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ABELIAN IDEALS AND COHOMOLOGY OF SYMPLECTIC TYPE

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Abstract. Let $\mathfrak{b}$ be a Borel subalgebra of the symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ and let $\mathfrak{n}$ be the corresponding maximal nilpotent subalgebra. We find a connection between the abelian ideals of $\mathfrak{b}$ and the cohomology of $\mathfrak{n}$ with trivial coefficients. Using this connection, we are able to enumerate the number of abelian ideals of $\mathfrak{b}$ with given dimension via the Poincaré polynomials of Weyl groups of types A_{n-1} and C_n.

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Suppose that $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ is a Borel subalgebra, where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the maximal nilpotent subalgebra of $\mathfrak{g}$. Note that $\mathfrak{n}$ is also the nilradical of $\mathfrak{b}$ under the Killing form. Needless to say, the structure of the subalgebras $\mathfrak{b}$ and $\mathfrak{n}$ is important for the study of $\mathfrak{g}$ itself.

The study of abelian ideals in the Boral subalgebra $\mathfrak{b}$ can be traced back to the work by Schur [9] (1905). It has recently drawn considerable attention. Kostant [5] (1998) mentioned Peterson's 2^r-theorem saying that the number of abelian ideals in $\mathfrak{b}$ is exactly 2^r, where $r = \dim \mathfrak{h}$ is the rank of $\mathfrak{g}$. Moreover, Kostant found a relation between abelian ideals of a Borel subalgebra and the discrete series representations of the Lie group. Spherical orbits were described by Panyushev and Röhrle [8] (2001) in terms of abelian ideals. Furthermore, Panyushev [7] (2003) discovered a correspondence of maximal abelian ideals of a Borel subalgebra to long positive roots. Suter [10] (2004) determined the maximal dimension among abelian subalgebras of a finite-dimensional simple Lie algebra purely in terms of certain invariants and gave a uniform explanation for Panyushev's result. Kostant [6] (2004) showed that the powers of the Euler product and the abelian ideals of a Borel subalgebra are intimately related. Cellini and Papi [2] (2004) had a detailed study of certain remarkable posets which form a natural partition of all abelian ideals of a Borel subalgebra.

It is remarkable that the affine Weyl group $\widehat{\mathcal{W}}$ associated with $\mathfrak{g}$ is used in the proof of the 2^r-theorem. On the other hand, Bott [1] gave a celebrated theorem which shows that the Betti numbers b_i of $\mathfrak{n}$ (i.e. the dimension of $H^i(\mathfrak{n})$) can be

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expressed by the Weyl group $\mathcal{W}$ associated with $\mathfrak{g}$. Later Kostant [4] generalized his result to the nilradical of any parabolic subalgebras $\mathfrak{p}$ of $\mathfrak{g}$. ($\mathfrak{n}$ is the nilradical of $\mathfrak{b}$.)

It seems that the Weyl group $\mathcal{W}$ (and its affine group $\widehat{\mathcal{W}}$) can be a bridge for connecting the cohomology of $\mathfrak{n}$ to the abelian ideals of $\mathfrak{b}$. But no one has given an explicit relationship between these two objects so far. In this paper, we shall construct this relationship in the case of $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$.

In the following text, we let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ be the symplectic Lie algebra. Our main theorem of this paper is as follows.

Main Theorem. *Let $\mathcal{I}$ be the set of all abelian ideals of $\mathfrak{b}$ and let S_n be the permutation group on n elements. In terms of a certain map $L : S_n \times \mathcal{I} \to \wedge \mathfrak{n}^*$ defined in (2.16), the cohomology group $H(\mathfrak{n}) = \bigoplus_{(\sigma,I)\in S_n\times\mathcal{I}} \mathbb{C}[L(\sigma, I)]$, where $[L(\sigma, I)]$ is the cohomology class defined by the harmonic cocycle $L(\sigma, I)$.*

An interesting application of this theorem is to compute the number of abelian ideals of $\mathfrak{b}$ with given dimension via the Poincaré polynomials of Weyl groups of types A and C. That is,

Corollary. *The number of abelian ideals of $\mathfrak{b}$ with dimension i is equal to the coefficient of t^i in $\prod_{r=1}^{n}(1 + t^r)$.*

2. Proof of the main result

Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ be the symplectic Lie algebra. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Denote by $\mathcal{W} \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ its Weyl group, by $\Phi = \{\pm 2\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid 1 \le i \ne j \le n\}$ its root system with positive roots $\Phi_+ = \{2\epsilon_i, \epsilon_i \pm \epsilon_j \mid i < j\}$ and by $\pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$ its simple roots. We have the Cartan root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha)$. Then $\mathfrak{b} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha)$ is the associated Borel subalgebra and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ is its nilradical.

