A STEADY-STATE EXTERIOR NAVIER-STOKES PROBLEM THAT IS NOT WELL-POSED

GIOVANNI P. GALDI

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Abstract. We prove that the exterior Navier-Stokes problem with zero velocity at infinity is not well-posed in homogeneous Sobolev spaces. This result complements and clarifies well-known previous results obtained by various authors.

1. Introduction

Let $\Omega$ be the complement of the closure of a bounded domain, $\Omega_0$, of $\mathbb{R}^3$ of class $C^2$. The objective of this paper is to investigate the well-posedness of the following Navier-Stokes boundary value problem:

\[
\begin{align*}
-\nu \Delta u + u \cdot \nabla u &= -\nabla p + f, \\
\text{div } u &= 0,
\end{align*}
\]

in $\Omega$, $\nabla u = 0$ at $\partial \Omega$, and $\lim_{|x| \to \infty} u(x) = 0$.

In homogeneous Sobolev spaces. We recall that (1.1) governs the steady-state motion of a viscous liquid, $L$, in the exterior of the “rigid obstacle” $\Omega_0$. In particular, $u$ and $p$ are velocity and pressure fields, respectively, and $\nu > 0$ is the (constant) kinematical viscosity of $L$, while $f$ is the prescribed body force acting on $L$.

In order to describe our results, we denote by $D^{1,q}_0(\Omega)$, $1 < q < \infty$, the homogeneous Sobolev space defined as the completion of smooth vector functions with compact support in $\Omega$, $C^\infty(\Omega)$, in the Dirichlet norm $|\cdot|_{1,q} := \left(\int_{\Omega} |\nabla \cdot |_{q}\right)^{1/q}$, and by $D^{-1,q'}_0(\Omega)$ its (strong) dual with corresponding norm $|\cdot|_{-1,q'}$ ($q' := q/(q-1)$); see, e.g. §II.5, §II.6. We also indicate by $D^{1,q}_{0,\sigma}(\Omega)$ the subspace of $D^{1,q}_0(\Omega)$ of solenoidal functions, $v$, namely, satisfying $\text{div } v = 0$ in $\Omega$.

It is well known—basically, since the work of J. Leray [10]—that for each $f \in D^{-1,2}_0(\Omega)$, (1.1) has at least one solution (in the sense of distributions) $u \in D^{1,2}_{0,\sigma}(\Omega)$, with corresponding $p \in L^2(\omega)$, for an arbitrary bounded domain $\omega \subset \Omega$. Moreover,
if $f$ is sufficiently smooth and decays “fast enough” at large distances, then $u$ belongs also to $D_{0,\sigma}^{1,3/2}(\Omega)$, for all $q > 2$ [13, 14].

The interesting question that has attracted the attention of several mathematicians is the solvability of (1.1) in the class of those $u \in D_{0,\sigma}^{1,q}(\Omega) \cap D_{0,\sigma}^{1,2}(\Omega)$, when $q < 2$; see [3, 8, 2, 11, 9, 7]. The results proved in these papers are many-fold, and we would like to recall the most relevant. In the first place, because of the particular structure of the nonlinear term, $u \cdot \text{grad } u$, one has to restrict to the case $q = 3/2$. Furthermore, if $\Omega = \mathbb{R}^3$ (namely, $\Omega_0 = \emptyset$), then under the assumption $f \in D_{0,\sigma}^{1,3/2}(\mathbb{R}^3) \cap D_{0,\sigma}^{1,2}(\mathbb{R}^3)$ of “sufficiently small” magnitude, solutions do exist in the class where $u \in D_{0,\sigma}^{1,3/2}(\mathbb{R}^3) \cap D_{0,\sigma}^{1,2}(\mathbb{R}^3)$. By the standard theory on representation of functionals on homogeneous Sobolev spaces [3, Theorem III.5.2], it then follows that $p \in L^{3/2}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. In addition, these solutions are also unique and depend continuously upon the data. On the other side, if $\Omega_0 \neq \emptyset$, we have that, under the assumption $f \in D_{0,\sigma}^{1,3/2}(\Omega) \cap D_{0,\sigma}^{1,2}(\Omega) \equiv Y^*(\Omega)$, a solution $u \in D_{0,\sigma}^{1,3/2}(\Omega) \cap D_{0,\sigma}^{1,2}(\Omega) \equiv X_1(\Omega)$, with associated $p \in L^{3/2}(\Omega) \cap L^2(\Omega) \equiv X_2(\Omega)$, can exist only if $u$ and $f$ satisfy the nonlocal compatibility condition

$$\begin{align*}
0 &= \int_{\partial \Omega} [\nu(\text{grad } u + (\text{grad } u)^\top) - pI] \cdot n + \int_{\Omega} F \cdot n = -\mathcal{F}_1 - \mathcal{F}_2,
\end{align*}$$

in a distributional sense. In this equation, $\top$ denotes transpose, $I$ is the identity matrix, $n$ is the unit outer normal to $\partial \Omega$, and $\text{div } F = f$.

