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THE WEIL–PETERSSON GEOMETRY OF THE MODULI SPACE OF RIEMANN SURFACES

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ABSTRACT. In 2007, Z. Huang showed that in the thick part of the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g, the sectional curvature of the Weil–Petersson metric is bounded below by a constant depending on the injectivity radius, but independent of the genus g. In this article, we prove this result by a different method. We also show that the same result holds for Ricci curvature. For the universal Teichmüller space equipped with a Hilbert structure induced by the Weil–Petersson metric, we prove that its sectional curvature is bounded below by a universal constant.

1. INTRODUCTION

There have been many studies on the geometry of the Weil–Petersson (WP) metric on moduli spaces of Riemann surfaces, especially regarding its curvature properties [1, 10, 16, 14, 11, 6, 9, 17, 18, 15, 7, 8, 3, 4]. In a pioneering work, Ahlfors [1] showed that the Ricci, holomorphic sectional and scalar curvatures of the WP metric are all negative. Later, Royden [10] showed that the holomorphic sectional curvature is bounded above by a negative constant, and he conjectured that on the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g, this constant is equal to $\frac{-1}{2\pi(g-1)}$. By deriving more compact expressions for the Riemann tensors of the WP metric, Wolpert [16] verified Royden's conjecture. He also showed that the Ricci curvature is bounded above by $\frac{-1}{2\pi(g-1)}$, and the scalar curvature is bounded above by $-\frac{3(3g-2)}{4\pi}$. In the communications between Wolpert and Tromba and between Wolpert and Royden, it was proved that the sectional curvature of the WP metric is also negative. A detailed proof of this result was given by Wolpert in [16] and by Tromba in [14]. Regarding the upper bound, it was proved in [3] that the sectional curvature does not have a negative upper bound.

Lower bounds of the curvatures have received less attention. There are some results obtained by [11, 13, 5, 4]. The results of [11, 13, 5] showed that the sectional curvature is not bounded below on the moduli space $\mathcal{M}_g$. Therefore, attention must be shifted to find lower bounds of the sectional curvature on compact subsets of the moduli space $\mathcal{M}_g$. This problem was studied by Huang in [4]. To describe his result

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in more detail, we need to introduce some notation first. A point on the moduli space $\mathcal{M}_g$ can be considered as a compact Riemann surface X of genus g endowed with a unique metric of constant curvature -1, called the hyperbolic metric. The injectivity radius of X at a point $z \in X$, $inj(X;z)$, is defined as the supremum over r for which the open set $U_z^r = \{w \in X : d(z,w) < r\}$ is isometric to a disc. The injectivity radius of X, $inj(X)$, is defined to be the infimum of $inj(X;z)$, $z \in X$. By a well-known result, $inj(X)$ is equal to one half of the length of the shortest closed geodesic of X. Given a positive constant r_0, the thick part of the moduli space $\mathcal{M}_g$ (with respect to r_0) is defined as the subset of $\mathcal{M}_g$ consisting of those points where the injectivity radius of the corresponding Riemann surfaces is greater than r_0. In [4], Huang showed that on the thick part of the moduli space $\mathcal{M}_g$, the holomorphic sectional and sectional curvatures of the WP metric are both bounded below by negative constants $-C_1$ and $-C_2$ depending on r_0, but independent of the genus g. As a result, the Ricci and scalar curvatures are bounded below by $-C_3g$ and $-C_4g^2$ respectively, where C_3 and C_4 are two positive constants depending on r_0, but independent of g. The main tool used by Huang is the analysis of harmonic maps between hyperbolic surfaces. In the present article, we are going to give a different proof of Huang's result without using harmonic maps. Moreover, we are going to improve the bounds $-C_3g$ and $-C_4g^2$ for Ricci and scalar curvatures to $-C_3$ and $-C_4g$ respectively. Explicit dependence of the constants C_1, C_2, C_3, C_4 on the injectivity radius r_0 is given.

In [12], we have defined a Hilbert structure on the universal Teichmüller space $T(1)$, so that the Weil–Petersson metric is a well-defined metric on $T(1)$. We have also obtained an explicit formula for the Riemann curvature tensor of the WP metric, which is a generalization of the result of Wolpert [16]. In this article, we are going to show that the sectional curvature of the WP metric on $T(1)$ is bounded below by a universal negative constant. We also show that it does not have a negative upper bound.

