

A STEADY-STATE EXTERIOR NAVIER-STOKES PROBLEM THAT IS NOT WELL-POSED

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ABSTRACT. We prove that the exterior Navier-Stokes problem with zero velocity at infinity is not well-posed in homogeneous Sobolev spaces. This result complements and clarifies well-known previous results obtained by various authors.

1. INTRODUCTION

Let Ω be the complement of the closure of a bounded domain, Ω_0 , of \mathbb{R}^3 of class C^2 . The objective of this paper is to investigate the well-posedness of the following Navier-Stokes boundary value problem:

$$(1.1) \quad \left. \begin{aligned} -\nu \Delta \mathbf{u} + \mathbf{u} \cdot \text{grad } \mathbf{u} &= -\text{grad } p + \mathbf{f} \\ \text{div } \mathbf{u} &= 0 \end{aligned} \right\} \text{ in } \Omega,$$

$$\mathbf{u} = \mathbf{0} \quad \text{at } \partial\Omega,$$

$$\lim_{|x| \rightarrow \infty} \mathbf{u}(x) = \mathbf{0},$$

in homogeneous Sobolev spaces. We recall that (1.1) governs the steady-state motion of a viscous liquid, \mathfrak{L} , in the exterior of the “rigid obstacle” Ω_0 . In particular, \mathbf{u} and p are velocity and pressure fields, respectively, and $\nu > 0$ is the (constant) kinematical viscosity of \mathfrak{L} , while \mathbf{f} is the prescribed body force acting on \mathfrak{L} .

In order to describe our results, we denote by $D_0^{1,q}(\Omega)$, $1 < q < \infty$, the *homogeneous* Sobolev space defined as the completion of smooth vector functions with compact support in Ω , $C_0^\infty(\Omega)$, in the Dirichlet norm $|\cdot|_{1,q} := (\int_\Omega |\text{grad } \cdot|^q)^{1/q}$, and by $D_0^{-1,q'}(\Omega)$ its (strong) dual with corresponding norm $|\cdot|_{-1,q'}$ ($q' := q/(q-1)$); see, e.g. [3, § II.5, § II.6]. We also indicate by $D_{0,\sigma}^{1,q}(\Omega)$ the subspace of $D_0^{1,q}(\Omega)$ of solenoidal functions, \mathbf{v} , namely, satisfying $\text{div } \mathbf{v} = 0$ in Ω .

It is well known—basically, since the work of J. Leray [10]—that for each $\mathbf{f} \in D_0^{-1,2}(\Omega)$, (1.1) has at least one solution (in the sense of distributions) $\mathbf{u} \in D_{0,\sigma}^{1,2}(\Omega)$, with corresponding $p \in L^2(\omega)$, for an arbitrary bounded domain $\omega \subset \Omega$. Moreover,

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if \mathbf{f} is sufficiently smooth and decays “fast enough” at large distances, then \mathbf{u} belongs also to $D_{0,\sigma}^{1,q}(\Omega)$, for all $q > 2$ [13, 4].

The interesting question that has attracted the attention of several mathematicians is the solvability of (1.1) in the class of those $\mathbf{u} \in D_{0,\sigma}^{1,q}(\Omega) \cap D_{0,\sigma}^{1,2}(\Omega)$, when $q < 2$; see [5, 8, 2, 11, 9, 7]. The results proved in these papers are many-fold, and we would like to recall the most relevant. In the first place, because of the particular structure of the nonlinear term, $\mathbf{u} \cdot \text{grad } \mathbf{u}$, one has to restrict to the case $q = 3/2$. Furthermore, if $\Omega = \mathbb{R}^3$ (namely, $\Omega_0 = \emptyset$), then under the assumption $\mathbf{f} \in D_0^{-1,3/2}(\mathbb{R}^3) \cap D_0^{-1,2}(\mathbb{R}^3)$ of “sufficiently small” magnitude, solutions do exist in the class where $\mathbf{u} \in D_{0,\sigma}^{1,3/2}(\mathbb{R}^3) \cap D_{0,\sigma}^{1,2}(\mathbb{R}^3)$. By the standard theory on representation of functionals on homogeneous Sobolev spaces [3, Theorem III.5.2], it then follows that $p \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. In addition, these solutions are also unique and depend continuously upon the data. On the other side, if $\Omega_0 \neq \emptyset$, we have that, under the assumption $\mathbf{f} \in D_0^{-1,3/2}(\Omega) \cap D_0^{-1,2}(\Omega) \equiv Y^*(\Omega)$, a solution $\mathbf{u} \in D_{0,\sigma}^{1,3/2}(\Omega) \cap D_{0,\sigma}^{1,2}(\Omega) \equiv X_1(\Omega)$, with associated $p \in L^{3/2}(\Omega) \cap L^2(\Omega) \equiv X_2(\Omega)$, can exist *only if* \mathbf{u} , p and \mathbf{f} satisfy the *nonlocal* compatibility condition