The objective of this paper is to show that, in fact, if $\Omega_0 \neq \emptyset$, problem (1.1) is not well-posed in the space $X(\Omega) \equiv X_1(\Omega) \times X_2(\Omega)$. More precisely, we prove that if, for a certain $\mathcal{F} \in Y^*(\Omega)$, (1.1) has a solution $\{ \mathbf{u}, p \} \in X(\Omega)$, then in any arbitrary $Y^*$-neighborhood of $\mathcal{F}$ we can find a “body force” $f$ such that problem (1.1) has no solution $\{ \mathbf{u}, p \} \in X(\Omega)$; see Theorem 6.1. We obtain this result by using classical properties of nonlinear Fredholm maps with negative index, due to S. Smale [12], that we recall in the following section.

The physical interpretation of our result goes as follows. Because of (1.1), the obstacle $\Omega_0$ is at rest. This implies that the total force, $\mathcal{F}$, acting on $\Omega_0$ must vanish. In general, $\mathcal{F}$ is the sum of three contributions: $\mathcal{F}_1$, due to the action of the liquid; $\mathcal{F}_2$, due to the body force acting on the liquid, and $\mathcal{F}_3$, representing the external force directly applied to $\Omega_0$. Clearly, for any $\mathcal{F}_1$ and $\mathcal{F}_2$, we can always find $\mathcal{F}_3$ such that $\mathcal{F}_1 + \mathcal{F}_2 + \mathcal{F}_3 = 0$, so that $\Omega_0$ is “kept in place”. However, condition (1.2) tells us that, in the class $X(\Omega)$, $\mathcal{F}_3$ is necessarily zero and, consequently, the obstacle $\Omega_0$ must be kept in place only by the contribution due to the body force, $f$, acting on the liquid. Our result then states that forces $f$ for which this happens are “rare”. Notice that, of course, the case $\Omega_0 = \emptyset$ does not present such a problem.

In conclusion, we wish to mention that, as shown in [6, 7], problem (1.1) is well-posed (for “small” $f$) in appropriate function spaces other than $X(\Omega)$, where condition (1.2) does not necessarily hold.

2. Some preliminary results

In this section we recall some standard properties of nonlinear Fredholm maps.

Let $X$ and $Z$ be separable Banach spaces, with norms $|| \cdot ||_X$ and $|| \cdot ||_Z$, respectively, and let $M$ be a map defined on the whole $X$ with range $R(M) \subset Z$. For $z \in Z$, we put $M(z) = \{ x \in X : M(x) = z \}$ (the solution set of the map $M$ at $z$) and $N(M) := \{ x \in X : M(x) = 0 \}$ (the null set of the map $M$). Furthermore, we shall
write $M \in C^k(X,Z)$, $k$ a nonnegative integer, if, at each $x \in X$, $M$ has continuous derivatives, in the sense of Fréchet, up to the order $k$ included. The derivative of $M$ at $x$ is denoted by $M'(x)$.

A map $M \in C^1(X,Z)$ is said to be Fredholm if and only if the integers $\alpha := \dim N[M'(x)]$ and $\beta := \codim R[M'(x)]$ are both finite. The integer $\ind(M) := \alpha - \beta$ is then independent of the particular $x \in X$ [14, §5.15] and is called the index of $M$.

For a given $M \in C^1(X,Z)$, a point $x \in X$ is a regular point iff $M'(x)$ is surjective. A point $z \in Z$ is called a regular value iff either $\sigma_M(z) = \emptyset$ or $\sigma_M(z)$ is constituted only by regular points.

The following well-known result is due to Smale [12].

**Lemma 2.1.** Let $M \in C^k(X,Z)$ be a Fredholm map with $k > \max\{\ind(M),0\}$. Then, the set of regular values of $M$, $R$, is dense in $Z$. More specifically, $Z - R$ is of Baire first category.

An immediate, and fundamental to our aims, consequence of Lemma 2.1 is given by the following corollary, whose simple proof we include for the reader’s convenience.

