}} W_1^{'}, V_2^{'} \cup_{S_2^{'}} W_2^{'}, \cdots, V_n^{'} \cup_{S_n^{'}} W_n^{'}$ along the components contained in M_2 of $\bigcup F_i^{'}$ to obtain a Heegaard splitting $V_3 \cup_{S_3} W_3$ such that the following conditions are satisfied:

(1) $V_3 \cap M_1 = V_j^* \cap M_1$ and $W_3 \cap M_1 = W_j^* \cap M_1$;
(2) $S_3 \cap F = S_j^* \cap F$ and $S_3 \cap M_1 = S_j^* \cap M_1$;
(3) only one component of $S_3 \cap M_2$ is compressible in V_3, denoted by $Q^{''}$, and other incompressible components are just the components of $S_j^* \cap M_2$.

Then any component of $V_3 \backslash F$ and $W_3 \backslash F$ is a compression body. Let $U^{'}$ be the component of $V_3 \backslash F$ which contains $Q^{''}$ and $U^{'*}$ be the component of $W_3 \backslash F$ which contains $Q^{''}$. Then by the similar arguments of Theorem 3.1, the 3-manifold $U^{'} \cup_{Q^{''}} U^{'*}$ is homeomorphic to M_2. Let $C^{'} = U^{'} \cup N(F \cap U^{'*}, U^{'*})$ and $C^{'*} = U^{'*} \backslash N(F \cap U^{'*}, U^{'*})$. Then $C^{'}$ and $C^{'*}$ are compression bodies, $\partial_+ C^{'} = \partial_+ C^{'*} = S^{'*}$. So $C^{'} \cup_{S^{'*}} C^{'*}$ is a Heegaard splitting of M_2. We compare $g(S^{'*})$ with $g(M_2)$.

Obviously, $g(S_3) \leqslant g(S^{'}) \leqslant g(M_1) + g(M_2) - g(F)$,

$$2-\chi(S_3 \cap M_1) = 2-\chi(S_j^* \cap M_1) \geqslant D(S_1) \geqslant 2g(M_1)+1,$$

$$-\chi(S^{'*}) \leqslant -\chi(S_3 \cap M_2) - \chi(F) = -\chi(S_3) + \chi(S_j^* \cap M_1) - \chi(F).$$

So

$$2g(S^{'*}) - 2 \leqslant 2g(M_1) + 2g(M_2) - 2g(F) - 2 + \chi(S_j^* \cap M_1) + 2g(F) - 2 \leqslant 2g(M_2) - 3.$$

Thus we have $g(S^{'*}) < g(M_2)$, a contradiction.

Hence $V^{'} \cup_{S^{'}} W^{'}$ is isotopic to the amalgamation of $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ and $g(M) = g(S^{'}) = g(M_1) + g(M_2) - g(F)$. This completes the proof of Theorem 1.1. □

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