PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 2, February 2009, Pages 679–684
S 0002-9939(08)09658-5
Article electronically published on August 15, 2008

A STEADY-STATE EXTERIOR NAVIER-STOKES PROBLEM THAT IS NOT WELL-POSED

GIOVANNI P. GALDI

(Communicated by Walter Craig)

Abstract. We prove that the exterior Navier-Stokes problem with zero velocity at infinity is not well-posed in homogeneous Sobolev spaces. This result complements and clarifies well-known previous results obtained by various authors.

1. Introduction

Let Ω be the complement of the closure of a bounded domain, Ω_0, of $\mathbb{R}^3$ of class C^2. The objective of this paper is to investigate the well-posedness of the following Navier-Stokes boundary value problem:

$$\left.\begin{aligned} &-\nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\operatorname{grad}\boldsymbol{u} = -\operatorname{grad}p + \boldsymbol{f} \\ &\operatorname{div}\boldsymbol{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega,$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{at } \partial\Omega\,, \tag{1.1}$$

$$\lim_{|x|\to\infty}\boldsymbol{u}(x) = \boldsymbol{0}\,,$$

in homogeneous Sobolev spaces. We recall that (1.1) governs the steady-state motion of a viscous liquid, $\mathfrak{L}$, in the exterior of the "rigid obstacle" Ω_0. In particular, $\boldsymbol{u}$ and p are velocity and pressure fields, respectively, and $\nu > 0$ is the (constant) kinematical viscosity of $\mathfrak{L}$, while $\boldsymbol{f}$ is the prescribed body force acting on $\mathfrak{L}$.

In order to describe our results, we denote by $D_0^{1,q}(\Omega)$, $1 < q < \infty$, the *homogeneous* Sobolev space defined as the completion of smooth vector functions with compact support in Ω, $C_0^\infty(\Omega)$, in the Dirichlet norm $|\cdot|_{1,q} := \left(\int_\Omega |\operatorname{grad}\cdot|^q\right)^{1/q}$, and by $D_0^{-1,q'}(\Omega)$ its (strong) dual with corresponding norm $|\cdot|_{-1,q'}$ ($q' := q/(q-1)$); see, *e.g.* [3, § II.5, § II.6]. We also indicate by $D_{0,\sigma}^{1,q}(\Omega)$ the subspace of $D_0^{1,q}(\Omega)$ of solenoidal functions, $\boldsymbol{v}$, namely, satisfying $\operatorname{div}\boldsymbol{v} = 0$ in Ω.

It is well known–basically, since the work of J. Leray [10]–that for each $\boldsymbol{f} \in D_0^{-1,2}(\Omega)$, (1.1) has at least one solution (in the sense of distributions) $\boldsymbol{u} \in D_{0,\sigma}^{1,2}(\Omega)$, with corresponding $p \in L^2(\omega)$, for an arbitrary bounded domain $\omega \subset \Omega$. Moreover,

Received by the editors January 9, 2008.
2000 *Mathematics Subject Classification.* Primary 76D05, 76D03; Secondary 76D07.
Key words and phrases. Navier-Stokes equations, exterior problem, homogeneous Sobolev spaces.
This work was supported in part by NSF Grant #DMS-0707281.

if $\boldsymbol{f}$ is sufficiently smooth and decays "fast enough" at large distances, then $\boldsymbol{u}$ belongs also to $D^{1,q}_{0,\sigma}(\Omega)$, for all $q > 2$ [13, 4].