**Corollary 2.1.** Suppose $M$ satisfies the assumption of Lemma 2.1 and that, for some $\varphi \in Z$, $\sigma_M(\varphi) \neq \emptyset$. Then, if $\ind(M) < 0$, the problem $M(x) = z$ is not well-posed, in the sense that the solution $x$ cannot depend continuously on the data $z$. Precisely, for any $\varepsilon > 0$, we can find $z \in Z$ such that $\|z - \varphi\|_Z < \varepsilon$ and the equation $M(x) = z$ has no solution.

**Proof.** For the given $\varepsilon$, by Lemma 2.1 we may choose $z$ to be a regular value. Now, if we suppose, by contradiction, $\sigma_M(z) \neq \emptyset$, we would have that $M'(x)$ is surjective, for all $x \in \sigma_M(z)$, which would imply $\ind(M) = \dim N[M'(x)] \geq 0$, in contrast with the assumption. \hfill $\square$

An equivalent way of phrasing Corollary 2.1 is that, under the stated assumptions on $M$, the interior of $R(M)$ is empty.

### 3. Application to the exterior Navier-Stokes problem

We begin to rewrite [14] as a nonlinear equation in a suitable Banach space. We set $Y = Y(\Omega) := D_0^{1,3}(\Omega) + D_0^{1,2}(\Omega)$ equipped with the norm

$$
\|\varphi\|_Y := \inf \left\{ |\varphi_1|_{1,3} + |\varphi_2|_{1,2} : \varphi = \varphi_1 + \varphi_2, \varphi_1 \in D_0^{1,3}(\Omega), \varphi_2 \in D_0^{1,2}(\Omega) \right\}.
$$

Since both $D_0^{1,3}(\Omega)$ and $D_0^{1,2}(\Omega)$ are reflexive, it follows that for any $\varphi \in Y$ there exist $\varphi_1 \in D_0^{1,3}(\Omega)$ and $\varphi_2 \in D_0^{1,2}(\Omega)$ such that

$$
\|\varphi\|_Y = |\varphi_1|_{1,3} + |\varphi_2|_{1,2}.
$$

Also, since $\{ \varphi \in C_0^{\infty}(\Omega) : \div \varphi = 0 \}$ is dense in $D_0^{1,3}(\Omega) \cap D_0^{1,2}(\Omega)$ [3, Exercise III.6.2], we have that the (strong) dual, $Y^*$, of $Y$ can be isomorphically represented as $Y^* = D_0^{-1,3/2}(\Omega) \cap D_0^{-1,2}(\Omega)$ with associated norm $\|\cdot\|_{Y^*} := \|\cdot\|_{1,3/2} + |\cdot|_{1,2}$; see [1]. Moreover, $Y^*$ is separable [3, Exercise II.5.1].

If we now multiply, formally, (1.1) by $\varphi \in Y$ and integrate by parts over $\Omega$, we find:

$$
\nu(\grad u, \grad \varphi) - (p, \div \varphi) - (u \cdot \grad \varphi, u) = \langle f, \varphi \rangle,
$$

where $\nu \in L^2(\Omega)$ is a viscosity coefficient.
where $(\cdot, \cdot)$ and $(\cdot, \cdot)$ represent the $L^2$-scalar product and duality pairing between $Y^*$ and $Y$, respectively. Set
\[
X_1 = X_1(\Omega) := D^{1,3/2}_0(\Omega) \cap D^{1,2}_0(\Omega), \quad \| \cdot \|_{X_1} := \| \cdot \|_{1,3/2} + \| \cdot \|_{1,2} \\
X_2 = X_2(\Omega) := L^{3/2}(\Omega) \cap L^2(\Omega), \quad \| \cdot \|_{X_2} := \| \cdot \|_{3/2} + \| \cdot \|_{2} \\
X = X(\Omega) := X_1 \times X_2, \quad \| (u,p) \|_X := \| u \|_{X_1} + \| p \|_{X_2}.
\]
The space $X$ is separable [3, Exercise II.5.1]. Because of the continuous embeddings
\begin{equation}
D^{1,3/2}_0(\Omega) \subset L^3(\Omega), \quad D^{1,2}_0(\Omega) \subset L^6(\Omega)
\end{equation}
(see [3, Theorem II.5.1]), it is immediately checked (by the Hölder inequality) that, for any $(u, p) \in X$, the left-hand side of (3.2) defines two linear functionals, $A(u, p)$ (Stokes operator) and $M(u)$, on $Y$ as follows:
\begin{equation}
(A(u, p), \varphi) := \nu(\text{grad } u, \text{grad } \varphi) - (p, \text{div } \varphi), \quad (M(u), \varphi) := -(u \cdot \text{grad } \varphi, u).
\end{equation}
Therefore, (3.2) can be rewritten in the following operator equation form:
\begin{equation}
N(u, p) = f \quad \text{in } Y^*,
\end{equation}
where the map $N$ is defined as
\[
N : \{ u, p \} \in X \mapsto A(u, p) + M(u) \in Y^*.
\]
Set $B_a(y) := \{ f \in Y^* : \| f - y \|_{Y^*} < a \}$, $a > 0$. We have the following.

