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INCOMPARABLE PRIME IDEALS IN COMMUTATIVE RADICAL FRÉCHET ALGEBRAS

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ABSTRACT. Let R be a commutative radical Fréchet algebra having a non-nilpotent element a with $a \in \overline{Ra}$. Then R contains a continuum of incomparable prime ideals.

In [3], J. Esterle proved the following result; it is a main ingredient in his proof that epimorphisms from $\mathcal{C}_0(\Omega)$ onto Banach algebras are continuous.

Theorem (Esterle)**.** *Let R be a commutative radical Banach algebra. Suppose that there exists a non-nilpotent element $a \in R$ with $a \in \overline{Ra}$. Then the set of prime ideals in R, ordered by inclusion, does not form a chain.*

Thus, each algebra R as in the theorem contains at least two prime ideals which are incomparable. In [1], Bouloussa extended the result to commutative radical Fréchet algebras. In this paper, we shall extend this by producing a continuum of pairwise incomparable prime ideals. This will be proved as a consequence of a result on ideals in (not necessarily commutative) radical Fréchet algebras and the existence of a continuum of "almost disjoint" subsets of $\mathbb{N}$ due to Sierpinski.

1. Preliminaries

More details of the following can be found, for example, in [2].

A Fréchet algebra is a topological algebra A whose topology is determined by a sequence of algebra seminorms (p_n) such that

$$d(a,b) = \sum_{n=1}^{\infty} \frac{\min\{p_n(a-b), 1\}}{2^n} \quad (a, b \in A)$$

is a complete metric.

Let A be an algebra. Denote by $A^{\#}$ the *conditional unitization* of A: adjoining an identity in the case where A is non-unital.

Let S be a subset of an algebra A. For $n \in \mathbb{N}$, denote by S^n the linear span of $\{a_1 \cdots a_n : a_i \in S\}$ (and $A^{(n)}$ the n-fold Cartesian product of A).

For clarity, we shall use boldface characters to denote tuples of elements; for example, we set

$$\boldsymbol{x} = (x_1, \ldots, x_m) \quad \text{or} \quad \boldsymbol{y} = (y_1, \ldots, y_n)\,.$$

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2. A continuum of incomparable prime ideals

Lemma 1. *Let R be a radical Fréchet algebra. Let $m, n \in \mathbb{N}$, and let $(k_1, \ldots, k_n) \subset \mathbb{N}$. Let I be a left ideal in R satisfying that $I^{(m)} = \overline{R \cdot I^{(m)}}$. Suppose that p is a continuous algebra seminorm on R such that $I^{2^k-1} \not\subset \ker p$, where $k = k_1 + \cdots + k_n$. Then, for each function $\phi : R \to \mathbb{R}^+$, the set of all $(\boldsymbol{x}, \boldsymbol{y}) \in I^{(m)} \times I^{(n)}$ satisfying*

$$\sum_{i=1}^{m} \phi(vx_i) < p(vy_1^{k_1} \cdots y_n^{k_n}),$$

for some $v \in R^{\#}$, is dense in $I^{(m+n)}$.

Proof. Denote by U the set under consideration. First, let $(\boldsymbol{a}, \boldsymbol{b})$ be arbitrary in $I^{(m+n)}$ with $\boldsymbol{a} \in R \cdot I^{(m)}$ and $p(b_1^{k_1} \cdots b_n^{k_n}) \neq 0$. Let $\boldsymbol{a}' = (a'_1, \ldots, a'_m) \in I^{(m)}$ and $c \in R$ such that $c \cdot \boldsymbol{a}' = \boldsymbol{a}$. Since $p(b_1^{k_1} \cdots b_n^{k_n}) \neq 0$, we see that (cf. [1, Lemme 1.1]), for each $r \in \mathbb{N}$, there exists $\lambda_r \in \mathbb{C}$ such that

$$0 < |\lambda_r| < \frac{1}{r} \quad \text{and that} \quad p((\lambda_r + c)^{-1} b_1^{k_1} \cdots b_n^{k_n}) > \sum_{i=1}^{m} \phi(a'_i).$$

Set $v_r = \lambda_r + c \in R^{\#}$ and $\boldsymbol{x}_r = v_r \cdot \boldsymbol{a}' \in I^{(m)}$. Then

$$\sum_{i=1}^{m} \phi(v_r^{-1} x_{r,i}) = \sum_{i=1}^{m} \phi(a'_i) < p(v_r^{-1} b_1^{k_1} \cdots b_n^{k_n}),$$

so that $(\boldsymbol{x}_r, \boldsymbol{b}) \in U$ $(r \in \mathbb{N})$. We have $\lim \boldsymbol{x}_r = \boldsymbol{a}$, so $(\boldsymbol{a}, \boldsymbol{b}) \in \overline{U}$.