$$(1.2) \quad \mathbf{0} = \int_{\partial\Omega} [\nu(\text{grad } \mathbf{u} + (\text{grad } \mathbf{u})^\top) - p\mathbf{I}] \cdot \mathbf{n} + \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \equiv -\mathfrak{F}_1 - \mathfrak{F}_2,$$

in a distributional sense. In this equation, $^\top$ denotes transpose, \mathbf{I} is the identity matrix, \mathbf{n} is the unit outer normal to $\partial\Omega$, and $\text{div } \mathbf{F} = \mathbf{f}$.

The objective of this paper is to show that, in fact, if $\Omega_0 \neq \emptyset$, problem (1.1) is not well-posed in the space $X(\Omega) \equiv X_1(\Omega) \times X_2(\Omega)$. More precisely, we prove that if, for a certain $\overline{\mathbf{f}} \in Y^*(\Omega)$, (1.1) has a solution $\{\overline{\mathbf{u}}, \overline{p}\} \in X(\Omega)$, then in any arbitrary Y^* -neighborhood of $\overline{\mathbf{f}}$ we can find a “body force” \mathbf{f} such that problem (1.1) has *no* solution $\{\mathbf{u}, p\} \in X(\Omega)$; see Theorem 3.1. We obtain this result by using classical properties of nonlinear Fredholm maps with negative index, due to S. Smale [12], that we recall in the following section.

The physical interpretation of our result goes as follows. Because of (1.1)₄, the obstacle Ω_0 is at rest. This implies that the total force, \mathfrak{F} , acting on Ω_0 must vanish. In general, \mathfrak{F} is the sum of three contributions: \mathfrak{F}_1 , due to the action of the liquid; \mathfrak{F}_2 , due to the body force acting on the liquid, and \mathfrak{F}_3 , representing the external force *directly* applied to Ω_0 . Clearly, for any \mathfrak{F}_1 and \mathfrak{F}_2 , we can always find \mathfrak{F}_3 such that $\mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 = \mathbf{0}$, so that Ω_0 is “kept in place”. However, condition (1.2) tells us that, in the class $X(\Omega)$, \mathfrak{F}_3 is necessarily zero and, consequently, the obstacle Ω_0 must be kept in place *only* by the contribution due to the body force, \mathbf{f} , acting on the liquid. Our result then states that forces \mathbf{f} for which this happens are “rare”. Notice that, of course, the case $\Omega_0 = \emptyset$ does not present such a problem.

In conclusion, we wish to mention that, as shown in [6, 7], problem (1.1) *is* well-posed (for “small” \mathbf{f}) in appropriate function spaces other than $X(\Omega)$, where condition (1.2) does not necessarily hold.

2. SOME PRELIMINARY RESULTS

In this section we recall some standard properties of nonlinear Fredholm maps.

Let X and Z be separable Banach spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively, and let M be a map defined on the whole X with range $\mathbf{R}(M) \subset Z$. For $z \in Z$, we put $\sigma_M(z) = \{x \in X : M(x) = z\}$ (the solution set of the map M at z) and $\mathbf{N}(M) := \{x \in X : M(x) = 0\}$ (the null set of the map M). Furthermore, we shall

write $M \in C^k(X, Z)$, k a nonnegative integer, if, at each $x \in X$, M has continuous derivatives, in the sense of Fréchet, up to the order k included. The derivative of M at x is denoted by $M'(x)$.

