

ABELIAN IDEALS AND COHOMOLOGY OF SYMPLECTIC TYPE

LI LUO

(Communicated by Gail R. Letzter)

ABSTRACT. Let \mathfrak{b} be a Borel subalgebra of the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ and let \mathfrak{n} be the corresponding maximal nilpotent subalgebra. We find a connection between the abelian ideals of \mathfrak{b} and the cohomology of \mathfrak{n} with trivial coefficients. Using this connection, we are able to enumerate the number of abelian ideals of \mathfrak{b} with given dimension via the Poincaré polynomials of Weyl groups of types A_{n-1} and C_n .

1. INTRODUCTION

Let \mathfrak{g} be a finite-dimensional simple Lie algebra and let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Suppose that $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ is a Borel subalgebra, where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the maximal nilpotent subalgebra of \mathfrak{g} . Note that \mathfrak{n} is also the nilradical of \mathfrak{b} under the Killing form. Needless to say, the structure of the subalgebras \mathfrak{b} and \mathfrak{n} is important for the study of \mathfrak{g} itself.

The study of abelian ideals in the Borel subalgebra \mathfrak{b} can be traced back to the work by Schur [9] (1905). It has recently drawn considerable attention. Kostant [5] (1998) mentioned Peterson's 2^r -theorem saying that the number of abelian ideals in \mathfrak{b} is exactly 2^r , where $r = \dim \mathfrak{h}$ is the rank of \mathfrak{g} . Moreover, Kostant found a relation between abelian ideals of a Borel subalgebra and the discrete series representations of the Lie group. Spherical orbits were described by Panyushev and Röhrle [8] (2001) in terms of abelian ideals. Furthermore, Panyushev [7] (2003) discovered a correspondence of maximal abelian ideals of a Borel subalgebra to long positive roots. Suter [10] (2004) determined the maximal dimension among abelian subalgebras of a finite-dimensional simple Lie algebra purely in terms of certain invariants and gave a uniform explanation for Panyushev's result. Kostant [6] (2004) showed that the powers of the Euler product and the abelian ideals of a Borel subalgebra are intimately related. Cellini and Papi [2] (2004) had a detailed study of certain remarkable posets which form a natural partition of all abelian ideals of a Borel subalgebra.

It is remarkable that the affine Weyl group \widehat{W} associated with \mathfrak{g} is used in the proof of the 2^r -theorem. On the other hand, Bott [1] gave a celebrated theorem which shows that the Betti numbers b_i of \mathfrak{n} (i.e. the dimension of $H^i(\mathfrak{n})$) can be

Received by the editors January 24, 2008.

2000 *Mathematics Subject Classification*. Primary 17B05, 17B56; Secondary 17B20, 17B30.

Key words and phrases. Abelian ideal, cohomology, symplectic Lie algebra, Weyl group, Poincaré polynomial.

©2008 American Mathematical Society
Reverts to public domain 28 years from publication

expressed by the Weyl group \mathcal{W} associated with \mathfrak{g} . Later Kostant [4] generalized his result to the nilradical of any parabolic subalgebras \mathfrak{p} of \mathfrak{g} . (\mathfrak{n} is the nilradical of \mathfrak{b} .)

It seems that the Weyl group \mathcal{W} (and its affine group $\widehat{\mathcal{W}}$) can be a bridge for connecting the cohomology of \mathfrak{n} to the abelian ideals of \mathfrak{b} . But no one has given an explicit relationship between these two objects so far. In this paper, we shall construct this relationship in the case of $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$.

In the following text, we let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ be the symplectic Lie algebra. Our main theorem of this paper is as follows.

