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CENTER TYPE PERFORMANCE OF DIFFERENTIABLE VECTOR FIELDS IN THE PLANE

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Abstract. Suppose that X is a planar vector field whose linearization outside some compact set is nonsingular and has pure imaginary spectrum. Then by adding to X a constant vector, one obtains center behavior at infinity: the flow is conjugate to a rotation flow outside some compact set.

1. Introduction and statement of the results

In the qualitative theory of planar differential systems, there are many cases where the local phase portrait at a singular point has been characterized (see for instance [21, 5, 16, 2, 7, 8, 20]). On the other hand, we know that some global phase portraits were described in [9, 6]. These theorems may be paraphrased as follows: "the known singular point of Y is a global attractor as long as all the linear parts of Y are asymptotically stable" (see also [17, 4]). This study has motivated the present paper, which is closely related to the results concerning the behavior of a system near infinity [13, 12, 10, 1]. Note that, in order to understand a global phase portrait it is absolutely necessary to research its behavior in a neighborhood of infinity.

For every $\sigma > 0$, let $\overline{D}_\sigma = \{z \in \mathbb{R}^2 : ||z|| \leq \sigma\}$. Thus, $V = (\mathbb{R}^2 \setminus \overline{D}_\sigma) \cup \{\infty\}$ is the topological subspace of the Riemann sphere on the neighborhood of infinity obtained from $\mathbb{R}^2 \setminus \overline{D}_\sigma$. We consider a differentiable vector field $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ whose Jacobian determinant $\det(DX)$ is always different from zero, and we denote by DX_z the linearization of X at $z \in \mathbb{R}^2 \setminus \overline{D}_\sigma$. In this context, we obtain the following result: "if the eigenvalues of DX_z are purely imaginary, then by adding to X some constant $v \in \mathbb{R}^2$ one obtains center behavior at infinity, that is: $X + v$ has a periodic trajectory $\Gamma \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ such that Γ is enclosing $\overline{D}_\sigma$, and in the unbounded component of $(\mathbb{R}^2 \setminus \overline{D}_\sigma) \setminus \Gamma$, all the solutions of $X + v$ are periodic trajectories". Notice that the vector field $X : (V, \infty) \to (\mathbb{R}^2, 0)$ is differentiable in $V \setminus \{\infty\}$, but not necessarily continuous at ∞. In the case of global vector fields $Y : \mathbb{R}^2 \to \mathbb{R}^2$ with $Y(0) = 0$, such an eigenvalue condition implies the topological equivalency of Y with the linear vector field $(x, y) \to (-y, x)$. Observe that as

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we study vector fields with nonzero eigenvalues, our assumptions imply that the divergence of the vector field is zero.

The next subsection gives the most basic concepts and notation of the theory of differential equations. This description is necessary because the system X is supposed to be just differentiable (as in our previous papers [6, 10]); thus the eigenvalues of DX_p do not depend continuously on p. However, in Subsection 2.1, we will see that the solutions of system X are unique as long as X satisfies our eigenvalue condition. In the C^1 case, the eigenvalue assumption is not necessarily open in the C^1-Whitney topology. Therefore, a C^1-vector field whose linearization is nonsingular and has pure imaginary spectrum might not have an approximation by a smooth vector field which also satisfies such eigenvalue conditions.

1.1. **Differentiable vector fields.** Let $U \subset \mathbb{R}^2$ be an open subset, and suppose that $X : U \to \mathbb{R}^2$ is a differentiable vector field. We consider the following autonomous differential equation:

$$z' = X(z). \tag{1.1}$$

Since each point on the domain can be an initial condition, such a point jointly with the system (1.1) gives an *initial value problem* which may have many solutions defined on their maximal interval of existence. Nevertheless, for each of those trajectories—through the same point, kept fixed—all their local funnel sections are compact connected sets (see [14]). Moreover, each trajectory has its two limit sets, α and ω respectively, which are well defined in the sense that they only depend on the solution. Notice that, for us, a trajectory is the curve determined by any solution defined on its maximal interval of existence.