A map $M \in C^1(X, Z)$ is said to be *Fredholm* if and only if the integers $\alpha := \dim N[M'(x)]$ and $\beta := \text{codim } R[M'(x)]$ are both finite. The integer $\text{ind}(M) := \alpha - \beta$ is then independent of the particular $x \in X$ [14, §5.15] and is called the *index* of M .

For a given $M \in C^1(X, Z)$, a point $x \in X$ is a *regular point* iff $M'(x)$ is surjective. A point $z \in Z$ is called a *regular value* iff either $\sigma_M(z) = \emptyset$ or $\sigma_M(z)$ is constituted only by regular points.

The following well-known result is due to Smale [12].

Lemma 2.1. *Let $M \in C^k(X, Z)$ be a Fredholm map with $k > \max\{\text{ind}(M), 0\}$. Then, the set of regular values of M , \mathfrak{R} , is dense in Z . More specifically, $Z - \mathfrak{R}$ is of Baire first category.*

An immediate, and fundamental to our aims, consequence of Lemma 2.1 is given by the following corollary, whose simple proof we include for the reader's convenience.

Corollary 2.1. *Suppose M satisfies the assumption of Lemma 2.1 and that, for some $\bar{z} \in Z$, $\sigma_M(\bar{z}) \neq \emptyset$. Then, if $\text{ind}(M) < 0$, the problem $M(x) = z$ is not well-posed, in the sense that the solution x cannot depend continuously on the data z . Precisely, for any $\varepsilon > 0$, we can find $z \in Z$ such that $\|z - \bar{z}\|_Z < \varepsilon$ and the equation $M(x) = z$ has no solution.*

Proof. For the given ε , by Lemma 2.1 we may choose z to be a regular value. Now, if we suppose, by contradiction, $\sigma_M(z) \neq \emptyset$, we would have that $M'(x)$ is surjective, for all $x \in \sigma_M(z)$, which would imply $\text{ind}(M) = \dim N[M'(x)] \geq 0$, in contrast with the assumption. □

An equivalent way of phrasing Corollary 2.1 is that, under the stated assumptions on M , the interior of $R(M)$ is empty.

3. APPLICATION TO THE EXTERIOR NAVIER-STOKES PROBLEM

We begin to rewrite (1.1) as a nonlinear equation in a suitable Banach space. We set $Y = Y(\Omega) := D_0^{1,3}(\Omega) + D_0^{1,2}(\Omega)$ equipped with the norm

$$\|\varphi\|_Y := \inf \left\{ |\varphi_1|_{1,3} + |\varphi_2|_{1,2} : \varphi = \varphi_1 + \varphi_2, \varphi_1 \in D_0^{1,3}(\Omega), \varphi_2 \in D_0^{1,2}(\Omega) \right\}.$$

Since both $D_0^{1,3}(\Omega)$ and $D_0^{1,2}(\Omega)$ are reflexive, it follows that for any $\varphi \in Y$ there exist $\varphi_1 \in D_0^{1,3}(\Omega)$ and $\varphi_2 \in D_0^{1,2}(\Omega)$ such that

$$(3.1) \quad \|\varphi\|_Y = |\varphi_1|_{1,3} + |\varphi_2|_{1,2}.$$

Also, since $\{\varphi \in C_0^\infty(\Omega) : \text{div } \varphi = 0\}$ is dense in $D_0^{1,3}(\Omega) \cap D_0^{1,2}(\Omega)$ [3, Exercise III.6.2], we have that the (strong) dual, Y^* , of Y can be isomorphically represented as $Y^* = D_0^{-1,3/2}(\Omega) \cap D_0^{-1,2}(\Omega)$ with associated norm $\|\cdot\|_{Y^*} := |\cdot|_{-1,3/2} + |\cdot|_{-1,2}$; see [1]. Moreover, Y^* is separable [3, Exercise II.5.1].