Denote by $\mathcal{I}$ the set of all abelian ideals of $\mathfrak{b}$. As mentioned in [10], there is a bijection as follows:

$$\begin{array}{rcl} \Upsilon := \{\Psi \subset \Phi_+ \mid \Psi \dot{+} \Phi_+ \subset \Psi, \Psi \dot{+} \Psi = \emptyset\} & \leftrightarrow & \mathcal{I} \\ \Psi & \mapsto & I_\Psi := \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha, \end{array} \tag{2.1}$$

where $\Psi \dot{+} \Phi_+ := (\Psi + \Phi_+) \cap \Phi_+$ and $\Psi \dot{+} \Psi := (\Psi + \Psi) \cap \Phi_+$.

Denote $\Phi_+^0 := \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\}$.

Lemma 2.1. *Equation $\Psi \cap \Phi_+^0 = \emptyset$ holds for any $\Psi \in \Upsilon$.*

Proof. Suppose that there is an $\epsilon_i - \epsilon_j \in \Psi$. Since $2\epsilon_j \in \Phi_+$ and $(\epsilon_i - \epsilon_j) + 2\epsilon_j = \epsilon_i + \epsilon_j \in \Phi_+$, we have $\epsilon_i + \epsilon_j \in \Psi$. But $(\epsilon_i - \epsilon_j) + (\epsilon_i + \epsilon_j) = 2\epsilon_i \in \Phi_+$, a contradiction. □

Define a partial ordering on

$$\Phi_+^1 := \Phi_+ \setminus \Phi_+^0 = \{\epsilon_i + \epsilon_j \mid 1 \le i \le j \le n\}$$

by

$$\epsilon_{i_1} + \epsilon_{j_1} \prec \epsilon_{i_2} + \epsilon_{j_2} \Leftrightarrow i_1 \ge i_2, j_1 \ge j_2, \tag{2.2}$$

where $i_1 \le j_1, i_2 \le j_2$. In fact, this partial ordering is just the one induced from the partial ordering on the positive roots. A subset $\Psi \subset \Phi_+^1$ is called an increasing subset if for any $x, y \in \Phi_+^1$, the conditions $x \in \Psi$ and $x \prec y$ imply $y \in \Psi$.

The following lemma is obvious.

Lemma 2.2. *The set Υ is the set of all increasing subsets of Φ_+^1.*

We shall show that these increasing subsets also appear in the cohomology of $\mathfrak{n}$ with trivial coefficients.

Choose a nonzero element e_α in $\mathfrak{g}_\alpha$ for each $\alpha \in \Phi_+$. Hence $\{e_\alpha \mid \alpha \in \Phi_+\}$ is a basis of $\mathfrak{n}$. Define a linear function $f_\alpha \in \mathfrak{n}^*$ by $f_\alpha(e_\beta) = \delta_{\alpha,\beta}$. Then $\{f_\alpha \mid \alpha \in \Phi_+\}$ is a basis of $\mathfrak{n}^*$.

Recall the following theorem:

Theorem 2.3 (Bott-Kostant; cf. [1, 4])**.** *The cohomology group*

$$H(\mathfrak{n}) = \bigoplus_{w \in \mathcal{W}} \mathbb{C}[\bigwedge_{\alpha \in \Phi_w} f_\alpha], \tag{2.3}$$

where $\Phi_w = w(-\Phi_+) \cap \Phi_+$, and $[\bigwedge_{\alpha\in\Phi_w} f_\alpha]$ is the cohomology class defined by the (harmonic) cocycle $\bigwedge_{\alpha\in\Phi_w} f_\alpha$.

We see in Theorem 2.3 that Φ_w plays an important role in cohomology. So we shall get some more information about Φ_w.