Let γ_q denote a trajectory passing through a point $q \in U$; thus γ_q^+ (resp. γ_q^-) is the positive (resp. negative) semi-trajectory of X contained in γ_q and starting at q. In this way $\gamma_q = \gamma_q^- \cup \gamma_q^+$. As usual, a point $p \in U$ in which $X(p) = 0$ is called a *singular point* or a singularity of X. When a trajectory γ_q is defined on $\mathbb{R}$ and there exist $\tau > 0$ such that $\gamma_q(t+\tau) = \gamma_q(t)$ for all $t \in \mathbb{R}$, this γ_q is said to be a *periodic trajectory* and such a q is called a periodic point. If $X(p) = 0$, this singular point is said to be a *center* if it admits a punctured neighborhood $\mathcal{A} \setminus \{p\}$ covered with periodic trajectories. The maximal punctured neighborhood of a center is called its *period annulus.*

Given a differentiable vector field $X : U \to \mathbb{R}^2$, we shall denote by $\mathrm{Spc}(X) = \{\text{eigenvalues of } DX_z : z \in U\}$. Our first result is the following.

Theorem A. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field. If $Y(0) = 0$ and $\mathrm{Spc}(Y) \subset \{z \in \mathbb{C} : \Re(z) = 0\} \setminus \{0\}$, then for any $p \in \mathbb{R}^2$ there is a unique trajectory starting at p and the origin is a center whose period annulus is $\mathbb{R}^2 \setminus \{0\}$.*

Theorem A describes the global phase portrait of Y and complements [6, 9, 17].

1.2. **Vector fields defined on a neighborhood of infinity.** In this subsection we present more vector fields with center behavior. To this end, we denote by $\overline{D}(\Gamma)$ (resp. $D(\Gamma)$) the compact disc (resp. open disc) enclosed by a topological circle $\Gamma \subset \mathbb{R}^2$.

Definition 1.1. We will say that the differentiable vector field $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ has a center type performance at infinity if enclosing the origin there exists a periodic trajectory $\Gamma \subset \mathbb{R}^2 \setminus \overline{D}_\sigma$ such that for each $p \in \mathbb{R}^2 \setminus D(\Gamma)$: (1) all the solutions γ_p, passing through the point p, are periodic trajectories and (2) these trajectories also surround the origin, that is, $\overline{D}_\sigma \subset \overline{D}(\Gamma) \subset \overline{D}(\gamma_p)$.

The vector fields of Theorem A have center type performance at infinity. We are now ready to state our second result.

Theorem B. *Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable vector field. If $\mathrm{Spc}(X) \subset \{z \in \mathbb{C} : \Re(z) = 0\} \setminus \{0\}$, then for any $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$ there exists a unique trajectory starting at p. Moreover, there is a constant $v \in \mathbb{R}^2$ such that $X + v$ has a center type performance at infinity.*

Theorem B complements [12] where the authors study the asymptotic stability at infinity of C^1-vector fields.

The present paper is organized as follows: Section 2 presents some preparatory results about the behavior of the dynamics at infinity, Section 3 includes the proof of Theorem A, and Section 4 is dedicated to proving Theorem B by using Theorem A.

2. Preliminary results

This section is devoted to some preliminary results which will be used in the proof of the main theorems.

2.1. **Uniqueness without the C^1 condition.**

The next lemma shows that our eigenvalue condition gives the uniqueness of the trajectories.

Lemma 2.1. *If $Y : \mathbb{R}^2 \to \mathbb{R}^2$ is a differentiable vector field and $\mathrm{Spc}(Y) \subset \{z \in \mathbb{C} : \Re(z) = 0\} \setminus \{0\}$, then there exists a unique trajectory of Y starting at $p \in \mathbb{R}^2$.*

Proof. The assumption over the eigenvalues implies that $\mathrm{Spc}(Y) = \mathrm{Spc}(-Y)$. Since any positive semi-trajectory of Y is a negative semi-trajectory of $-Y$, it is not difficult to obtain the lemma after proving the next affirmation.

(a) For any $p \in \mathbb{R}^2$, there is a unique positive semi-trajectory starting at p.