The layout of this article is as follows. In Section 2 we review some necessary facts. In Section 3 we obtain the lower bounds of the curvatures of the WP metric on the moduli space $\mathcal{M}_g$ as a function of the injectivity radius. In Section 4 we find the lower bound of the sectional curvature of the WP metric on the universal Teichmüller space.

2. Background

In this section, we present some necessary facts. Let $T(X)$ and $\mathcal{M}(X)$ be respectively the Teichmüller space and the moduli space of a compact Riemann surface X of genus g, where $g \geq 2$. The Teichmüller space $T(X)$ has a complex analytic model described as follows. Let $\mathbb{D}$ and $\mathbb{D}^*$ be respectively the unit disc and its exterior. There is a Fuchsian group $\Gamma \in \text{PSU}(1,1)$ such that the quotient of the unit disc $\mathbb{D}$ by the action of Γ is X, i.e., $X \simeq \Gamma\backslash\mathbb{D}$. The space of bounded Beltrami differentials on X can be identified with the space of bounded Γ-automorphic $(-1,1)$ differentials on $\mathbb{D}$, denoted by $\mathcal{A}^{-1,1}(\mathbb{D},\Gamma)$, which consists of bounded functions μ on $\mathbb{D}$ satisfying

$$\mu(\gamma(z))\frac{\overline{\gamma'(z)}}{\gamma'(z)} = \mu(z).$$

Let $\mathcal{B}^{-1,1}(\mathbb{D},\Gamma)$ be the unit ball of $\mathcal{A}^{-1,1}(\mathbb{D},\Gamma)$ with respect to the sup-norm:

$$\|\mu\|_\infty = \sup_{z\in\mathbb{D}} |\mu(z)|.$$

Given a Beltrami differential $\mu \in \mathcal{B}^{-1,1}(\mathbb{D},\Gamma)$, extend it to $\hat{\mathbb{C}} := \mathbb{C}\cup\{\infty\}$ by reflection:

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)}\frac{z^2}{\bar{z}^2}, \qquad z \in \mathbb{D}^*. \tag{2.1}$$

There is a unique quasiconformal mapping $w_\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which fixes the points $-1, -i, 1$ and satisfies the Beltrami equation $(w_\mu)_{\bar{z}} = \mu(w_\mu)_z$. The conjugation of Γ by w_μ, $\Gamma_\mu = w_\mu \circ \Gamma \circ w_\mu^{-1}$, is again a Fuchsian group. The corresponding quotient surface $X_\mu = \Gamma_\mu\backslash\mathbb{D}$ is a Riemann surface having the same type as X, but with different complex structure. Define an equivalence relation on $\mathcal{B}^{-1,1}(\mathbb{D},\Gamma)$ so that $\mu \sim \nu$ if and only if $w_\mu = w_\nu$ on the unit circle S^1. Real analytically, the Teichmüller space $T(X)$ is isomorphic to $\mathcal{B}^{-1,1}(\mathbb{D},\Gamma)/\sim$.

Denote by w^μ the corresponding quasiconformal mapping if we extend $\mu \in \mathcal{B}^{-1,1}(\mathbb{D},\Gamma)$ to $\hat{\mathbb{C}}$ by setting it equal to zero outside $\mathbb{D}$. w^μ is holomorphic on $\mathbb{D}^*$ and $\Gamma^\mu = w^\mu \circ \Gamma \circ (w^\mu)^{-1}$ is no longer a Fuchsian group, but a quasi–Fuchsian group. The corresponding Riemann surface $X^\mu = \Gamma^\mu\backslash w^\mu(\mathbb{D})$ is biholomorphic to X_μ. Complex analytically, the Teichmüller space is the quotient $\mathcal{B}^{-1,1}(\mathbb{D},\Gamma)/\sim$, where $\mu \sim \nu$ if and only if $w^\mu = w^\nu$ on the unit circle. For two compact Riemann surfaces X and Y having the same genus g, their Teichmüller spaces $T(X)$ and $T(Y)$ are naturally isomorphic, and we use T_g to denote the Teichmüller space of compact Riemann surfaces of genus g.