Main Theorem. *Let \mathcal{I} be the set of all abelian ideals of \mathfrak{b} and let S_n be the permutation group on n elements. In terms of a certain map $L : S_n \times \mathcal{I} \rightarrow \wedge \mathfrak{n}^*$ defined in (2.16), the cohomology group $H(\mathfrak{n}) = \bigoplus_{(\sigma, I) \in S_n \times \mathcal{I}} \mathbb{C}[L(\sigma, I)]$, where $[L(\sigma, I)]$ is the cohomology class defined by the harmonic cocycle $L(\sigma, I)$.*

An interesting application of this theorem is to compute the number of abelian ideals of \mathfrak{b} with given dimension via the Poincaré polynomials of Weyl groups of types A and C . That is,

Corollary. *The number of abelian ideals of \mathfrak{b} with dimension i is equal to the coefficient of t^i in $\prod_{r=1}^n (1 + t^r)$.*

2. PROOF OF THE MAIN RESULT

Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ be the symplectic Lie algebra. Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} . Denote by $\mathcal{W} \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ its Weyl group, by $\Phi = \{\pm 2\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) \mid 1 \leq i \neq j \leq n\}$ its root system with positive roots $\Phi_+ = \{2\epsilon_i, \epsilon_i \pm \epsilon_j \mid i < j\}$ and by $\pi = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \dots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$ its simple roots. We have the Cartan root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha)$. Then $\mathfrak{b} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha)$ is the associated Borel subalgebra and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha$ is its nilradical.

Denote by \mathcal{I} the set of all abelian ideals of \mathfrak{b} . As mentioned in [10], there is a bijection as follows:

$$(2.1) \quad \begin{aligned} \Upsilon := \{ \Psi \subset \Phi_+ \mid \Psi \dot{+} \Phi_+ \subset \Psi, \Psi \dot{+} \Psi = \emptyset \} &\leftrightarrow \mathcal{I} \\ \Psi &\mapsto I_\Psi := \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha, \end{aligned}$$

where $\Psi \dot{+} \Phi_+ := (\Psi + \Phi_+) \cap \Phi_+$ and $\Psi \dot{+} \Psi := (\Psi + \Psi) \cap \Phi_+$.

Denote $\Phi_+^0 := \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\}$.

Lemma 2.1. *Equation $\Psi \cap \Phi_+^0 = \emptyset$ holds for any $\Psi \in \Upsilon$.*

Proof. Suppose that there is an $\epsilon_i - \epsilon_j \in \Psi$. Since $2\epsilon_j \in \Phi_+$ and $(\epsilon_i - \epsilon_j) + 2\epsilon_j = \epsilon_i + \epsilon_j \in \Phi_+$, we have $\epsilon_i + \epsilon_j \in \Psi$. But $(\epsilon_i - \epsilon_j) + (\epsilon_i + \epsilon_j) = 2\epsilon_i \in \Phi_+$, a contradiction. \square

Define a partial ordering on

$$\Phi_+^1 := \Phi_+ \setminus \Phi_+^0 = \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq j \leq n\}$$

by

$$(2.2) \quad \epsilon_{i_1} + \epsilon_{j_1} \prec \epsilon_{i_2} + \epsilon_{j_2} \Leftrightarrow i_1 \geq i_2, j_1 \geq j_2,$$

where $i_1 \leq j_1, i_2 \leq j_2$. In fact, this partial ordering is just the one induced from the partial ordering on the positive roots. A subset $\Psi \subset \Phi_+^1$ is called an increasing subset if for any $x, y \in \Phi_+^1$, the conditions $x \in \Psi$ and $x \prec y$ imply $y \in \Psi$.

The following lemma is obvious.

Lemma 2.2. *The set Υ is the set of all increasing subsets of Φ_+^1 .*

We shall show that these increasing subsets also appear in the cohomology of \mathfrak{n} with trivial coefficients.

Choose a nonzero element e_α in \mathfrak{g}_α for each $\alpha \in \Phi_+$. Hence $\{e_\alpha \mid \alpha \in \Phi_+\}$ is a basis of \mathfrak{n} . Define a linear function $f_\alpha \in \mathfrak{n}^*$ by $f_\alpha(e_\beta) = \delta_{\alpha,\beta}$. Then $\{f_\alpha \mid \alpha \in \Phi_+\}$ is a basis of \mathfrak{n}^* .