Suppose by way of contradiction the existence of some $p \in \mathbb{R}^2$ which has at least two positive semi-trajectories $\gamma_p^+ \subset \mathbb{R}^2$ and $\sigma_p^+ \subset \mathbb{R}^2$ of $Y = (f, g)$. By using the orthogonal vector field $Y^* = (-g, f)$, as in [17], we can see that there are $q_1 \in \gamma_p^+ \setminus \{p\}$, $q_2 \in \sigma_p^+ \setminus \{p\}$ and a small compact oriented arc $[q_1, q_2]^*$ which is tangent to Y^* at any point. Moreover, since $Y^*(p) \neq 0$ we can assume that $||Y^*(z)|| > 0$ for all $z \in [q_1, q_2]^*$.

We consider the closed region B whose boundary is the union of $[p, q_1] \subset \gamma_p^+$, $[q_1, q_2]^*$ and $[p, q_2] \subset \sigma_p^+$ and apply the Green's formula [18] to the map $z \mapsto Y(z)$. Thus (by using the unitary outer normal vector and the arc length element ds, in the line integral) we obtain that

$$\int_{[q_1,q_2]^*} ||Y|| ds = \int_B \mathrm{Trace}(DY) dx \wedge dy.$$

Since the eigenvalue condition shows $\mathrm{Trace}(DY) = 0$, both integrals are zero. This contradiction proves (a) and concludes the proof. □

Remark 2.2. This lemma remains true if we consider a differentiable vector field $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ for which $\mathrm{Spc}(X) \subset \{z \in \mathbb{C} : \Re(z) = 0\} \setminus \{0\}$.

2.2. **Pseudo-hyperbolic sector at infinity.** In this subsection we describe, in a proposition, some qualitative properties at infinity of the vector field $Y = (f, g)$. To this end, we take $Y^* = (-g, f)$ and consider the region $\mathcal{S} = S(p_1, p_2; q_1, q_2, \{\sigma_i\})$ whose boundary $\partial\mathcal{S}$ is made up of two unbounded semi-trajectories $[q_1, \infty)$ and $(\infty, q_2]$ of Y, a compact arc of trajectories $[p_1, p_2]$ of Y, two arcs of trajectory $[p_1, q_1]^*$, $[p_2, q_2]^*$ of Y^*, and a set at most countable (which may be empty) of pairwise disjoint trajectories $\sigma_1, \sigma_2, \ldots, \sigma_i, \ldots$ that start and end at infinity.

We call such a region a *pseudo-hyperbolic sector of* Y if the following conditions are satisfied:

(1) for each $z \in [p_1, q_1)^*$, there exists an arc of trajectory $[z, \pi(z)) \subset \mathcal{S}$ of Y starting at $z \in [p_1, q_1)^*$ and ending at $\pi(z) \in [p_2, q_2]^*$, and
(2) the closure $\overline{\bigcup_{z\in[p_1,q_1)^*} [z, \pi(z)]}$ is all of $\mathcal{S}$.

This concept of pseudo-hyperbolic sector at infinity has previously been used in our paper [10] (see also [1, 15]).

Proposition 2.3. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field whose Jacobian determinant is always different from zero. Suppose that:*

i) *There are constants $s_1 > 0$ and $c > 0$ for which $||z|| > s_1$ implies that $||Y(z)|| \geq c$.*
ii) *There is $s_2 > 0$ such that for all z with $||z|| > s_2$, the eigenvalues of DY_z are purely imaginary.*

Then Y has no pseudo-hyperbolic sector at infinity.

Proof. Suppose by way of contradiction the existence of a pseudo-hyperbolic sector of Y. We can change the boundary of the sector and assume that

(a) the pseudo-hyperbolic sector $\mathcal{S} = S(p_1, p_2; q_1, q_2, \{\sigma_i\})$ is a subset of $\mathbb{R}^2\backslash\overline{D}_s$, where $s > \max\{s_1, s_2\}$, that is, $\mathcal{S} \subset \mathbb{R}^2 \setminus \overline{D}_s$.

Therefore, for any $z \in \mathcal{S}$ we have $||Y(z)|| \geq c$, and the forward Poincaré map $T : [p_1, q_1)^* \to [p_2, q_2]^*$ is well-defined (subsection 2.1).

(b.1) We claim that, for each $q \in [q_1, \infty)$ there are $p \in [p_1, p_2]$ and an arc of trajectory $[p, q]^* \subset \mathcal{S}$ of Y^* starting from p and ending at q.