The set

$$\left\{ \boldsymbol{b} \in I^{(n)} : \; p(b_1^{k_1} \cdots b_n^{k_n}) \neq 0 \right\}$$

must be dense in $I^{(n)}$. For otherwise, there exist open subsets B_i of I $(1 \le i \le n)$ such that $p(b_1^{k_1} \cdots b_n^{k_n}) = 0$ whenever $(b_1, \ldots, b_n) \in \prod_{i=1}^{n} B_i$. Then we see that $p(b^k) = 0$ whenever $b \in I$. It then follows from Nagata-Higman's theorem (see, for example, [2, Theorem 1.3.33]) that $I^{2^k-1} \subset \ker p$, contradicting the hypothesis.

Hence U is dense in $I^{(m+n)}$ as claimed. □

The following theorem extends [1, Theorem 1.2]; in the commutative case, it is possible to extend the proof in [1] to yield the same result.

Theorem 2. *Let R be a radical Fréchet algebra. Suppose that there exists a non-nilpotent element $a \in R$ such that $a \in \overline{Ra}$. Then there exists a sequence (a_n) in R such that $\overline{Ra_n} = \overline{Ra}$ $(n \in \mathbb{N})$ and such that, for each Fréchet algebra A containing R as a topological subalgebra,*

$$a_{j_1}^{k_1} \cdots a_{j_n}^{k_n} \notin a_{i_1} A + \ldots + a_{i_m} A$$

for every finite sequences $\boldsymbol{i} = (i_1, \ldots, i_m)$, $\boldsymbol{j} = (j_1, \ldots, j_n)$, and $\boldsymbol{k} = (k_1, \ldots, k_n)$ in $\mathbb{N}$ such that $\boldsymbol{i}$ and $\boldsymbol{j}$ are disjoint.

Proof. Set $I = \overline{Ra}$. We see that $0 \neq a^m \in I^m$ and

$$I^{(m)} = \overline{R \cdot I^{(m)}} \quad (m \in \mathbb{N}).$$

For each $k \in \mathbb{N}$, fix a continuous algebra seminorm q_k of R such that

$$I^{2^k-1} \not\subset \ker q_k.$$

Without loss of generality, we can assume that (q_k) is an increasing sequence of seminorms defining the topology of R. Denote by Ω the product space $I^{\mathbb{N}}$; its topology is defined by a complete metric.

Let d be any complete metric defining the topology of R. For each $n \in \mathbb{N}$, set $V_n = \{x \in I : \ d(a, vx) < 1/n \text{ for some } v \in R\}$. Then V_n is an open subset of I. We see that $a \in V_n$, and V_n is closed under multiplication on the left by elements in $R^{\#} \setminus R$. Hence, V_n is dense in $\overline{Ra} = I$.

For each $m, n \in \mathbb{N}$, set

$$V_{n,m} = \{(x_r) \in \Omega : \ x_r \in V_n \ (1 \leq r \leq m)\}.$$

From the previous paragraph, we see that $V_{n,m}$ is an open dense subset of Ω.

For each $\boldsymbol{s} = (l, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ in $\mathbb{N} \times \mathbb{N}^m \times \mathbb{N}^n \times \mathbb{N}^n$, with $\boldsymbol{i}$ and $\boldsymbol{j}$ being disjoint, let $U_{\boldsymbol{s}}$ be the set of all $(x_r) \in \Omega$ with the property that

$$l^2 \sum_{t=1}^{m} q_l(v x_{i_t}) < q_k(v x_{j_1}^{k_1} \ldots x_{j_n}^{k_n})$$

for some $v \in R^{\#}$, where $k = k_1 + \ldots + k_n$. By Lemma 1, this is a dense (open) subset of Ω.