The interesting question that has attracted the attention of several mathematicians is the solvability of (1.1) in the class of those $\boldsymbol{u} \in D^{1,q}_{0,\sigma}(\Omega) \cap D^{1,2}_{0,\sigma}(\Omega)$, *when* $q < 2$; see [5, 8, 2, 11, 9, 7]. The results proved in these papers are many-fold, and we would like to recall the most relevant. In the first place, because of the particular structure of the nonlinear term, $\boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{u}$, one has to restrict to the case $q = 3/2$. Furthermore, if $\Omega = \mathbb{R}^3$ (namely, $\Omega_0 = \emptyset$), then under the assumption $\boldsymbol{f} \in D^{-1,3/2}_0(\mathbb{R}^3) \cap D^{-1,2}_0(\mathbb{R}^3)$ of "sufficiently small" magnitude, solutions do exist in the class where $\boldsymbol{u} \in D^{1,3/2}_{0,\sigma}(\mathbb{R}^3) \cap D^{1,2}_{0,\sigma}(\mathbb{R}^3)$. By the standard theory on representation of functionals on homogeneous Sobolev spaces [3, Theorem III.5.2], it then follows that $p \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. In addition, these solutions are also unique and depend continuously upon the data. On the other side, if $\Omega_0 \neq \emptyset$, we have that, under the assumption $\boldsymbol{f} \in D^{-1,3/2}_0(\Omega) \cap D^{-1,2}_0(\Omega) \equiv Y^*(\Omega)$, a solution $\boldsymbol{u} \in D^{1,3/2}_{0,\sigma}(\Omega) \cap D^{1,2}_{0,\sigma}(\Omega) \equiv X_1(\Omega)$, with associated $p \in L^{3/2}(\Omega) \cap L^2(\Omega) \equiv X_2(\Omega)$, can exist *only if* $\boldsymbol{u}$, p and $\boldsymbol{f}$ satisfy the *nonlocal* compatibility condition

$$\mathbf{0} = \int_{\partial\Omega} [\nu(\operatorname{grad} \boldsymbol{u} + (\operatorname{grad} \boldsymbol{u})^\top) - p\boldsymbol{I}] \cdot \boldsymbol{n} + \int_{\partial\Omega} \boldsymbol{F} \cdot \boldsymbol{n} \equiv -\mathfrak{F}_1 - \mathfrak{F}_2, \tag{1.2}$$

in a distributional sense. In this equation, $^\top$ denotes transpose, $\boldsymbol{I}$ is the identity matrix, $\boldsymbol{n}$ is the unit outer normal to $\partial\Omega$, and $\operatorname{div} \boldsymbol{F} = \boldsymbol{f}$.

The objective of this paper is to show that, in fact, if $\Omega_0 \neq \emptyset$, problem (1.1) is not well-posed in the space $X(\Omega) \equiv X_1(\Omega) \times X_2(\Omega)$. More precisely, we prove that if, for a certain $\overline{\boldsymbol{f}} \in Y^*(\Omega)$, (1.1) has a solution $\{\overline{\boldsymbol{u}}, \overline{p}\} \in X(\Omega)$, then in any arbitrary Y^*-neighborhood of $\overline{\boldsymbol{f}}$ we can find a "body force" $\boldsymbol{f}$ such that problem (1.1) has *no* solution $\{\boldsymbol{u}, p\} \in X(\Omega)$; see Theorem 3.1. We obtain this result by using classical properties of nonlinear Fredholm maps with negative index, due to S. Smale [12], that we recall in the following section.

The physical interpretation of our result goes as follows. Because of $(1.1)_4$, the obstacle Ω_0 is at rest. This implies that the total force, $\mathfrak{F}$, acting on Ω_0 must vanish. In general, $\mathfrak{F}$ is the sum of three contributions: $\mathfrak{F}_1$, due to the action of the liquid; $\mathfrak{F}_2$, due to the body force acting on the liquid, and $\mathfrak{F}_3$, representing the external force *directly* applied to Ω_0. Clearly, for any $\mathfrak{F}_1$ and $\mathfrak{F}_2$, we can always find $\mathfrak{F}_3$ such that $\mathfrak{F}_1 + \mathfrak{F}_2 + \mathfrak{F}_3 = \mathbf{0}$, so that Ω_0 is "kept in place". However, condition (1.2) tells us that, in the class $X(\Omega)$, $\mathfrak{F}_3$ is necessarily zero and, consequently, the obstacle Ω_0 must be kept in place *only* by the contribution due to the body force, $\boldsymbol{f}$, acting on the liquid. Our result then states that forces $\boldsymbol{f}$ for which this happens are "rare". Notice that, of course, the case $\Omega_0 = \emptyset$ does not present such a problem.

In conclusion, we wish to mention that, as shown in [6, 7], problem (1.1) *is* well-posed (for "small" $\boldsymbol{f}$) in appropriate function spaces other than $X(\Omega)$, where condition (1.2) does not necessarily hold.

2. Some preliminary results

In this section we recall some standard properties of nonlinear Fredholm maps.