Recall that for type C_n, the Weyl group $\mathcal{W}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. Precisely, $\mathcal{W}$ can be realized by the composite of all permutations of $\{1, 2, \ldots, n\}$ and $\langle r_i \mid i = 1, 2, \ldots, n\rangle$, where $r_i(j) = -\delta_{i,j} i + (1 - \delta_{i,j})j$ $(j = 1, 2, \ldots, n)$. In this paper, we always use $(i_1, i_2, \ldots, i_n)$ to denote the permutation $j \mapsto i_j$ $(j = 1, 2, \ldots, n)$.

Each element in $\mathcal{W}$ can be expressed in the form

$$w = r_{j_1} r_{j_2} \cdots r_{j_k} (i_1, i_2, \ldots, i_n) \ (0 \le k \le n), \tag{2.4}$$

where $(i_1, i_2, \ldots, i_n)(j_1) < (i_1, i_2, \ldots, i_n)(j_2) < \cdots < (i_1, i_2, \ldots, i_n)(j_k)$. We call it the *standard form* of w.

The following two lemmas are well-known and follow from Kostant's classic paper [4]. Since we only consider the special case of type C, we include the proofs in this case for completeness.

Lemma 2.4. *The set $\Phi_\sigma = \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n, \sigma^{-1}(i) > \sigma^{-1}(j)\}$ for any $\sigma \in S_n \subset \mathcal{W}$. Hence if $\Phi_{\sigma_1} = \Phi_{\sigma_2} (\sigma_1, \sigma_2 \in S_n)$, then $\sigma_1 = \sigma_2$.*

Proof. It is obvious by a direct calculation. □

Lemma 2.5. *For any $w \in \mathcal{W}$, there is a unique element $\eta_w \in S_n$ such that $\Phi_w \cap \Phi_+^0 = \Phi_{\eta_w}$. Precisely, if $w = r_{j_1} r_{j_2} \cdots r_{j_k} (i_1, i_2, \ldots, i_n)$ is in the standard form, then*

$$\eta_w = (i_1, \ldots, \widehat{j_1}, \ldots, \widehat{j_2}, \ldots, \ldots, \widehat{j_k}, \ldots, i_n, j_k, j_{k-1}, \ldots, j_1), \tag{2.5}$$

where $(i_1, \ldots, j_1, \ldots, j_2, \ldots, \ldots, j_k, \ldots, i_n) = (i_1, i_2, \ldots, i_n)$ and the sign $\widehat{\ }$ indicates that the argument below it must be omitted.

Proof. The uniqueness of η_w follows from Lemma 2.4.

Denote $\sigma = (i_1, \ldots, \widehat{j_1}, \ldots, \widehat{j_2}, \ldots, \ldots, \widehat{j_k}, \ldots, i_n, j_k, j_{k-1}, \ldots, j_1)$. We have to show $\Phi_w \cap \Phi_+^0 = \Phi_\sigma$. In fact, for any $\epsilon_i - \epsilon_j \in \Phi_w \cap \Phi_+^0$, we have $w^{-1}(\epsilon_i - \epsilon_j) =$

$\epsilon_{w^{-1}(i)} - \epsilon_{w^{-1}(j)} \in -\Phi_+$. What we need to do is to check $\sigma^{-1}(i) > \sigma^{-1}(j)$. There are four cases as follows.

Case 1: $i, j \notin \{j_1, j_2, \ldots, j_k\}$. Then $w^{-1}(\epsilon_i - \epsilon_j) = \epsilon_{\sigma_0^{-1}(i)} - \epsilon_{\sigma_0^{-1}(j)}$, where $\sigma_0 = (i_1, i_2, \ldots, i_n)$. Hence $\sigma_0^{-1}(i) > \sigma_0^{-1}(j)$. So $\sigma^{-1}(i) > \sigma^{-1}(j)$ by the definition of σ.

Case 2: $i \in \{j_1, j_2, \ldots, j_k\}, j \notin \{j_1, j_2, \ldots, j_k\}$. Then $w^{-1}(\epsilon_i - \epsilon_j) = -\epsilon_{\sigma_0^{-1}(i)} - \epsilon_{\sigma_0^{-1}(j)}$, which is always in $-\Phi_+$. We also do have $\sigma^{-1}(i) > \sigma^{-1}(j)$ by the definition of σ.