Recall the following theorem:

Theorem 2.3 (Bott-Kostant; cf. [1, 4]). *The cohomology group*

$$(2.3) \quad H(\mathfrak{n}) = \bigoplus_{w \in \mathcal{W}} \mathbb{C} \left[\bigwedge_{\alpha \in \Phi_w} f_\alpha \right],$$

where $\Phi_w = w(-\Phi_+) \cap \Phi_+$, and $[\bigwedge_{\alpha \in \Phi_w} f_\alpha]$ is the cohomology class defined by the (harmonic) cocycle $\bigwedge_{\alpha \in \Phi_w} f_\alpha$.

We see in Theorem 2.3 that Φ_w plays an important role in cohomology. So we shall get some more information about Φ_w .

Recall that for type C_n , the Weyl group \mathcal{W} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \times S_n$. Precisely, \mathcal{W} can be realized by the composite of all permutations of $\{1, 2, \dots, n\}$ and $\langle r_i \mid i = 1, 2, \dots, n \rangle$, where $r_i(j) = -\delta_{i,j}i + (1 - \delta_{i,j})j$ ($j = 1, 2, \dots, n$). In this paper, we always use (i_1, i_2, \dots, i_n) to denote the permutation $j \mapsto i_j$ ($j = 1, 2, \dots, n$).

Each element in \mathcal{W} can be expressed in the form

$$(2.4) \quad w = r_{j_1} r_{j_2} \cdots r_{j_k} (i_1, i_2, \dots, i_n) \quad (0 \leq k \leq n),$$

where $(i_1, i_2, \dots, i_n)(j_1) < (i_1, i_2, \dots, i_n)(j_2) < \cdots < (i_1, i_2, \dots, i_n)(j_k)$. We call it the *standard form* of w .

The following two lemmas are well-known and follow from Kostant's classic paper [4]. Since we only consider the special case of type C , we include the proofs in this case for completeness.

Lemma 2.4. *The set $\Phi_\sigma = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n, \sigma^{-1}(i) > \sigma^{-1}(j)\}$ for any $\sigma \in S_n \subset \mathcal{W}$. Hence if $\Phi_{\sigma_1} = \Phi_{\sigma_2}$ ($\sigma_1, \sigma_2 \in S_n$), then $\sigma_1 = \sigma_2$.*

Proof. It is obvious by a direct calculation. □

Lemma 2.5. *For any $w \in \mathcal{W}$, there is a unique element $\eta_w \in S_n$ such that $\Phi_w \cap \Phi_+^0 = \Phi_{\eta_w}$. Precisely, if $w = r_{j_1} r_{j_2} \cdots r_{j_k} (i_1, i_2, \dots, i_n)$ is in the standard form, then*

$$(2.5) \quad \eta_w = (i_1, \dots, \widehat{j_1}, \dots, \widehat{j_2}, \dots, \dots, \widehat{j_k}, \dots, i_n, j_k, j_{k-1}, \dots, j_1),$$

where $(i_1, \dots, \widehat{j_1}, \dots, \widehat{j_2}, \dots, \dots, \widehat{j_k}, \dots, i_n) = (i_1, i_2, \dots, i_n)$ and the sign $\widehat{}$ indicates that the argument below it must be omitted.

Proof. The uniqueness of η_w follows from Lemma 2.4.

Denote $\sigma = (i_1, \dots, \widehat{j_1}, \dots, \widehat{j_2}, \dots, \dots, \widehat{j_k}, \dots, i_n, j_k, j_{k-1}, \dots, j_1)$. We have to show $\Phi_w \cap \Phi_+^0 = \Phi_\sigma$. In fact, for any $\epsilon_i - \epsilon_j \in \Phi_w \cap \Phi_+^0$, we have $w^{-1}(\epsilon_i - \epsilon_j) =$

$\epsilon_{w^{-1}(i)} - \epsilon_{w^{-1}(j)} \in -\Phi_+$. What we need to do is to check $\sigma^{-1}(i) > \sigma^{-1}(j)$. There are four cases as follows.