If we now multiply, formally, (1.1)₁ by $\varphi \in Y$ and integrate by parts over Ω , we find:

$$(3.2) \quad \nu(\text{grad } \mathbf{u}, \text{grad } \varphi) - (p, \text{div } \varphi) - (\mathbf{u} \cdot \text{grad } \varphi, \mathbf{u}) = \langle \mathbf{f}, \varphi \rangle,$$

where (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ represent the L^2 -scalar product and duality pairing between Y^* and Y , respectively. Set

$$\begin{aligned} X_1 &= X_1(\Omega) := D_{0,\sigma}^{1,3/2}(\Omega) \cap D_{0,\sigma}^{1,2}(\Omega), \quad \|\cdot\|_{X_1} := \|\cdot\|_{1,3/2} + \|\cdot\|_{1,2} \\ X_2 &= X_2(\Omega) := L^{3/2}(\Omega) \cap L^2(\Omega), \quad \|\cdot\|_{X_2} := \|\cdot\|_{3/2} + \|\cdot\|_2 \\ X &= X(\Omega) := X_1 \times X_2, \quad \|\{\mathbf{u}, p\}\|_X := \|\mathbf{u}\|_{X_1} + \|p\|_{X_2}. \end{aligned}$$

The space X is separable [3, Exercise II.5.1]. Because of the continuous embeddings

$$(3.3) \quad D_{0,\sigma}^{1,3/2}(\Omega) \subset L^3(\Omega), \quad D_{0,\sigma}^{1,2}(\Omega) \subset L^6(\Omega)$$

(see [3, Theorem II.5.1]), it is immediately checked (by the Hölder inequality) that, for any $\{\mathbf{u}, p\} \in X$, the left-hand side of (3.2) defines two linear functionals, $\mathbf{A}(\mathbf{u}, p)$ (Stokes operator) and $\mathbf{M}(\mathbf{u})$, on Y as follows:

$$(3.4) \quad \langle \mathbf{A}(\mathbf{u}, p), \varphi \rangle := \nu(\operatorname{grad} \mathbf{u}, \operatorname{grad} \varphi) - (p, \operatorname{div} \varphi), \quad \langle \mathbf{M}(\mathbf{u}), \varphi \rangle := -(\mathbf{u} \cdot \operatorname{grad} \varphi, \mathbf{u}).$$

Therefore, (3.2) can be rewritten in the following operator equation form:

$$(3.5) \quad \mathbf{N}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } Y^*,$$

where the map \mathbf{N} is defined as

$$\mathbf{N} : \{\mathbf{u}, p\} \in X \mapsto \mathbf{A}(\mathbf{u}, p) + \mathbf{M}(\mathbf{u}) \in Y^*.$$

Set $B_a(\mathbf{y}) := \{\mathbf{f} \in Y^* : \|\mathbf{f} - \mathbf{y}\|_{Y^*} < a\}$, $a > 0$. We have the following.

Theorem 3.1. *Let $\Omega_0 \neq \emptyset$. Assume that (3.5) has a solution $\{\bar{\mathbf{u}}, \bar{p}\} \in X$ corresponding to some $\bar{\mathbf{f}} \in Y^*$. Then, for any $\varepsilon > 0$, there exists $\mathbf{f} \in B_\varepsilon(\bar{\mathbf{f}})$ such that (3.5) does not have a solution.*

Proof. In view of Corollary 2.1, it suffices to show that \mathbf{N} is a Fredholm map of negative index. In order to reach this goal, we begin to observe that $\mathbf{N} \in C^1(X, Y)$ and that

$$[\mathbf{N}'(\mathbf{u}, p)](\mathbf{w}, \tau) = \mathbf{A}(\mathbf{w}, \tau) + [\mathbf{M}'(\mathbf{u})](\mathbf{w}),$$

where

$$(3.6) \quad \langle [\mathbf{M}'(\mathbf{u})](\mathbf{w}), \varphi \rangle = -(\mathbf{u} \cdot \operatorname{grad} \varphi, \mathbf{w}) - (\mathbf{w} \cdot \operatorname{grad} \varphi, \mathbf{u}), \quad \varphi \in Y.$$