Case 3: $i \notin \{j_1, j_2, \ldots, j_k\}, j \in \{j_1, j_2, \ldots, j_k\}$. Then $w^{-1}(\epsilon_i - \epsilon_j) = \epsilon_{\sigma_0^{-1}(i)} + \epsilon_{\sigma_0^{-1}(j)}$, which is always in Φ_+. There should be no such i, j with $\sigma^{-1}(i) > \sigma^{-1}(j)$. It is the case by the definition of σ.

Case 4: $i, j \in \{j_1, j_2, \ldots, j_k\}$. Then $w^{-1}(\epsilon_i - \epsilon_j) = \epsilon_{\sigma_0^{-1}(j)} - \epsilon_{\sigma_0^{-1}(i)}$. Hence $\sigma_0^{-1}(j) > \sigma_0^{-1}(i)$. So $\sigma^{-1}(i) > \sigma^{-1}(j)$ by the definition of σ. □

By this lemma, we can define a map

$$\eta : \mathcal{W} \to S_n, \quad w \mapsto \eta_w. \tag{2.6}$$

Denote

$$\sigma_l := (n, n-1, \ldots, 1) \in S_n, \tag{2.7}$$

which is the longest element in S_n. It is clear that $\sigma_l = \sigma_l^{-1}$ and

$$x \prec y \Leftrightarrow \sigma_l(y) \prec \sigma_l(x) \ (x, y \in \Phi_+^1). \tag{2.8}$$

Lemma 2.6. *We have* $\sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1) \in \Upsilon$.

Proof. Write $w = r_{j_1} r_{j_2} \cdots r_{j_k} \sigma_0$ in the standard form with $\sigma_0 = (i_1, i_2, \ldots, i_n)$. Take any $\epsilon_i + \epsilon_j \in \Phi_w \cap \Phi_+^1$. Then $w^{-1}(\epsilon_i + \epsilon_j) = \epsilon_{w^{-1}(i)} + \epsilon_{w^{-1}(j)} \in -\Phi_+$. There are three cases as follows:

Case 1: $i, j \notin \{j_1, j_2, \ldots, j_k\}$. Then $w^{-1}(\epsilon_i + \epsilon_j) = \epsilon_{\sigma_0^{-1}(i)} + \epsilon_{\sigma_0^{-1}(j)}$, which cannot be in $-\Phi_+$.

Case 2: Either i or j, but not both, is in $\{j_1, j_2, \ldots, j_k\}$. Without loss of generality, we assume $i \notin \{j_1, j_2, \ldots, j_k\}$ and $j \in \{j_1, j_2, \ldots, j_k\}$. Note that $w^{-1}(\epsilon_i + \epsilon_j) = \epsilon_{\sigma_0^{-1}(i)} - \epsilon_{\sigma_0^{-1}(j)}$, which is in $-\Phi_+$ if and only if $\sigma_0^{-1}(i) > \sigma_0^{-1}(j)$.

Case 3: $i, j \in \{j_1, j_2, \ldots, j_k\}$. In this case, we have $w^{-1}(\epsilon_i + \epsilon_j) = -\epsilon_{\sigma_0^{-1}(i)} - \epsilon_{\sigma_0^{-1}(j)}$, which is always in $-\Phi_+$.

All the above three cases imply that for any i and j with $\sigma_0(i) < \sigma_0(j)$, $\epsilon_i + \epsilon_j \in \Phi_w \cap \Phi_+^1$ if and only if $i \in \{j_1, j_2, \ldots, j_k\}$. So we have that for any $t \in \{1, 2, \ldots, k\}$, if $\epsilon_{j_t} + \epsilon_j \in \Phi_w \cap \Phi_+^1$, then (1) $\epsilon_{j_t} + \epsilon_{j'} \in \Phi_w \cap \Phi_+^1$ for all j' with $\sigma_0(j') > \sigma_0(j)$; and (2) $\epsilon_{j_{t'}} + \epsilon_j \in \Phi_w \cap \Phi_+^1$ for all $t' < t$. Hence by (2.5), we obtain that if $\eta_w^{-1}(x) \in \Phi_w \cap \Phi_+^1$, then $\eta_w^{-1}(y) \in \Phi_w \cap \Phi_+^1$ for any $y \in \Phi_+^1$ with $\eta_w^{-1}(y) \prec \eta_w^{-1}(x)$. Thus $\sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1) \in \Upsilon$ by (2.8). □