In order to prove (b.1) we select a sequence $z_n \in [p_1, q_1]^*$ such that $z_n \to q_1$ and assume that there are arcs of trajectories $[z_n, \pi(z_n)] \subset \mathcal{S}$ whose union satisfies $\overline{\bigcup_{z_n\in[p_1,q_1)^*} [z_n, \pi(z_n)]} = \mathcal{S}$. From this we define A_n as the *compact* set bounded by the union of the arcs $[p_1, z_n]^* \subset [p_1, q_1]^*$, $[z_n, \pi(z_n)]$, $[p_2, \pi(z_n)]^* \subset [p_2, q_2]^*$ and $[p_1, p_2]$.

We consider γ_q^-, a negative semi-trajectory of Y^* starting at $q \in [q_1, \infty)$. Thus, for some arc $[z_n, \pi(z_n)]$ with n large enough, γ_q^- intersects the interior of A_n. Since Y^* is free of singularities in the simply connected set $A_n \subset \mathcal{S}$ we obtain (b.1) from the Poincaré–Bendixson Theory [14, p. 156], which claims that there is no semi-trajectory of Y^* contained in the compact A_n. (Notice that, the oriented arc $[z_n, \pi(z_n)]$ is transverse to γ_q^-.) Therefore, (b.1) is true.

(b.2) We claim that there exists a constant $K > 0$, such that the arc length $\ell([p, q]^*)$ of any $[p, q]^*$ is bounded by K. That is, for each $[p, q]^*$ as in (b.1) we have that $\ell([p, q]^*) < K$.

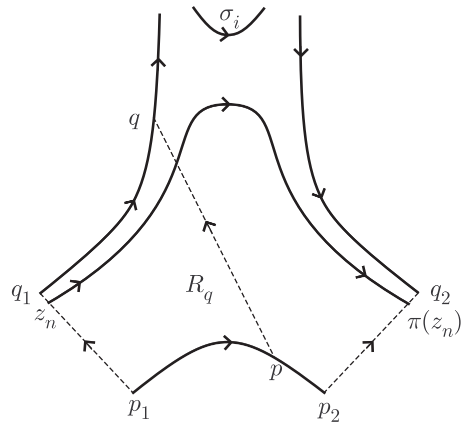

FIGURE 1. Pseudo-hyperbolic sector at infinity

In order to prove (b.2) we denote by R_q the compact subset of $\mathcal{S}$ enclosed by the union of the arcs $[p_1, q_1]^* \subset \partial\mathcal{S}$, $[q_1, q] \subset [q_1, \infty)$, $[p, q]^*$ as in (b.1), and $[p_1, p] \subset [p_1, p_2]$ (see Figure 1). By using the Green's formula [18] in R_q and the arc length element ds, we obtain that

$$\left| \int_{[p,q]^*} ||Y(s)||ds \right| = \left| \int_{[p_1,q_1]^*} ||Y(s)||ds \right|. \tag{2.1}$$

On the other hand, assumption (i) implies that $c.\ell([p,q]^*) \leq \left| \int_{[p,q]^*} ||Y(s)||ds \right|$. Thus, if we define $d = \max\{||Y(z)|| : z \in [p_1, q_1]^*\}$ the equality of (2.1) shows that

$$c.\ell([p,q]^*) \leq \left| \int_{[p_1,q_1]^*} ||Y(s)||ds \right| \leq d\ell([p_1, q_1]^*).$$

As $c > 0$, statement (b.2) holds.

From (b.1) and (b.2) we obtain that the arc $[q_1, \infty)$ is bounded, because the distance from any point $q \in [q_1, \infty)$ to the compact arc $[p_1, p_2]$ is smaller than K. This contradiction gives the proposition. □

Corollary 2.4. *For every Y which satisfies the conditions of Proposition* 2.3*, both vector fields $-Y$ and Y have no pseudo-hyperbolic sector at infinity.*

Proof. We refer the reader to Proposition 2.3. □

3. Centers with global period annulus

In this section we present the proof of Theorem A. Thus, we will need the next lemma, which can be deduced from Theorem 2.1 of [11].