(The proof of these properties is completely standard, and, therefore, it will be omitted.) We prove, next, that $\mathbf{M}'(\mathbf{u})$ is compact at each $\mathbf{u} \in X_1$. Let $\{\mathbf{w}_m\}$ be a sequence in X_1 such that

$$(3.7) \quad \|\mathbf{w}_m\|_{X_1} \leq M_1,$$

where M_1 is independent of the integer m . Since $D_{0,\sigma}^{1,3/2}(\Omega)$ and $D_{0,\sigma}^{1,2}(\Omega)$ are reflexive, we can select a subsequence (again denoted by $\{\mathbf{w}_m\}$) and find $\mathbf{w} \in X_1$ such that

$$(3.8) \quad \mathbf{w}_m \rightharpoonup \mathbf{w} \quad \text{weakly in } D_{0,\sigma}^{1,3/2}(\Omega) \text{ and in } D_{0,\sigma}^{1,2}(\Omega).$$

From (3.6) we find that

$$(3.9) \quad \langle [\mathbf{M}'(\mathbf{u})](\mathbf{v}_m), \varphi \rangle = -(\mathbf{u} \cdot \operatorname{grad} \varphi, \mathbf{v}_m) - (\mathbf{v}_m \cdot \operatorname{grad} \varphi, \mathbf{u}), \quad \varphi \in Y,$$

where $\mathbf{v}_m := \mathbf{w} - \mathbf{w}_m$. For sufficiently large $R > 0$, we set $\Omega_R = \Omega \cap \{|x| < R\}$, $\Omega^R = \Omega \cap \{|x| > R\}$ and denote by $\|\cdot\|_{r,A}$ the $L^r(A)$ -norm. Recalling that

$\varphi = \varphi_1 + \varphi_2$, where $\varphi_i, i = 1, 2$, satisfy (3.1), with the help of the Hölder inequality we find

$$\begin{aligned}
 |(\mathbf{u} \cdot \text{grad } \varphi_1, \mathbf{v}_m)| &\leq \|\mathbf{u}\|_3 \|\mathbf{v}_m\|_{3,\Omega_R} |\varphi_1|_{1,3} + \|\mathbf{u}\|_{3,\Omega^R} \|\mathbf{v}_m\|_{3,\Omega^R} |\varphi_1|_{1,3} \\
 &\leq (\|\mathbf{u}\|_3 \|\mathbf{v}_m\|_{3,\Omega_R} + M \|\mathbf{u}\|_{3,\Omega^R}) \|\varphi\|_Y, \\
 |(\mathbf{u} \cdot \text{grad } \varphi_2, \mathbf{v}_m)| &\leq \|\mathbf{u}\|_6 \|\mathbf{v}_m\|_{3,\Omega_R} |\varphi_2|_{1,2} + \|\mathbf{u}\|_{6,\Omega^R} \|\mathbf{v}_m\|_{3,\Omega^R} |\varphi_2|_{1,2} \\
 &\leq (\|\mathbf{u}\|_6 \|\mathbf{v}_m\|_{3,\Omega_R} + M_2 \|\mathbf{u}\|_{6,\Omega^R}) \|\varphi\|_Y,
 \end{aligned}
 \tag{3.10}$$

where M_2 denotes an upper bound for $\|\mathbf{v}_m\|_X$. Set $M_3 = \max\{\|\mathbf{u}\|_3, \|\mathbf{u}\|_6, M_2\}$. Collecting (3.9) and (3.10), we thus obtain

$$\|[M'(\mathbf{u})](\mathbf{v}_m)\|_{Y^*} \leq M_3 (\|\mathbf{v}_m\|_{3,\Omega_R} + \|\mathbf{u}\|_{3,\Omega^R} + \|\mathbf{u}\|_{6,\Omega^R}) .
 \tag{3.11}$$