Let X and Z be separable Banach spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Z$, respectively, and let M be a map defined on the whole X with range $\mathsf{R}(M) \subset Z$. For $z \in Z$, we put $\sigma_M(z) = \{x \in X : \; M(x) = z\}$ (the solution set of the map M at z) and $\mathsf{N}(M) := \{x \in X : M(x) = 0\}$ (the null set of the map M). Furthermore, we shall write $M \in C^k(X, Z)$, k a nonnegative integer, if, at each $x \in X$, M has continuous derivatives, in the sense of Fréchet, up to the order k included. The derivative of M at x is denoted by $M'(x)$.

A map $M \in C^1(X, Z)$ is said to be *Fredholm* if and only if the integers $\alpha := \dim \mathsf{N}[M'(x)]$ and $\beta := \operatorname{codim} \mathsf{R}[M'(x)]$ are both finite. The integer $\operatorname{ind}(M) := \alpha - \beta$ is then independent of the particular $x \in X$ [14, §5.15] and is called the *index* of M.

For a given $M \in C^1(X, Z)$, a point $x \in X$ is a *regular point* iff $M'(x)$ is surjective. A point $z \in Z$ is called *a regular value* iff either $\sigma_M(z) = \emptyset$ or $\sigma_M(z)$ is constituted only by regular points.

The following well-known result is due to Smale [12].

Lemma 2.1. *Let $M \in C^k(X, Z)$ be a Fredholm map with $k > \max\{\operatorname{ind}(M), 0\}$. Then, the set of regular values of M, $\mathfrak{R}$, is dense in Z. More specifically, $Z - \mathfrak{R}$ is of Baire first category.*

An immediate, and fundamental to our aims, consequence of Lemma 2.1 is given by the following corollary, whose simple proof we include for the reader's convenience.

Corollary 2.1. *Suppose M satisfies the assumption of* Lemma 2.1 *and that, for some $\overline{z} \in Z$, $\sigma_M(\overline{z}) \neq \emptyset$. Then, if* $\operatorname{ind}(M) < 0$*, the problem $M(x) = z$ is not well-posed, in the sense that the solution x cannot depend continuously on the data z. Precisely, for any $\varepsilon > 0$, we can find $z \in Z$ such that $\|z - \overline{z}\|_Z < \varepsilon$ and the equation $M(x) = z$ has no solution.*

Proof. For the given ε, by Lemma 2.1 we may choose z to be a regular value. Now, if we suppose, by contradiction, $\sigma_M(z) \neq \emptyset$, we would have that $M'(x)$ is surjective, for all $x \in \sigma_M(z)$, which would imply $\operatorname{ind}(M) = \dim \mathsf{N}[M'(x)] \geq 0$, in contrast with the assumption. □

An equivalent way of phrasing Corollary 2.1 is that, under the stated assumptions on M, the interior of $\mathsf{R}(M)$ is empty.

3. Application to the exterior Navier-Stokes problem

We begin to rewrite (1.1) as a nonlinear equation in a suitable Banach space. We set $Y = Y(\Omega) := D_0^{1,3}(\Omega) + D_0^{1,2}(\Omega)$ equipped with the norm

$$\|\boldsymbol{\varphi}\|_Y := \inf \left\{ |\boldsymbol{\varphi}_1|_{1,3} + |\boldsymbol{\varphi}_2|_{1,2} : \boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2 ,\ \boldsymbol{\varphi}_1 \in D_0^{1,3}(\Omega), \boldsymbol{\varphi}_2 \in D_0^{1,2}(\Omega) \right\}.$$

Since both $D_0^{1,3}(\Omega)$ and $D_0^{1,2}(\Omega)$ are reflexive, it follows that for any $\boldsymbol{\varphi} \in Y$ there exist $\boldsymbol{\varphi}_1 \in D_0^{1,3}(\Omega)$ and $\boldsymbol{\varphi}_2 \in D_0^{1,2}(\Omega)$ such that

$$\|\boldsymbol{\varphi}\|_Y = |\boldsymbol{\varphi}_1|_{1,3} + |\boldsymbol{\varphi}_2|_{1,2} . \tag{3.1}$$

Also, since $\{\boldsymbol{\varphi} \in C_0^\infty(\Omega) : \operatorname{div} \boldsymbol{\varphi} = 0\}$ is dense in $D_0^{1,3}(\Omega) \cap D_0^{1,2}(\Omega)$ [3, Exercise III.6.2], we have that the (strong) dual, Y^*, of Y can be isomorphically represented as $Y^* = D_0^{-1,3/2}(\Omega) \cap D_0^{-1,2}(\Omega)$ with associated norm $\|\cdot\|_{Y^*} := |\cdot|_{-1,3/2} + |\cdot|_{-1,2}$; see [1]. Moreover, Y^* is separable [3, Exercise II.5.1].