Lemma 3.1. *If Y is as in Theorem A, the map associated to Y is globally injective.*

An important consequence of this lemma is that the injective vector fields of Theorem A are free of singularities in a neighborhood of infinity.

Proposition 3.2. *Take $Y : \mathbb{R}^2 \to \mathbb{R}^2$ to be a local homeomorphism with $Y(0) = 0$. If there exists a constant $\sigma > 0$ for which the restriction $Y|_{\mathbb{R}^2 \setminus \overline{D}_\sigma}$ is differentiable and $\operatorname{Spc}(Y|_{\mathbb{R}^2 \setminus \overline{D}_\sigma}) \subset \{z \in \mathbb{C} : \Re(z) = 0\} \setminus \{0\}$, then, for any pair of trajectories γ_p and σ_p of Y both contained in $\mathbb{R}^2 \setminus \overline{D}_\sigma$ and passing through the same point $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$, we obtain that $\gamma_p = \sigma_p$. Moreover, Y has a center type performance at infinity.*

Proof. The uniqueness of the trajectories contained in $\mathbb{R}^2 \setminus \overline{D}_\sigma$ follows directly from Remark 2.2. Therefore we only consider the second affirmation.

By Lemma 3.1, the associated map $Y : \mathbb{R}^2 \to \mathbb{R}^2$ is injective and $Y(0) = 0$. Therefore, we can apply Corollary 2.4 and obtain that:

(a) The injective vector fields $-Y$ and Y have no pseudo-hyperbolic sector at infinity.

From this we can prove the next

Assertion: *There exists an unbounded sequence of periodic trajectories.*

Proof of the assertion. Suppose that Y has no unbounded sequence of periodic trajectories. Thus,

(b.1) there is $s > \sigma$ such that Y has neither singularities nor periodic trajectories in $\mathbb{R}^2 \setminus \overline{D}_s$.

We can apply the results of [14, pp. 166-174] to the (fixed point free) flow induced by $Y|_{\mathbb{R}^2 \setminus \overline{D}_s}$ and obtain that for any circle $C \subset \mathbb{R}^2 \setminus \overline{D}_s$ surrounding the origin with a finite number of tangencies, the Brouwer degree $\deg(Y|_C)$ satisfies

$$\deg(Y|_C) = \frac{2 - n^e(Y,C) + n^i(Y,C)}{2},$$

where $n^e(Y,C)$ (resp. $n^i(Y,C)$) is the number of tangent points of C with Y that are "external" to $D(C)$ (resp. "internal" to $D(C)$).

(b.2) We claim that if $C_s \subset \mathbb{R}^2 \setminus \overline{D}_s$ *minimizes* $n^i(Y, C_s)$, then every internal tangency in C_s gives a pseudo-hyperbolic sector at infinity.

For every internal tangency $q \in C_s$ we consider the forward Poincaré map $T : [p,q)_s \subset C_s \to C_s$ induced by Y (if $T : (q,r]_s \subset C_s \to C_s$, we obtain a pseudo-hyperbolic sector of $-Y$) where $[p,q)_s \subset C_s$ is the maximal connected domain of definition of T on which this first return map is continuous.

If the arc of trajectory $(p,\infty) \subset \gamma_p^+$ intersects C_s, we can apply Lemma 24 and Lemma 25 of [10], so we can deform C_s in a new circle $C^1 \subset \mathbb{R}^2 \setminus \overline{D}_s$ such that the number of internal tangencies of C^1 with Y is (strictly) smaller than that of C_s. This is a contradiction. Therefore $(p,\infty) \subset \gamma_p^+$ is disjoint from C_s. By using this and our selection of $[p,q)_s \subset C_s$, it is not difficult to check that there is a pseudo-hyperbolic sector of Y whose boundary intersects $\gamma_p^+ \supset (p,\infty)$. Thus, (b.2) holds.

(b.3) We claim that there exists a circle C_s such that C_s is transverse to Y and $D_s \subset D(C_s)$,

Take a circle C_s as in (b.2). By using (a) we have that $\deg(Y|_{C_s}) = 1$ and $n^i(Y, C_c) = 0$. Therefore, we obtain (b.3) from the last formula of the Brouwer degree.