By the Baire category theorem, there exists (a_r) belonging to all $U_{\boldsymbol{s}}$ and $V_{n,m}$ above. Since $(a_r) \in V_{n,m}$ $(n, m \in \mathbb{N})$, it follows that $a \in \overline{Ra_r}$, and so $\overline{Ra} = \overline{Ra_r}$ $(r \in \mathbb{N})$.

Let A be a Fréchet algebra containing R as a topological subalgebra. Let $m, n \in \mathbb{N}$, and let $\boldsymbol{i} = (i_1, \ldots, i_m)$, $\boldsymbol{j} = (j_1, \ldots, j_n)$, and $\boldsymbol{k} = (k_1, \ldots, k_n)$ be finite sequences in $\mathbb{N}$ such that $\boldsymbol{i}$ and $\boldsymbol{j}$ are disjoint. It remains to prove that

$$a_{j_1}^{k_1} \cdots a_{j_n}^{k_n} \notin a_{i_1} A + ... + a_{i_m} A.$$

Indeed, assume toward a contradiction that $a_{j_1}^{k_1} \cdots a_{j_n}^{k_n} = a_{i_1} c_1 + ... + a_{i_m} c_m$ for some $c_r \in A$. Set $k = k_1 + \ldots + k_n$. The previous paragraph shows that there exists an element $v_l \in R^{\#}$ such that

$$\sum_{t=1}^{m} q_l(v_l a_{i_t}) < \frac{1}{l} \quad \text{and} \quad q_k(v_l a_{j_1}^{k_1} \ldots a_{j_n}^{k_n}) > l \quad (l \in \mathbb{N}).$$

We then see that $\lim_{l \to \infty} v_l a_{i_t} = 0$, so $\lim_{l \to \infty} v_l \sum_{t=1}^{m} a_{i_t} c_t = 0$, but $(v_l a_{j_1}^{k_1} \ldots a_{j_n}^{k_n} : l \in \mathbb{N})$ can never converge in R (and hence can never converge to 0 in A), a contradiction. $\square$

We now present the construction due to Sierpinski of a family $\{E_\alpha : \ \alpha \in \mathfrak{c}\}$ of infinite subsets of $\mathbb{N}$ satisfying the following properties (cf. [6]):

(i) $\mathbb{N} = \bigcup_{\alpha \in \mathfrak{c}} E_\alpha$, and

(ii) $E_\alpha \cap E_\beta$ is finite for each $\alpha \neq \beta \in \mathfrak{c}$.

The set $\mathbb{N}$ is isomorphic to

$$C = \bigcup_{n=1}^{\infty} \{f : \{1, \ldots, n\} \to \{1, 2\}\}.$$

For each $f : \mathbb{N} \to \{1, 2\}$, define

$$C_f = \{\text{the restrictions of } f \text{ to } \{1, \ldots, n\} : \ (n \in \mathbb{N})\}.$$

We see that $C = \bigcup_{f:\mathbb{N}\to\{1,2\}} C_f$ and that $C_f \cap C_g$ is finite for each $f \neq g$. We can then map back from C to $\mathbb{N}$.

Corollary 3. *Let R be a commutative radical Fréchet algebra. Suppose that there exists a non-nilpotent element $a \in R$ such that $a \in \overline{Ra}$. Then there exists a family of prime ideals $(P_\alpha : \alpha \in \mathfrak{c})$ in R such that $P_\alpha \not\subset P_\beta$ $(\alpha \neq \beta \in \mathfrak{c})$.*

Proof. Let (a_n) be a sequence in R as specified in the theorem. We then see that, for each $E \subset \mathbb{N}$, there exists a prime ideal Q_E in R such that $a_i \in Q_E$ $(i \in E)$ but $a_j \notin Q_E$ $(j \notin E)$. Set $P_\alpha = Q_{E_\alpha}$ $(\alpha \in \mathfrak{c})$, where $(E_\alpha : \alpha \in \mathfrak{c})$ is the Sierpinski family of subsets of $\mathbb{N}$ constructed in the previous paragraph. Then (P_α) is the desired collection of prime ideals in R. □

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