The tangent space and cotangent space at a point $[\mu]$ of the Teichmüller space $T(X)$ can be naturally identified with the space of harmonic Beltrami differentials and the space of holomorphic quadratic differentials of X^μ. The space of holomorphic quadratic differentials of X^μ can be identified with the space of Γ_μ-automorphic $(2,0)$ differentials $\Omega^{2,0}(\mathbb{D},\Gamma_\mu)$ on $\mathbb{D}$, which consists of holomorphic functions q on $\mathbb{D}$ satisfying

$$q(\gamma(z))\gamma'(z)^2 = q(z), \qquad \forall\, \gamma \in \Gamma_\mu.$$

Correspondingly, the space of harmonic Beltrami differentials of X^μ can be identified with $\Omega^{-1,1}(\mathbb{D},\Gamma_\mu)$, which is a subspace of $\mathcal{A}^{-1,1}(\mathbb{D},\Gamma_\mu)$ consisting of functions ν of the form

$$\nu(z) = \rho(z)^{-1}\overline{q(z)} = \frac{(1-|z|^2)^2}{4}\overline{q(z)},$$

where $q \in \Omega^{2,0}(\mathbb{D},\Gamma_\mu)$ and ρ is the hyperbolic metric density on $\mathbb{D}$. The moduli space $\mathcal{M}(X)$ is the quotient of the Teichmüller space $T(X)$ under the action of the mapping class group. The Weil–Petersson metric on $T(X)$, which is defined by

$$\langle \nu_\alpha, \nu_\beta\rangle_{WP} = \iint_{\Gamma_\mu\backslash\mathbb{D}} \nu_\alpha(z)\overline{\nu_\beta(z)}\rho(z)d^2z$$

on the tangent space of $T(X)$ at $[\mu]$, is modular invariant and hence descends to a well-defined metric on the moduli space $\mathcal{M}(X)$.

The universal Teichmüller space $T(1)$ can be defined similarly with the group Γ being the trivial group consisting of only the identity element, i.e., $\Gamma = \{\mathrm{id}\}$. More precisely, $T(1) = \mathcal{B}^{-1,1}(\mathbb{D})/\sim$, where $\mathcal{B}^{-1,1}(\mathbb{D})$ is the space of bounded functions on $\mathbb{D}$ with sup-norm less than one, and $\mu \sim \nu$ if and only if $w_\mu \sim w_\nu$ on S^1. The cotangent space at any point of $T(1)$ is naturally isomorphic to the Banach space

$$A_\infty(\mathbb{D}) = \left\{ q \text{ holomorphic on } \mathbb{D} \; : \; \|q\|_\infty = \sup_{z\in\mathbb{D}} \rho(z)^{-1}|q(z)| < \infty \right\},$$

while the tangent space is identified with the Banach space

$$\Omega^{-1,1}(\mathbb{D}) = \left\{ \rho^{-1}\bar{q} \; : \; q \in A_\infty(\mathbb{D}) \right\}$$

of harmonic Beltrami differentials on $\mathbb{D}$. Obviously, the inner product

$$\langle \nu_\alpha, \nu_\beta \rangle = \iint_{\mathbb{D}} \nu_\alpha(z)\overline{\nu_\beta(z)}\rho(z) d^2z \tag{2.2}$$

is not well defined on $\Omega^{-1,1}(\mathbb{D})$. In [12], we showed that we can define a Hilbert structure on $T(1)$ so that at any point, its tangent space is isomorphic to

$$H^{-1,1}(\mathbb{D}) = \left\{ \rho^{-1}\bar{q} \; : \; q \in A_2(\mathbb{D}) \right\},$$

where

$$A_2(\mathbb{D}) = \left\{ q \text{ holomorphic on } \mathbb{D} \; : \; \|q\|_2^2 = \iint_{\mathbb{D}} |q(z)|^2 \rho(z)^{-1} d^2z < \infty \right\}.$$

We denote the Teichmüller space with this Hilbert structure as $T_H(1)$. The inner product (2.2) is well defined on the tangent space $H^{-1,1}(\mathbb{D})$, and we call the resulting metric on $T_H(1)$ the Weil–Petersson metric.