Remark 2.7. In fact, the proof of Lemma 2.6 also yields the explicit expression of $\sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1)$. That is, for any $w = r_{j_1} r_{j_2} \cdots r_{j_k} \sigma_0 \in \mathcal{W}$ $(\sigma_0 \in S_n)$,

$$\sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1) = \{\epsilon_i + \epsilon_j \mid 1 \le i \le k, i \le j \le n + 1 - \sigma_0^{-1}(i)\}. \tag{2.9}$$

Therefore, we can define a map

$$\xi : \mathcal{W} \to \Upsilon, \quad w \mapsto \sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1). \tag{2.10}$$

Lemma 2.8. *There exists a one-to-one correspondence*

$$\mathcal{W} \leftrightarrow S_n \times \Upsilon, \quad w \mapsto (\eta(w), \xi(w)). \tag{2.11}$$

Proof. Expressions (2.5) and (2.9) imply that $w \mapsto (\eta(w), \xi(w))$ is injective.

Take any $(\sigma, \Psi) \in S_n \times \Upsilon$ and consider $\sigma\sigma_l\Psi \subset \Phi_+^1$. We are going to determine an element $w \in \mathcal{W}$ such that $(\eta(w), \xi(w)) = (\sigma, \Psi)$.

Assume $\{j \mid 2\epsilon_j \in \sigma\sigma_l\Psi\} = \{j_i, j_2, \ldots, j_k\}$ with $\sigma^{-1}(j_1) > \sigma^{-1}(j_2) > \cdots > \sigma^{-1}(j_k)$. In other words, $j_t = \sigma\sigma_l(t)$ $(t = 1, 2, \ldots, k)$. Hence σ must be of the form $\sigma = (i_1, i_2, \ldots, i_{n-k}, j_k, j_{k-1}, \ldots, j_1)$, where $\{i_1, i_2, \ldots, i_{n-k}\}$ is a permutation of $\{1, 2, \ldots, n\}\backslash\{j_i, j_2, \ldots, j_k\}$. Assume $m_t = \max\{m \mid \epsilon_t + \epsilon_m \in \Psi\}$ $(t = 1, 2, \ldots, k)$. It is clear that $m_1 \geq m_2 \geq \cdots \geq m_k$.

Take

$$w = (i_1, \ldots, i_{n-m_1}, j_1, i_{n-m_1+1}, \ldots, i_{n-m_k}, j_k, i_{n-m_k+1}, \ldots, i_{n-k})$$

with

$$\begin{aligned}&(\ldots, i_{n-m_t}, j_t, i_{n-m_t+1}, \ldots, i_{n-m_s}, j_s, i_{n-m_s+1}, \ldots)\\ &:= (\ldots, i_{n-m_t}, j_t, j_{t+1}, \ldots, j_s, i_{n-m_s+1}, \ldots)\end{aligned}$$

if $m_{t-1} < m_t = m_{t+1} = \cdots = m_s < m_{s+1}$. Then we can check easily that $(\eta(w), \xi(w)) = (\sigma, \Psi)$. So $w \mapsto (\eta(w), \xi(w))$ is also surjective. $\square$

Remark 2.9. By (2.1) and (2.11), we can obtain $|\mathcal{I}| = |\Upsilon| = \frac{|(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n|}{|S_n|} = 2^n$ immediately. This is Peterson's 2^r-theorem for type C.