Case 1: $i, j \notin \{j_1, j_2, \dots, j_k\}$. Then $w^{-1}(\epsilon_i - \epsilon_j) = \epsilon_{\sigma_0^{-1}(i)} - \epsilon_{\sigma_0^{-1}(j)}$, where $\sigma_0 = (i_1, i_2, \dots, i_n)$. Hence $\sigma_0^{-1}(i) > \sigma_0^{-1}(j)$. So $\sigma^{-1}(i) > \sigma^{-1}(j)$ by the definition of σ .

Case 2: $i \in \{j_1, j_2, \dots, j_k\}, j \notin \{j_1, j_2, \dots, j_k\}$. Then $w^{-1}(\epsilon_i - \epsilon_j) = -\epsilon_{\sigma_0^{-1}(i)} - \epsilon_{\sigma_0^{-1}(j)}$, which is always in $-\Phi_+$. We also do have $\sigma^{-1}(i) > \sigma^{-1}(j)$ by the definition of σ .

Case 3: $i \notin \{j_1, j_2, \dots, j_k\}, j \in \{j_1, j_2, \dots, j_k\}$. Then $w^{-1}(\epsilon_i - \epsilon_j) = \epsilon_{\sigma_0^{-1}(i)} + \epsilon_{\sigma_0^{-1}(j)}$, which is always in Φ_+ . There should be no such i, j with $\sigma^{-1}(i) > \sigma^{-1}(j)$. It is the case by the definition of σ .

Case 4: $i, j \in \{j_1, j_2, \dots, j_k\}$. Then $w^{-1}(\epsilon_i - \epsilon_j) = \epsilon_{\sigma_0^{-1}(j)} - \epsilon_{\sigma_0^{-1}(i)}$. Hence $\sigma_0^{-1}(j) > \sigma_0^{-1}(i)$. So $\sigma^{-1}(i) > \sigma^{-1}(j)$ by the definition of σ . □

By this lemma, we can define a map

$$(2.6) \quad \eta : \mathcal{W} \rightarrow S_n, \quad w \mapsto \eta_w.$$

Denote

$$(2.7) \quad \sigma_l := (n, n - 1, \dots, 1) \in S_n,$$

which is the longest element in S_n . It is clear that $\sigma_l = \sigma_l^{-1}$ and

$$(2.8) \quad x \prec y \Leftrightarrow \sigma_l(y) \prec \sigma_l(x) \quad (x, y \in \Phi_+^1).$$

Lemma 2.6. *We have $\sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1) \in \Upsilon$.*

Proof. Write $w = r_{j_1} r_{j_2} \cdots r_{j_k} \sigma_0$ in the standard form with $\sigma_0 = (i_1, i_2, \dots, i_n)$. Take any $\epsilon_i + \epsilon_j \in \Phi_w \cap \Phi_+^1$. Then $w^{-1}(\epsilon_i + \epsilon_j) = \epsilon_{w^{-1}(i)} + \epsilon_{w^{-1}(j)} \in -\Phi_+$. There are three cases as follows:

Case 1: $i, j \notin \{j_1, j_2, \dots, j_k\}$. Then $w^{-1}(\epsilon_i + \epsilon_j) = \epsilon_{\sigma_0^{-1}(i)} + \epsilon_{\sigma_0^{-1}(j)}$, which cannot be in $-\Phi_+$.

Case 2: Either i or j , but not both, is in $\{j_1, j_2, \dots, j_k\}$. Without loss of generality, we assume $i \notin \{j_1, j_2, \dots, j_k\}$ and $j \in \{j_1, j_2, \dots, j_k\}$. Note that $w^{-1}(\epsilon_i + \epsilon_j) = \epsilon_{\sigma_0^{-1}(i)} - \epsilon_{\sigma_0^{-1}(j)}$, which is in $-\Phi_+$ if and only if $\sigma_0^{-1}(i) > \sigma_0^{-1}(j)$.