If we now multiply, formally, $(1.1)_1$ by $\boldsymbol{\varphi} \in Y$ and integrate by parts over Ω, we find:

$$\nu(\operatorname{grad} \boldsymbol{u}, \operatorname{grad} \boldsymbol{\varphi}) - (p, \operatorname{div} \boldsymbol{\varphi}) - (\boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{\varphi}, \boldsymbol{u}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle , \tag{3.2}$$

where $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ represent the L^2-scalar product and duality pairing between Y^* and Y, respectively. Set

$$\begin{aligned}
&X_1 = X_1(\Omega) := D^{1,3/2}_{0,\sigma}(\Omega) \cap D^{1,2}_{0,\sigma}(\Omega), \quad \|\cdot\|_{X_1} := |\cdot|_{1,3/2} + |\cdot|_{1,2}\\
&X_2 = X_2(\Omega) := L^{3/2}(\Omega) \cap L^2(\Omega), \quad \|\cdot\|_{X_2} := \|\cdot\|_{3/2} + \|\cdot\|_2\\
&X = X(\Omega) := X_1 \times X_2, \quad \|\{\boldsymbol{u}, p\}\|_X := \|\boldsymbol{u}\|_{X_1} + \|p\|_{X_2}.
\end{aligned}$$

The space X is separable [3, Exercise II.5.1]. Because of the continuous embeddings

$$D^{1,3/2}_{0,\sigma}(\Omega) \subset L^3(\Omega)\,, \quad D^{1,2}_{0,\sigma}(\Omega) \subset L^6(\Omega) \tag{3.3}$$

(see [3, Theorem II.5.1]), it is immediately checked (by the Hölder inequality) that, for any $\{\boldsymbol{u}, p\} \in X$, the left-hand side of (3.2) defines two linear functionals, $\boldsymbol{A}(\boldsymbol{u}, p)$ (Stokes operator) and $\boldsymbol{M}(\boldsymbol{u})$, on Y as follows:

$$\langle \boldsymbol{A}(\boldsymbol{u}, p), \boldsymbol{\varphi}\rangle := \nu(\operatorname{grad} \boldsymbol{u}, \operatorname{grad} \boldsymbol{\varphi}) - (p, \operatorname{div} \boldsymbol{\varphi})\,, \quad \langle \boldsymbol{M}(\boldsymbol{u}), \boldsymbol{\varphi}\rangle := -(\boldsymbol{u}\cdot \operatorname{grad} \boldsymbol{\varphi}, \boldsymbol{u})\,. \tag{3.4}$$

Therefore, (3.2) can be rewritten in the following operator equation form:

$$\boldsymbol{N}(\boldsymbol{u}, p) = \boldsymbol{f} \quad \text{in } Y^*, \tag{3.5}$$

where the map $\boldsymbol{N}$ is defined as

$$\boldsymbol{N} : \{\boldsymbol{u}, p\} \in X \mapsto \boldsymbol{A}(\boldsymbol{u}, p) + \boldsymbol{M}(\boldsymbol{u}) \in Y^*\,.$$

Set $B_a(\boldsymbol{y}) := \{\boldsymbol{f} \in Y^* : \|\boldsymbol{f} - \boldsymbol{y}\|_{Y^*} < a\}$, $a > 0$. We have the following.

Theorem 3.1. *Let $\Omega_0 \neq \emptyset$. Assume that* (3.5) *has a solution $\{\overline{\boldsymbol{u}}, \overline{p}\} \in X$ corresponding to some $\overline{\boldsymbol{f}} \in Y^*$. Then, for any $\varepsilon > 0$, there exists $\boldsymbol{f} \in B_\varepsilon(\overline{\boldsymbol{f}})$ such that* (3.5) *does not have a solution.*