Each $\sigma \in S_n \subset \mathcal{W}$ induces a linear transform on $\mathfrak{n}^*$ by

$$\sigma(f_{\epsilon_i \pm \epsilon_j}) = f_{\epsilon_{\sigma(i)} \pm \epsilon_{\sigma(j)}}. \tag{2.12}$$

Moreover, this map can extend to $\bigwedge \mathfrak{n}^*$ by

$$\sigma(f_1 \wedge f_2 \wedge \cdots \wedge f_k) = \sigma(f_1) \wedge \sigma(f_2) \wedge \cdots \wedge \sigma(f_k). \tag{2.13}$$

By Lemma 2.4, (2.7), (2.10), (2.12) and the fact that $\sigma_l^{-1} = \sigma_l$, we have

$$\mathbb{C} \bigwedge_{\alpha \in \Phi_w} f_\alpha = \mathbb{C}(\bigwedge_{\alpha \in \Phi_{\eta_w}} f_\alpha) \wedge \eta_w\sigma_l(\bigwedge_{\alpha \in \xi_w} f_\alpha). \tag{2.14}$$

Combining (2.1) and (2.14), we get

$$\mathbb{C} \bigwedge_{\alpha \in \Phi_w} f_\alpha = \mathbb{C}(\bigwedge_{\alpha \in \Phi_{\eta_w}} f_\alpha) \wedge \eta_w\sigma_l(\wedge^{\max} I_{\xi_w}^*), \tag{2.15}$$

where $I_{\xi_w}^* \subset \mathfrak{n}^*$ is the set of all linear functions on I_{ξ_w} and $\wedge^{\max} I_{\xi_w}^*$ is the unique element (up to nonzero scalar multiples) in $\wedge^{\dim I_{\xi_w}} I_{\xi_w}^* \subset \mathfrak{n}^*$.

Define

$$L : S_n \times \mathcal{I} \to \wedge\mathfrak{n}^*, \quad (\sigma, I) \mapsto (\bigwedge_{\alpha \in \Phi_\sigma} f_\alpha) \wedge \sigma\sigma_l(\wedge^{\max} I^*). \tag{2.16}$$

Then we can obtain the following main theorem by (2.1), (2.3), (2.11), (2.14) and (2.16).

Theorem 2.10. *The cohomology group*

$$H(\mathfrak{n}) = \bigoplus_{(\sigma, I) \in S_n \times \mathcal{I}} \mathbb{C}[L(\sigma, I)], \tag{2.17}$$

where $[L(\sigma, I)]$ is the cohomology class defined by the (harmonic) cocycle $L(\sigma, I)$.

The definition of L also implies that

$$\deg[L(\sigma, I)] = |\Phi_\sigma| + \dim I. \tag{2.18}$$

Therefore

$$\dim H^i(\mathfrak{n}) = \sum_{j+k=i} (|S_n^{(j)}| + |\mathcal{I}^{(k)}|) \tag{2.19}$$

with

$$S_n^{(j)} := \{\sigma \in S_n \mid |\Phi_\sigma| = j\} \tag{2.20}$$

and

$$\mathcal{I}^{(k)} := \{I \in \mathcal{I} \mid \dim I = k\}. \tag{2.21}$$

On the other hand, Theorem 2.3 implies

$$\dim H^i(\mathfrak{n}) = |((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{(i)}|, \tag{2.22}$$

where

$$((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{(i)} = \{w \in (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \mid |\Phi_w| = i\}. \tag{2.23}$$

The generating functions of $|((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{(i)}|$ and $|S_n^{(i)}|$ are just the so-called Poincaré polynomials of the Weyl groups of types C_n and A_{n-1}, respectively. These two polynomials can be found in [3]. We list them below:

$$\sum_{i=0}^{\infty} |((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{(i)}| t^i = \frac{\prod_{i=1}^n (1 - t^{2i})}{(1-t)^n}, \tag{2.24}$$

$$\sum_{i=0}^{\infty} |S_n^{(i)}| t^i = \frac{\prod_{i=1}^n (1 - t^{i})}{(1-t)^n}. \tag{2.25}$$

Thanks to (2.19) and (2.22), we get

$$\sum_{i=0}^{\infty} |\mathcal{I}^{(i)}| t^i = \frac{\frac{\prod_{i=1}^n (1-t^{2i})}{(1-t)^n}}{\frac{\prod_{i=1}^n (1-t^{i})}{(1-t)^n}} = \prod_{i=1}^n (1 + t^i). \tag{2.26}$$

That is,

Corollary 2.11. *The number of abelian ideals of* $\mathfrak{b}$ *with dimension* i *is equal to the coefficient of* t^i *in* $\prod_{r=1}^n (1 + t^r)$.

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