Let $\Delta = -\rho^{-1}\partial\bar{\partial}$ be the Laplace–Beltrami operator of the hyperbolic metric on X and let

$$G = \frac{1}{2}\left(\Delta + \frac{1}{2}\right)^{-1} \tag{2.3}$$

be one-half of the resolvent of Δ at $\lambda = -1/2$. In [16], Wolpert showed that the Riemann curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ of the Weil–Petersson metric at the tangent space of a point on the moduli space corresponds to the compact Riemann surface $X = \Gamma\backslash\mathbb{D}$ is given by[1]

$$R_{\alpha\bar{\beta}\lambda\bar{\delta}} = -\iint_{\Gamma\backslash\mathbb{D}} G(\nu_\alpha\bar{\nu}_\beta)(\nu_\lambda\bar{\nu}_\delta)\rho d^2z - \iint_{\Gamma\backslash\mathbb{D}} G(\nu_\alpha\bar{\nu}_\delta)(\nu_\lambda\bar{\nu}_\beta)\rho d^2z. \tag{2.4}$$

In [12], we generalized this result and showed that formula (2.4) is still valid on $T_H(1)$ if $\Gamma\backslash\mathbb{D}$ is replaced by $\mathbb{D}$.

3. Lower bounds of curvatures of the Weil–Petersson metric on moduli space of compact Riemann surfaces

In [16], Wolpert has shown that the holomorphic sectional, Ricci and scalar curvatures are bounded above by $-\frac{1}{2\pi(g-1)}$, $-\frac{1}{2\pi(g-1)}$ and $-\frac{3(3g-2)}{4\pi}$ respectively. These upper bounds depend on the genus g. On the other hand, Huang showed that the sectional curvature does not have a negative upper bound [3]. Here we would like to find lower bounds for the curvatures which only depend on the injectivity radius of the corresponding Riemann surface.

[1]Our convention differs from the convention of Wolpert by a sign.

Recall that the holomorphic sectional and Ricci curvatures at a point on the moduli space corresponding to the Riemann surface $X = \Gamma\backslash\mathbb{D}$ in the direction spanned by $\nu_\alpha \in \Omega^{-1,1}(\mathbb{D},\Gamma)$ with $\|\nu_\alpha\|_{WP} = 1$ are given respectively by [2, 16]:

$$s_\alpha = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = -2\iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\alpha|^2\rho d^2z \tag{3.1}$$

and

$$\begin{aligned}\mathcal{R}_{\alpha\bar{\alpha}} &= \sum_{\beta=1}^{3g-3} R_{\alpha\bar{\beta}\beta\bar{\alpha}} \\ &= -\sum_{\beta=1}^{3g-3}\left\{\iint_{\Gamma\backslash\mathbb{D}} G(\nu_\alpha\bar{\nu}_\beta)\bar{\nu}_\alpha\nu_\beta\rho d^2z + \iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2\rho d^2z\right\},\end{aligned} \tag{3.2}$$

where $\{\nu_1,\ldots,\nu_{3g-3}\}$ is an orthonormal basis of $\Omega^{-1,1}(\mathbb{D},\Gamma)$. The scalar curvature S is equal to the trace of the Ricci tensor:

$$S = \sum_{\alpha=1}^{3g-3}\mathcal{R}_{\alpha\bar{\alpha}} = -\sum_{\alpha=1}^{3g-3}\sum_{\beta=1}^{3g-3}\left\{\iint_{\Gamma\backslash\mathbb{D}} G(\nu_\alpha\bar{\nu}_\beta)\bar{\nu}_\alpha\nu_\beta\rho d^2z + \iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2\rho d^2z\right\}. \tag{3.3}$$