Case 3: $i, j \in \{j_1, j_2, \dots, j_k\}$. In this case, we have $w^{-1}(\epsilon_i + \epsilon_j) = -\epsilon_{\sigma_0^{-1}(i)} - \epsilon_{\sigma_0^{-1}(j)}$, which is always in $-\Phi_+$.

All the above three cases imply that for any i and j with $\sigma_0(i) < \sigma_0(j)$, $\epsilon_i + \epsilon_j \in \Phi_w \cap \Phi_+^1$ if and only if $i \in \{j_1, j_2, \dots, j_k\}$. So we have that for any $t \in \{1, 2, \dots, k\}$, if $\epsilon_{j_t} + \epsilon_j \in \Phi_w \cap \Phi_+^1$, then (1) $\epsilon_{j_t} + \epsilon_{j'}$ $\in \Phi_w \cap \Phi_+^1$ for all j' with $\sigma_0(j') > \sigma_0(j)$; and (2) $\epsilon_{j_{t'}} + \epsilon_j \in \Phi_w \cap \Phi_+^1$ for all $t' < t$. Hence by (2.5), we obtain that if $\eta_w^{-1}(x) \in \Phi_w \cap \Phi_+^1$, then $\eta_w^{-1}(y) \in \Phi_w \cap \Phi_+^1$ for any $y \in \Phi_+^1$ with $\eta_w^{-1}(y) \prec \eta_w^{-1}(x)$. Thus $\sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1) \in \Upsilon$ by (2.8). □

Remark 2.7. In fact, the proof of Lemma 2.6 also yields the explicit expression of $\sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1)$. That is, for any $w = r_{j_1} r_{j_2} \cdots r_{j_k} \sigma_0 \in \mathcal{W}$ ($\sigma_0 \in S_n$),

$$(2.9) \quad \sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1) = \{\epsilon_i + \epsilon_j \mid 1 \leq i \leq k, i \leq j \leq n + 1 - \sigma_0^{-1}(i)\}.$$

Therefore, we can define a map

$$(2.10) \quad \xi : \mathcal{W} \rightarrow \Upsilon, \quad w \mapsto \sigma_l \eta_w^{-1}(\Phi_w \cap \Phi_+^1).$$

Lemma 2.8. *There exists a one-to-one correspondence*

$$(2.11) \quad \mathcal{W} \leftrightarrow S_n \times \Upsilon, \quad w \mapsto (\eta(w), \xi(w)).$$

Proof. Expressions (2.5) and (2.9) imply that $w \mapsto (\eta(w), \xi(w))$ is injective.

Take any $(\sigma, \Psi) \in S_n \times \Upsilon$ and consider $\sigma\sigma_l\Psi \subset \Phi_+^1$. We are going to determine an element $w \in \mathcal{W}$ such that $(\eta(w), \xi(w)) = (\sigma, \Psi)$.

Assume $\{j \mid 2\epsilon_j \in \sigma\sigma_l\Psi\} = \{j_1, j_2, \dots, j_k\}$ with $\sigma^{-1}(j_1) > \sigma^{-1}(j_2) > \dots > \sigma^{-1}(j_k)$. In other words, $j_t = \sigma\sigma_l(t)$ ($t = 1, 2, \dots, k$). Hence σ must be of the form $\sigma = (i_1, i_2, \dots, i_{n-k}, j_k, j_{k-1}, \dots, j_1)$, where $\{i_1, i_2, \dots, i_{n-k}\}$ is a permutation of $\{1, 2, \dots, n\} \setminus \{j_1, j_2, \dots, j_k\}$. Assume $m_t = \max\{m \mid \epsilon_t + \epsilon_m \in \Psi\}$ ($t = 1, 2, \dots, k$). It is clear that $m_1 \geq m_2 \geq \dots \geq m_k$.