We now let $m \rightarrow \infty$ in (3.11). By (3.3), we find that X_1 continuously embeds in the Sobolev space $W^{1,2}(\Omega_R)$, for all $R > 0$. Thus, by (3.7), by (3.8) and by the Rellich theorem, the first term on the right-hand side of (3.11) tends to zero as $m' \rightarrow \infty$, for some $\{m'\} \subset \{m\}$. Successively, we let $R \rightarrow \infty$, which, again with the help of (3.3), causes the second and the third term to go to zero as well. We thus deduce $\|[M'(\mathbf{u})](\mathbf{v}_{m'})\|_{Y^*} \rightarrow 0$ as $m' \rightarrow \infty$, for each fixed $\mathbf{u} \in X_1$, which completes the proof of the compactness of the operator $M'(\mathbf{u})$. Our next and final objective is to show that the linear operator $\mathbf{A} : \{\mathbf{u}, p\} \in X \mapsto \mathbf{A}(\mathbf{u}, p) \in Y^*$ defined in (3.4) is Fredholm and that $\text{ind}(\mathbf{A}) = -3$; after that, from the definition of a Fredholm map and from the fact that the index of a (linear) Fredholm operator is left invariant by a compact perturbation [14, Theorem 5.E], we find $\text{ind}(\mathbf{N}) = -3$. Clearly, the operator \mathbf{A} is graph-closed. Moreover, from [3, p. 282 and Theorem V.5.1] it follows that

$$\mathbf{N}(\mathbf{A}) = \{\mathbf{0}\}, \quad \mathbf{R}(\mathbf{A}) = \{\mathbf{f} \in Y^* : \langle \mathbf{f}, \mathbf{h}^{(i)} \rangle = 0, i = 1, 2, 3\},
 \tag{3.12}$$

where $\mathbf{h}_i \in D_{0,\sigma}^{1,3}(\Omega) (\subset Y)$, $i = 1, 2, 3$, are three independent functions. It is now easy to show that there exist three independent elements of Y^* , $\mathbf{l}_k, k = 1, 2, 3$, such that, denoting by \mathbf{S} their linear span, we have

$$Y^* = \mathbf{R}(\mathbf{A}) \oplus \mathbf{S} .
 \tag{3.13}$$

Since $\dim(\mathbf{S}) = 3$, from (3.12) we then find $\text{ind}(\mathbf{A}) = \dim[\mathbf{N}(\mathbf{A})] - \text{codim}[\mathbf{R}(\mathbf{A})] = -3$. In order to prove (3.13), let $L_k, k = 1, 2, 3$, be the vector spaces generated by $\{\mathbf{h}^{(2)}, \mathbf{h}^{(3)}\}, \{\mathbf{h}^{(1)}, \mathbf{h}^{(3)}\}$ and $\{\mathbf{h}^{(1)}, \mathbf{h}^{(2)}\}$, respectively. Set $d_k := \inf_{\mathbf{h} \in L_k} \|\mathbf{h}^{(k)} - \mathbf{h}\|_Y (> 0)$. From a corollary to the Hahn-Banach theorem (see, e.g., [14, Proposition I.2.3]) we know that there exists $\mathbf{l}_k \in Y^*$ such that

$$\|\mathbf{l}_k\|_{Y^*} = d_k^{-1}, \quad \langle \mathbf{l}_k, \mathbf{h}^{(j)} \rangle = \delta_{kj} .
 \tag{3.14}$$

We claim the validity of (3.13), where \mathbf{S} is the vector space generated by $\{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\}$. In fact, obviously, $\mathbf{R}(\mathbf{A}) \cap \mathbf{S} = \emptyset$. Furthermore, for any $\mathbf{f} \in Y^*$ we have, with the help of (3.14), that

$$\mathbf{f} - \sum_{k=1}^3 \langle \mathbf{f}, \mathbf{h}^{(k)} \rangle \mathbf{l}_k \in \mathbf{R}(\mathbf{A})$$

and (3.13) follows. □

Remark 3.1. The assumption, in Theorem 3.1, that $\Omega_0 \neq \emptyset$ is crucial. In fact, if $\Omega = \mathbb{R}^3$, then $\text{ind}(\mathbf{A}) = 0$ [3, Theorem IV.2.2], and so, by the same argument used

in the proof of Theorem 3.1, we can show that $\text{ind}(\mathbf{N}) = 0$. This is consistent with the results of [8] that prove (local) well-posedness of problem (1.1) in the space $D_{0,\sigma}^{1,3/2}(\mathbb{R}^3) \cap D_{0,\sigma}^{1,2}(\mathbb{R}^3)$.

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