Proof. In view of Corollary 2.1, it suffices to show that $\boldsymbol{N}$ is a Fredholm map of negative index. In order to reach this goal, we begin to observe that $\boldsymbol{N} \in C^1(X, Y)$ and that

$$[\boldsymbol{N}'(\boldsymbol{u}, p)](\boldsymbol{w}, \tau) = \boldsymbol{A}(\boldsymbol{w}, \tau) + [\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{w})\,,$$

where

$$\langle [\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{w}), \boldsymbol{\varphi}\rangle = -(\boldsymbol{u}\cdot \operatorname{grad} \boldsymbol{\varphi}, \boldsymbol{w}) - (\boldsymbol{w}\cdot \operatorname{grad} \boldsymbol{\varphi}, \boldsymbol{u})\,, \quad \boldsymbol{\varphi} \in Y\,. \tag{3.6}$$

(The proof of these properties is completely standard, and, therefore, it will be omitted.) We prove, next, that $\boldsymbol{M}'(\boldsymbol{u})$ is compact at each $\boldsymbol{u} \in X_1$. Let $\{\boldsymbol{w}_m\}$ be a sequence in X_1 such that

$$\|\boldsymbol{w}_m\|_{X_1} \le M_1\,, \tag{3.7}$$

where M_1 is independent of the integer m. Since $D^{1,3/2}_{0,\sigma}(\Omega)$ and $D^{1,2}_{0,\sigma}(\Omega)$ are reflexive, we can select a subsequence (again denoted by $\{\boldsymbol{w}_m\}$) and find $\boldsymbol{w} \in X_1$ such that

$$\boldsymbol{w}_m \to \boldsymbol{w} \quad \text{weakly in } D^{1,3/2}_{0,\sigma}(\Omega) \text{ and in } D^{1,2}_{0,\sigma}(\Omega). \tag{3.8}$$

From (3.6) we find that

$$\langle [\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{v}_m), \boldsymbol{\varphi}\rangle = -(\boldsymbol{u}\cdot \operatorname{grad} \boldsymbol{\varphi}, \boldsymbol{v}_m) - (\boldsymbol{v}_m\cdot \operatorname{grad} \boldsymbol{\varphi}, \boldsymbol{u})\,, \quad \boldsymbol{\varphi} \in Y\,, \tag{3.9}$$

where $\boldsymbol{v}_m := \boldsymbol{w} - \boldsymbol{w}_m$. For sufficiently large $R > 0$, we set $\Omega_R = \Omega \cap \{|x| < R\}$, $\Omega^R = \Omega \cap \{|x| > R\}$ and denote by $\|\cdot\|_{r,A}$ the $L^r(A)$-norm. Recalling that

$\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \boldsymbol{\varphi}_2$, where $\boldsymbol{\varphi}_i$, $i = 1, 2$, satisfy (3.1), with the help of the Hölder inequality we find

$$
\begin{aligned}
|(\boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{\varphi}_1, \boldsymbol{v}_m)| &\le \|\boldsymbol{u}\|_3 \|\boldsymbol{v}_m\|_{3,\Omega_R} |\boldsymbol{\varphi}_1|_{1,3} + \|\boldsymbol{u}\|_{3,\Omega^R} \|\boldsymbol{v}_m\|_{3,\Omega^R} |\boldsymbol{\varphi}_1|_{1,3} \\
&\le \left(\|\boldsymbol{u}\|_3 \|\boldsymbol{v}_m\|_{3,\Omega_R} + M \,\|\boldsymbol{u}\|_{3,\Omega^R} \right) \|\boldsymbol{\varphi}\|_Y , \\
|(\boldsymbol{u} \cdot \operatorname{grad} \boldsymbol{\varphi}_2, \boldsymbol{v}_m)| &\le \|\boldsymbol{u}\|_6 \|\boldsymbol{v}_m\|_{3,\Omega_R} |\boldsymbol{\varphi}_2|_{1,2} + \|\boldsymbol{u}\|_{6,\Omega^R} \|\boldsymbol{v}_m\|_{3,\Omega^R} |\boldsymbol{\varphi}_2|_{1,2} \\
&\le \left(\|\boldsymbol{u}\|_6 \|\boldsymbol{v}_m\|_{3,\Omega_R} + M_2 \,\|\boldsymbol{u}\|_{6,\Omega^R} \right) \|\boldsymbol{\varphi}\|_Y \, ,
\end{aligned}
\tag{3.10}
$$

where M_2 denotes an upper bound for $\|\boldsymbol{v}_m\|_X$. Set $M_3 = \max\{\|\boldsymbol{u}\|_3, \|\boldsymbol{u}\|_6, M_2\}$. Collecting (3.9) and (3.10), we thus obtain

$$
\|[\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{v}_m)\|_{Y^*} \le M_3 \left(\|\boldsymbol{v}_m\|_{3,\Omega_R} + \|\boldsymbol{u}\|_{3,\Omega^R} + \|\boldsymbol{u}\|_{6,\Omega^R} \right) . \tag{3.11}
$$