Take

$$w = (i_1, \dots, i_{n-m_1}, j_1, i_{n-m_1+1}, \dots, i_{n-m_k}, j_k, i_{n-m_k+1}, \dots, i_{n-k})$$

with

$$\begin{aligned} & (\dots, i_{n-m_t}, j_t, i_{n-m_t+1}, \dots, i_{n-m_s}, j_s, i_{n-m_s+1}, \dots) \\ & := (\dots, i_{n-m_t}, j_t, j_{t+1}, \dots, j_s, i_{n-m_s+1}, \dots) \end{aligned}$$

if $m_{t-1} < m_t = m_{t+1} = \dots = m_s < m_{s+1}$. Then we can check easily that $(\eta(w), \xi(w)) = (\sigma, \Psi)$. So $w \mapsto (\eta(w), \xi(w))$ is also surjective. \square

Remark 2.9. By (2.1) and (2.11), we can obtain $|\mathcal{I}| = |\Upsilon| = \frac{|(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n|}{|S_n|} = 2^n$ immediately. This is Peterson's 2^r -theorem for type C .

Each $\sigma \in S_n \subset \mathcal{W}$ induces a linear transform on \mathfrak{n}^* by

$$(2.12) \quad \sigma(f_{\epsilon_i \pm \epsilon_j}) = f_{\epsilon_{\sigma(i)} \pm \epsilon_{\sigma(j)}}.$$

Moreover, this map can extend to $\bigwedge \mathfrak{n}^*$ by

$$(2.13) \quad \sigma(f_1 \wedge f_2 \wedge \dots \wedge f_k) = \sigma(f_1) \wedge \sigma(f_2) \wedge \dots \wedge \sigma(f_k).$$

By Lemma 2.4, (2.7), (2.10), (2.12) and the fact that $\sigma_l^{-1} = \sigma_l$, we have

$$(2.14) \quad \mathbb{C} \bigwedge_{\alpha \in \Phi_w} f_\alpha = \mathbb{C} \left(\bigwedge_{\alpha \in \Phi_{\eta w}} f_\alpha \right) \wedge \eta_w \sigma_l \left(\bigwedge_{\alpha \in \xi_w} f_\alpha \right).$$

Combining (2.1) and (2.14), we get

$$(2.15) \quad \mathbb{C} \bigwedge_{\alpha \in \Phi_w} f_\alpha = \mathbb{C} \left(\bigwedge_{\alpha \in \Phi_{\eta w}} f_\alpha \right) \wedge \eta_w \sigma_l (\bigwedge^{\max} I_{\xi_w}^*),$$

where $I_{\xi_w}^* \subset \mathfrak{n}^*$ is the set of all linear functions on I_{ξ_w} and $\bigwedge^{\max} I_{\xi_w}^*$ is the unique element (up to nonzero scalar multiples) in $\bigwedge^{\dim I_{\xi_w}} I_{\xi_w}^* \subset \mathfrak{n}^*$.

Define

$$(2.16) \quad L : S_n \times \mathcal{I} \rightarrow \bigwedge \mathfrak{n}^*, \quad (\sigma, I) \mapsto \left(\bigwedge_{\alpha \in \Phi_\sigma} f_\alpha \right) \wedge \sigma\sigma_l (\bigwedge^{\max} I^*).$$

Then we can obtain the following main theorem by (2.1), (2.3), (2.11), (2.14) and (2.16).

Theorem 2.10. *The cohomology group*

$$(2.17) \quad H(\mathfrak{n}) = \bigoplus_{(\sigma, I) \in S_n \times \mathcal{I}} \mathbb{C}[L(\sigma, I)],$$

where $[L(\sigma, I)]$ is the cohomology class defined by the (harmonic) cocycle $L(\sigma, I)$.