On the other hand, given two orthogonal tangent vectors $\nu_\alpha, \nu_\beta \in \Omega^{-1,1}(\mathbb{D},\Gamma)$ with $\|\nu_\alpha\|_{WP} = \|\nu_\beta\|_{WP} = 1$, the sectional curvature of the plane spanned by the real tangent vectors corresponding to ν_α and ν_β is [2, 16]

$$\begin{aligned}K_{\alpha,\beta} &= \frac{1}{4}\left(R_{\alpha\bar{\beta}\beta\bar{\alpha}} + R_{\beta\bar{\alpha}\alpha\bar{\beta}} - R_{\alpha\bar{\beta}\alpha\bar{\beta}} - R_{\beta\bar{\alpha}\beta\bar{\alpha}}\right) \\ &= \operatorname{Re}\iint_{\Gamma\backslash\mathbb{D}} G(\nu_\alpha\bar{\nu}_\beta)\nu_\alpha\bar{\nu}_\beta\rho d^2z - \frac{1}{2}\iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2\rho d^2z \\ &\quad - \frac{1}{2}\iint_{\Gamma\backslash\mathbb{D}} G(\nu_\alpha\bar{\nu}_\beta)\bar{\nu}_\alpha\nu_\beta\rho d^2z.\end{aligned} \tag{3.4}$$

We have used the self-adjointness of G to obtain the last expression.

To obtain the lower bounds of curvatures, Huang [4] used harmonic maps to show that if a harmonic Beltrami differential has unit Weil–Petersson norm, then its sup-norm is bounded above by a constant depending on the injectivity radius of the underlying Riemann surface. Here we re-prove this result without resorting to harmonic maps, which better reveals its elementary nature and also allows us to generalize this result to the universal Teichmüller space later.

Proposition 3.1. *Let $X = \Gamma\backslash\mathbb{D}$ be a compact Riemann surface with injectivity radius r_X and let $\nu \in \Omega^{-1,1}(\mathbb{D},\Gamma)$ be a harmonic Beltrami differential of X. The ratio of the sup-norm of ν to the Weil–Petersson norm of ν is bounded above by a constant $C(r_X)$ depending only on r_X, i.e.,*

$$\|\nu\|_\infty \leq C(r_X)\|\nu\|_{WP}.$$

The constant $C(r_X)$ can be chosen to be equal to

$$C(r_X) = \left\{\frac{4\pi}{3}\left[1 - \left(\frac{4e^{r_X}}{(e^{r_X}+1)^2}\right)^3\right]\right\}^{-\frac{1}{2}}. \tag{3.5}$$

Proof. Let $z \in \mathbb{D}$ and let

$$\sigma_z(w) = \frac{z+w}{1+z\bar{w}}, \qquad w \in \mathbb{D},$$

be a linear transformation preserving $\mathbb{D}$ and mapping 0 to z. Notice that $\nu \circ \sigma_z$ is a harmonic Beltrami differential of the group $\sigma_z^{-1} \circ \Gamma \circ \sigma_z$, and $\|\nu \circ \sigma_z\|_{WP} = \|\nu\|_{WP}$, but $|\nu(z)| = |\nu \circ \sigma_z(0)|$. Therefore, it suffices to verify that there exists a constant $C(r_X)$ such that

$$|\nu(0)| \leq C(r_X)\|\nu\|_{WP}.$$

By definition, there exists $q \in \Omega^{2,0}(\mathbb{D}, \Gamma)$ such that $\nu = \rho^{-1}\bar{q}$. Being a holomorphic function on $\mathbb{D}$, q has a Taylor series expansion on $\mathbb{D}$ which can be written as

$$q(z) = \sum_{n=2}^{\infty} (n^3 - n) a_n z^{n-2}.$$

This implies that

$$\nu(0) = \rho(0)^{-1} q(0) = \frac{3a_2}{2},$$

whereas

$$\|\nu\|_{WP}^2 = \iint_{\Gamma\backslash\mathbb{D}} |q(z)|^2 \rho(z)^{-1} d^2 z.$$

By the definition of injectivity radius, we can choose a fundamental domain F for the action of Γ on $\mathbb{D}$ which contains a hyperbolic disc $D(0, r)$ with center at 0 and with radius r, for any r less than r_X. Elementary hyperbolic geometry gives us

$$D(0, r) = \left\{ z \in \mathbb{C} \, : \, |z| < \frac{e^r - 1}{e^r + 1} \right\}.$$