We now let $m \to \infty$ in (3.11). By (3.3), we find that X_1 continuously embeds in the Sobolev space $W^{1,2}(\Omega_R)$, for all $R > 0$. Thus, by (3.7), by (3.8) and by the Rellich theorem, the first term on the right-hand side of (3.11) tends to zero as $m' \to \infty$, for some $\{m'\} \subset \{m\}$. Successively, we let $R \to \infty$, which, again with the help of (3.3), causes the second and the third term to go to zero as well. We thus deduce $\|[\boldsymbol{M}'(\boldsymbol{u})](\boldsymbol{v}_{m'})\|_{Y^*} \to 0$ as $m' \to \infty$, for each fixed $\boldsymbol{u} \in X_1$, which completes the proof of the compactness of the operator $\boldsymbol{M}'(\boldsymbol{u})$. Our next and final objective is to show that the linear operator $\boldsymbol{A} : \{\boldsymbol{u}, p\} \in X \mapsto \boldsymbol{A}(\boldsymbol{u}, p) \in Y^*$ defined in (3.4) is Fredholm and that $\operatorname{ind}(\boldsymbol{A}) = -3$; after that, from the definition of a Fredholm map and from the fact that the index of a (linear) Fredholm operator is left invariant by a compact perturbation [14, Theorem 5.E], we find $\operatorname{ind}(\boldsymbol{N}) = -3$. Clearly, the operator $\boldsymbol{A}$ is graph-closed. Moreover, from [3, p. 282 and Theorem V.5.1] it follows that

$$
\mathsf{N}(\boldsymbol{A}) = \{\boldsymbol{0}\} , \quad \mathsf{R}(\boldsymbol{A}) = \{\boldsymbol{f} \in Y^* : \ \langle \boldsymbol{f}, \boldsymbol{h}^{(i)} \rangle = 0, \ i = 1, 2, 3\} , \tag{3.12}
$$

where $\boldsymbol{h}_i \in D^{1,3}_{0,\sigma}(\Omega)$ $(\subset Y)$, $i = 1, 2, 3$, are three independent functions. It is now easy to show that there exist three independent elements of Y^*, $\boldsymbol{l}_k$, $k = 1, 2, 3$, such that, denoting by S their linear span, we have

$$
Y^* = \mathsf{R}(\boldsymbol{A}) \oplus \mathsf{S} . \tag{3.13}
$$

Since $\dim(\mathsf{S}) = 3$, from (3.12) we then find $\operatorname{ind}(\boldsymbol{A}) = \dim[\mathsf{N}(\boldsymbol{A})] - \operatorname{codim}[\mathsf{R}(\boldsymbol{A})] = -3$. In order to prove (3.13), let L_k, $k = 1, 2, 3$, be the vector spaces generated by $\{\boldsymbol{h}^{(2)}, \boldsymbol{h}^{(3)}\}$, $\{\boldsymbol{h}^{(1)}, \boldsymbol{h}^{(3)}\}$ and $\{\boldsymbol{h}^{(1)}, \boldsymbol{h}^{(2)}\}$, respectively. Set $d_k := \inf_{\boldsymbol{h} \in L_k} \|\boldsymbol{h}^{(k)} - \boldsymbol{h}\|_Y$ (> 0). From a corollary to the Hahn-Banach theorem (see, *e.g.*, [14, Proposition I.2.3]) we know that there exists $\boldsymbol{l}_k \in Y^*$ such that

$$
\|\boldsymbol{l}_k\|_{Y^*} = d_k^{-1} , \quad \langle \boldsymbol{l}_k, \boldsymbol{h}^{(j)} \rangle = \delta_{kj} . \tag{3.14}
$$