The definition of L also implies that

$$(2.18) \quad \deg[L(\sigma, I)] = |\Phi_\sigma| + \dim I.$$

Therefore

$$(2.19) \quad \dim H^i(\mathfrak{n}) = \sum_{j+k=i} (|S_n^{(j)}| + |\mathcal{I}^{(k)}|)$$

with

$$(2.20) \quad S_n^{(j)} := \{\sigma \in S_n \mid |\Phi_\sigma| = j\}$$

and

$$(2.21) \quad \mathcal{I}^{(k)} := \{I \in \mathcal{I} \mid \dim I = k\}.$$

On the other hand, Theorem 2.3 implies

$$(2.22) \quad \dim H^i(\mathfrak{n}) = |((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{(i)}|,$$

where

$$(2.23) \quad ((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{(i)} = \{w \in (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \mid |\Phi_w| = i\}.$$

The generating functions of $|((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{(i)}|$ and $|S_n^{(i)}|$ are just the so-called Poincaré polynomials of the Weyl groups of types C_n and A_{n-1} , respectively. These two polynomials can be found in [3]. We list them below:

$$(2.24) \quad \sum_{i=0}^{\infty} |((\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n)^{(i)}| t^i = \frac{\prod_{i=1}^n (1 - t^{2i})}{(1 - t)^n},$$

$$(2.25) \quad \sum_{i=0}^{\infty} |S_n^{(i)}| t^i = \frac{\prod_{i=1}^n (1 - t^i)}{(1 - t)^n}.$$

Thanks to (2.19) and (2.22), we get

$$(2.26) \quad \sum_{i=0}^{\infty} |\mathcal{I}^{(i)}| t^i = \frac{\prod_{i=1}^n (1 - t^{2i})}{\frac{\prod_{i=1}^n (1 - t^i)}{(1 - t)^n}} = \prod_{i=1}^n (1 + t^i).$$

That is,

Corollary 2.11. *The number of abelian ideals of \mathfrak{b} with dimension i is equal to the coefficient of t^i in $\prod_{r=1}^n (1 + t^r)$.*

ACKNOWLEDGEMENTS

The author would like to thank his thesis advisor Prof. Xiaoping Xu, from whom he learned so much. He also thanks the referee for helpful suggestions on the exposition of this paper.

REFERENCES

1. R. Bott, *Homogeneous vector bundles*, Ann. of Math. (2) **66** (1957), 203–248. MR0089473 (19:681d)
2. P. Cellini and P. Papi, *Abelian ideals of Borel subalgebras and affine Weyl groups*, Adv. Math. **187** (2004), 320–361. MR2078340 (2005e:17012)
3. J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, 1990. MR1066460 (92h:20002)
4. B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem*, Ann. of Math. (2) **74** (1961), 329–387. MR0142696 (26:265)

5. B. Kostant, *The set of abelian ideals of a Borel subalgebras, Cartan decompositions, and discrete series representations*, Int. Math. Res. Not. **5** (1998), 225–252. MR1616913 (99c:17010)
6. B. Kostant, *Powers of the Euler product and commutative subalgebras of a complex Lie algebra*, Invent. Math. **158** (2004), 181–226. MR2090363 (2005m:17007)
7. D. Panyushev, *Abelian ideals of a Borel subalgebra and long positive roots*, Int. Math. Res. Not. (2003), 1889–1913. MR1995141 (2004g:17006)
8. D. Panyushev and G. Röhrle, *Spherical orbits and abelian ideals*, Adv. Math. **159** (2001), 229–246. MR1825058 (2002c:20073)
9. I. Schur, *Zur Theorie der vertauschbaren Matrizen*, J. Reine Angew. Math. **130** (1905), 66–76.
10. R. Suter, *Abelian ideals in a Borel subalgebra of a complex simple Lie algebra*, Invent. Math. **156** (2004), 175–221. MR2047661 (2005b:17020)

INSTITUTE OF MATHEMATICS, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

E-mail address: luoli@amss.ac.cn