We take the circle of (b.3) and consider the compact disk $\overline{D}(C_s)$. Now we apply the Green's formula of [18] to the map $z \mapsto Y(z)$. Thus

$$\int_{\overline{D}(C_s)} \mathrm{Trace}(DY) dx \wedge dy = \oint_{C_s} \langle Y(s), \eta(s) \rangle ds,$$

where $\eta(s)$ is the unitary outer normal vector to C_s and $\langle Y(s), \eta(s) \rangle$ is the inner product of $Y(s)$ with $\eta(s)$. Since C_s is transverse to Y we obtain

$$0 \neq \oint_{C_s} \langle Y(s), \eta(s) \rangle ds = \int_{\overline{D}(C_s)} \mathrm{Trace}(DY) dx \wedge dy.$$

But, the eigenvalue assumption shows that $\int_{\overline{D}(C_s)} \mathrm{Trace}(DY) dx \wedge dy = 0$. This contradiction gives the assertion. □

In order to conclude this proof we consider the sequence of the assertion. Since $Y(0) = 0$ and Y is injective, we obtain that:

(c) There exists an unbounded set of periodic trajectories $\{\Gamma, \Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots\}$ such that $\overline{D}_\sigma$ is a proper subset of $\overline{D}(\Gamma)$ and

$$\overline{D}(\Gamma) \subset \overline{D}(\Gamma_1) \subset \overline{D}(\Gamma_2) \subset \cdots \subset \overline{D}(\Gamma_n) \subset \cdots.$$

Moreover, we can assume that $\mathbb{R}^2 \setminus D(\Gamma)$ has no singular points of Y.

(d) We claim that the elements of $\mathbb{R}^2 \setminus D(\Gamma)$ are periodic points.

For each $p \in \mathbb{R}^2 \setminus D(\Gamma)$ there exists a compact annulus $\overline{A}_n = \overline{D}(\Gamma_n) \setminus D(\Gamma)$ containing p. Hence, the trajectory starting at p and its two limit sets are contained in $\overline{A}_n$, that is, $\gamma_p \subset \overline{A}_n$ and $\alpha(\gamma_p^-) \cup \omega(\gamma_p^+) \subset \overline{A}_n$. Therefore, the Poincaré–Bendixon Theorem implies that both limit sets $\alpha(\gamma_p^-)$ and $\omega(\gamma_p^+)$ are periodic trajectories of $Y = (f, g)$ surrounding $D(\Gamma)$.

To proceed we can assume that $\omega(\gamma_p^+)$ is clockwise oriented because, in the other case, the construction is similar.

By using the trajectory γ_q^* of $Y^* = (-g, f)$ for some $q \in \omega(\gamma_p^+)$, it is not difficult to see that there are two arcs of trajectories $[p_1, p_2] \subset \gamma_p^+$ and $[p_1, p_2]^* \subset \gamma_q^*$ (or $[p_2, p_1]^* \subset \gamma_q^*$ when γ_p^+ surrounds the disk bounded by $\omega(\gamma_p^+)$). These two arcs bound a compact set $B \subset \overline{A}_n$ where we can apply the Green's formula to the map $z \mapsto Y(z)$. This formula implies that

$$\left| \int_{[p_1, p_2]^*} ||Y(s)|| ds \right| = 0. \tag{3.1}$$

As $[p_1, p_2]^*$ is free of singularities, we have that $p_1 = p_2$. Therefore, γ_p is periodic. This implies (d). We conclude this proof from Definition 1.1. □

Corollary 3.3. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a globally injective local homeomorphism with $Y(0) = 0$. Suppose that there is $\sigma > 0$ such that $Y|_{\mathbb{R}^2 \setminus \overline{D}_\sigma}$ is differentiable and $\mathrm{Trace}(DY|_{\mathbb{R}^2 \setminus \overline{D}_\sigma}) = 0$. Then, the trajectories of $Y|_{\mathbb{R}^2 \setminus \overline{D}_\sigma}$ are unique and Y has a center type performance at infinity.*

Proof. There are constants $s_1 > 0$ and $c > 0$ for which $||z|| > s_1$ implies that $||Y(z)|| \geq c$. So, Corollary 2.4 shows that the injective vector fields $-Y$ and Y have no pseudo-hyperbolic sector at infinity. Therefore, we obtain this corollary from a slight change in the proof of Proposition 3.2. □

Before continuing, let us recall that the Local Inverse Function Theorem is also true for differentiable maps whose Jacobian determinant is always different from zero; see [3, 19] and its references.