**Theorem 3.1.** Let $\Omega_0 \neq \emptyset$. Assume that (3.5) has a solution $(\overline{\bm{u}}, \overline{\bm{\tau}}) \in X$ corresponding to some $\overline{f} \in Y^*$. Then, for any $\varepsilon > 0$, there exists $f \in B_\varepsilon(\overline{f})$ such that (3.3) does not have a solution.

**Proof.** In view of Corollary 2.1, it suffices to show that $N$ is a Fredholm map of negative index. In order to reach this goal, we begin to observe that $N \in C^1(X, Y)$ and that
\[
[N'(u, p)](w, \tau) = A(w, \tau) + [M'(u)](w),
\]
where
\begin{equation}
([M'(u)](w), \varphi) = -(u \cdot \text{grad } \varphi, w) - (w \cdot \text{grad } \varphi, u), \quad \varphi \in Y.
\end{equation}
(The proof of these properties is completely standard, and, therefore, it will be omitted.) We prove, next, that $M'(u)$ is compact at each $u \in X_1$. Let $\{ w_m \}$ be a sequence in $X_1$ such that
\begin{equation}
\| w_m \|_{X_1} \leq M_1,
\end{equation}
where $M_1$ is independent of the integer $m$. Since $D^{1,3/2}_0(\Omega)$ and $D^{1,2}_0(\Omega)$ are reflexive, we can select a subsequence (again denoted by $\{ w_m \}$) and find $w \in X_1$ such that
\begin{equation}
w_m \rightharpoonup w \quad \text{weakly in } D^{1,3/2}_0(\Omega) \quad \text{and in } D^{1,2}_0(\Omega).
\end{equation}
From (3.6) we find that
\begin{equation}
([M'(u)](v_m), \varphi) = -(u \cdot \text{grad } \varphi, v_m) - (v_m \cdot \text{grad } \varphi, u), \quad \varphi \in Y,
\end{equation}
where $v_m := w - w_m$. For sufficiently large $R > 0$, we set $\Omega_R = \Omega \cap \{|x| < R\}$, $\Omega^R = \Omega \cap \{|x| > R\}$ and denote by $\| \cdot \|_{r,A}$ the $L^r(A)$-norm. Recalling that...
\( \varphi = \varphi_1 + \varphi_2 \), where \( \varphi_i, i = 1, 2 \), satisfy (3.1), with the help of the Hölder inequality we find

\[
|(u \cdot \text{grad} \varphi_1, v_m)| \leq \|u\|_3 \|v_m\|_3,\Omega \|\varphi_1\|_{1,3} + \|u\|_{3,\Omega^N} \|v_m\|_{3,\Omega^N} |\varphi_1|_{1,3}
\]

(3.10)

\[
|u \cdot \text{grad} \varphi_2, v_m| \leq \|u\|_6 \|v_m\|_{3,\Omega^N} |\varphi_2|_{1,2} + \|u\|_{6,\Omega^N} \|v_m\|_{3,\Omega^N} |\varphi_2|_{1,2}
\]

and (3.10) follows. □

Remark 3.1. The assumption, in Theorem 5.3, that \( \Omega_0 \neq \emptyset \) is crucial. In fact, if \( \Omega = \mathbb{R}^3 \), then ind \((A) = 0 \) [3, Theorem IV.2.2], and so, by the same argument used.
in the proof of Theorem [3.1] we can show that \( \text{ind} (N) = 0 \). This is consistent with the results of [8] that prove (local) well-posedness of problem (1.1) in the space \( D^{1,3/2}_{0,\sigma} (\mathbb{R}^3) \cap D^{1,2}_{0,\sigma} (\mathbb{R}^3) \).

**References**


Department of Mechanical Engineering and Materials Science, University of Pittsburgh, Pittsburgh, Pennsylvania 15261

E-mail address: galdi@engr.pitt.edu