We claim the validity of (3.13), where S is the vector space generated by $\{\boldsymbol{l}_1, \boldsymbol{l}_2, \boldsymbol{l}_3\}$. In fact, obviously, $\mathsf{R}(\boldsymbol{A}) \cap \mathsf{S} = \emptyset$. Furthermore, for any $\boldsymbol{f} \in Y^*$ we have, with the help of (3.14), that

$$
\boldsymbol{f} - \sum_{k=1}^{3} \langle \boldsymbol{f}, \boldsymbol{h}^{(k)} \rangle \boldsymbol{l}_k \in \mathsf{R}(\boldsymbol{A})
$$

and (3.13) follows. □

Remark 3.1. The assumption, in Theorem 3.1, that $\Omega_0 \neq \emptyset$ is crucial. In fact, if $\Omega = \mathbb{R}^3$, then $\operatorname{ind}(\boldsymbol{A}) = 0$ [3, Theorem IV.2.2], and so, by the same argument used in the proof of Theorem 3.1, we can show that ind $(\boldsymbol{N}) = 0$. This is consistent with the results of [8] that prove (local) well-posedness of problem (1.1) in the space $D^{1,3/2}_{0,\sigma}(\mathbb{R}^3) \cap D^{1,2}_{0,\sigma}(\mathbb{R}^3)$.

References

1. N. Aronszajn and E. Gagliardo, *Interpolation Spaces and Interpolation Methods*, Ann. Mat. Pura Appl., **68**, 1965, 51–117. MR0226361 (37:1951)
2. W. Borchers and T. Miyakawa, *On Stability of Exterior Stationary Navier-Stokes Flows*, Acta Math., **174**, 1995, 311–382. MR1351321 (96j:35186)
3. G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. I. Linearized Steady Problems.* Springer Tracts in Natural Philosophy, vol. 38. Springer-Verlag, New York, 1998 (Revised Edition). MR1284205 (95i:35216a)
4. G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. II. Nonlinear Steady Problems.* Springer Tracts in Natural Philosophy, vol. 39. Springer-Verlag, New York, 1998 (Revised Edition). MR1284206 (95i:35216b)
5. G. P. Galdi and M. Padula, *Existence of Steady Incompressible Flows Past an Obstacle*, Mathematical Analysis of Phenomena in Fluid and Plasma Dynamics, RIMS Kokyuroku, Kyoto, vol. 745, 1991, 87–101.
6. G. P. Galdi and C.G. Simader, *New Estimates for the Steady-State Stokes Problem in Exterior Domains with Applications to the Navier-Stokes Problem*, Differential Integral Equations, **7**, 1994, 847–861. MR1270107 (95c:35192)
7. H. Kozono and M. Yamazaki, *Exterior Problem for the Stationary Navier-Stokes Equations in the Lorentz Space*, Math. Ann., **310**, 1998, 279–305. MR1602012 (98m:35159)
8. H. Kozono and H. Sohr, *On Stationary Navier-Stokes Equations in Unbounded Domains*, Ricerche Mat., **42**, 1993, 69–86. MR1283806 (95d:35128)
9. H. Kozono, H. Sohr, and M. Yamazaki, *Representation Formula, Net Force and Energy Relation to the Stationary Navier-Stokes Equations in 3-Dimensional Exterior Domains*, Kyushu J. Math., **51**, 1997, 239–260. MR1437320 (98g:35163)
10. J. Leray, *Étude de Diverses Équations Intégrales non Linéaires et de Quelques Problèmes que Pose l'Hydrodynamique*, J. Math. Pures Appl., **12**, 1933, 1–82.
11. T. Miyakawa, *On Uniqueness of Steady Navier-Stokes Flows in an Exterior Domain*, Adv. Math. Sci. Appl., **5**, 1995, 411–420. MR1360998 (97f:35164)
12. S. Smale, *An Infinite Dimensional Version of Sard's Theorem*, Amer. J. Math., **87**, 1965, 861–866. MR0185604 (32:3067)
13. H. Sohr, *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*, Birkhäuser Advanced Texts, Birkhäuser Verlag, Basel, 2001. MR1928881 (2004b:35265)
14. E. Zeidler, *Applied Functional Analysis: Main Principles and Their Applications*, Applied Math. Sci., vol. 109, Springer-Verlag, 1995. MR1347692 (96i:00006)

Department of Mechanical Engineering and Materials Science, University of Pittsburgh, Pittsburgh, Pennsylvania 15261

E-mail address: galdi@engr.pitt.edu