3.1. **Proof of Theorem A.** By Proposition 3.2, it suffices to prove that all the elements in $\overline{D}(\Gamma) \setminus \{0\}$ are periodic points, where Γ is the periodic trajectory given by Definition 1.1.

(a) We claim that for each $p \in \overline{D}(\Gamma) \setminus \{0\}$, at least one of its limit sets is a periodic trajectory.

Suppose, by contradiction, the existence of some $p \in \overline{D}(\Gamma) \setminus \{0\}$ for which (a) is false. Since the compact set $\overline{D}(\Gamma)$ contains the trajectory at every one of its points, Lemma 3.1 and the Poincaré–Bendixson Theorem imply that $\omega(\gamma_p^+) = \alpha(\gamma_p^-) = 0$. Let $Y = (f, g)$, consider the orthogonal vector field $Y^* = (-g, f)$ and a trajectory of Y^* starting at p, say γ_p^*. We take the semi-trajectory of γ_p^* which goes into the compact set bounded by $\gamma_p \cup \{0\}$. To proceed we can suppose that this semi-trajectory is the positive one, that is, $(\gamma_p^*)^+$. Thus the Poincaré–Bendixson Theory [14, p. 151] implies that $\omega((\gamma_p^*)^+) = \{0\}$. Hence, $\gamma_p^- \cup (\gamma_p^+)^* \cup \{0\}$ bounds a compact set where we can apply the Green's formula to the map $z \mapsto Y(z)$. This formula implies that

$$\left| \int_{(\gamma_p^*)^+} ||Y(s)|| ds \right| = 0.$$

This contradiction proves (a).

In order to conclude the proof we consider $p \in \overline{D}(\Gamma) \setminus \{0\}$. From (a), $\omega(\gamma_p)$ or $\alpha(\gamma_p)$ is periodic. Since $\overline{D}(\Gamma) \setminus \{0\}$ is free of singular points of Y we can proceed as in the proof of (3.1) and obtain that p is periodic. Therefore, Theorem A holds.

Corollary 3.4. *Let $Y : \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable vector field with* $\det(DY) \neq 0$*. Suppose that Y is globally injective, $Y(0) = 0$ and* $\operatorname{Trace}(DY) = 0$*. Then the origin is a center whose period annulus is $\mathbb{R}^2 \setminus \{0\}$.*

Proof. We refer the reader to Corollary 3.3 and the proof of Theorem A. □

4. Center type performance at infinity

This section is devoted to extending the result of the previous section to a vector field defined on a neighborhood of infinity which can be a proper subset of the plane. To this end, we shall need the following result contained in [11].

Theorem 4.1. *Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable vector field. If for some $\varepsilon > 0$,* $\operatorname{Spc}(X) \cap (-\varepsilon, +\infty) = \emptyset$*, then there exists $s \geq \sigma$ such that $X|_{\mathbb{R}^2 \setminus \overline{D}_s}$ can be extended to a globally injective local homeomorphism $\tilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$.*

As an application of this theorem and the results of Section 3 we obtain the next theorem.

Theorem B. *Let $X : \mathbb{R}^2 \setminus \overline{D}_\sigma \to \mathbb{R}^2$ be a differentiable vector field. If* $\operatorname{Spc}(X) \subset \{z \in \mathbb{C} : \Re(z) = 0\} \setminus \{0\}$*, then for any $p \in \mathbb{R}^2 \setminus \overline{D}_\sigma$ there exists a unique trajectory starting at p. Moreover, there is a constant $v \in \mathbb{R}^2$ such that $X + v$ has a center type performance at infinity.*

Proof. Let $\tilde{X} : \mathbb{R}^2 \to \mathbb{R}^2$ be the topological embedding given in Theorem 4.1. Set $v = -\tilde{X}(0)$ and consider the global injective map $\tilde{X} + v$ which sends the origin into itself. Therefore, $Y := \tilde{X} + v$ satisfies Proposition 3.2. Since $Y = X + v$ in a neighborhood of infinity, we conclude the proof. □

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