Therefore, for any $r \in (0, r_X)$, we have

$$\begin{aligned}
\|\nu\|_{WP}^2 =& \iint_F |q(z)|^2 \rho(z)^{-1} d^2 z \\
\geq& \iint_{D(0,r)} |q(z)|^2 \rho(z)^{-1} d^2 z \\
=& \int_0^{\frac{e^r-1}{e^r+1}} \int_0^{2\pi} \left| \sum_{n=2}^{\infty} (n^3 - n) a_n u^{n-2} e^{i(n-2)\theta} \right|^2 \frac{(1-u^2)^2}{4} d\theta u du \\
=& 2\pi \int_0^{\frac{e^r-1}{e^r+1}} \sum_{n=2}^{\infty} (n^3 - n)^2 |a_n|^2 u^{2n-4} \frac{(1-u^2)^2}{4} u du \\
\geq& 18\pi |a_2|^2 \int_0^{\frac{e^r-1}{e^r+1}} (1-u^2)^2 u du \\
=& \frac{4\pi}{3} |\nu(0)|^2 \left\{ 1 - \left(\frac{4e^r}{(e^r+1)^2} \right)^3 \right\}.
\end{aligned} \tag{3.6}$$

Since this is true for all $r \in (0, r_X)$, we can replace r in (3.6) by r_X. Therefore, we have proved the proposition with $C(r_X)$ equal to

$$C(r_X) = \left\{ \frac{4\pi}{3} \left[1 - \left(\frac{4e^{r_X}}{(e^{r_X}+1)^2} \right)^3 \right] \right\}^{-\frac{1}{2}}. \qquad \square$$

Notice that $C(r_X)$ is a decreasing function of r_X. As r_X approaches infinity, it approaches $\sqrt{3/(4\pi)}$. On the other hand, as $r_X \to 0$, it behaves like

$$C(r_X) \sim \frac{1}{\sqrt{\pi} r_X} + O(1). \tag{3.7}$$

Before computing the lower bounds for the curvature, we state a useful lemma here.

Lemma 3.1. *Let G be the positive self-adjoint operator on X defined by* (2.3)*,*
A. *For any $f \in L^2(X, \mathbb{R})$*

$$\iint_{\Gamma\backslash\mathbb{D}} G(f)\bar{f}\rho d^2z \geq 0.$$

B. *For any $f \in L^2(X, \mathbb{R})$,*

$$\iint_{\Gamma\backslash\mathbb{D}} G(f)\rho d^2z = \iint_{\Gamma\backslash\mathbb{D}} f\rho d^2z.$$

C. *If $f \in L^2(X, \mathbb{R})$ is such that $f \geq 0$, then $G(f) \geq 0$.*
D. *For any $f, g \in L^2(X, \mathbb{R})$, $|G(fg)| \leq G(f^2)^{1/2} G(g^2)^{1/2}$.*

Proof. Lemma 3.1(A) is an immediate consequence of the positivity of G. B is proved using the self-adjointness of G and the fact that $G(1) = 1$. C follows from the fact that the kernel $G(z, w)$ of G is a positive function for all z and w (see [12]). D is Lemma 4.3 in [16]. $\square$

Proposition 3.2. *Let $X = \Gamma\backslash\mathbb{D}$ be a compact Riemann surface of genus g with injectivity radius r_X. At the point on the moduli space $\mathcal{M}_g$ corresponding to X, the holomorphic sectional and sectional curvatures of the Weil–Petersson metric are bounded below by $-2C(r_X)^2$.*

Proof. The proof follows closely the proofs to obtain lower and upper bounds given in [16, 4]. For completeness, we repeat it here. We consider the holomorphic sectional curvature first. Given $\nu_\alpha \in \Omega^{-1,1}(\mathbb{D}, \Gamma)$ with $\|\nu_\alpha\|_{WP} = 1$, we find from (3.1), Proposition 3.1 and B and C of Lemma 3.1 that

$$\begin{aligned} -s_\alpha &\leq 2C(r_X)^2 \iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)\rho d^2z \\ &= 2C(r_X)^2 \iint_{\Gamma\backslash\mathbb{D}} |\nu_\alpha|^2 \rho d^2z = 2C(r_X)^2. \end{aligned}$$

This proves the statement for holomorphic sectional curvature. For the sectional curvature, we use formula (3.4). Notice that the Cauchy–Schwarz inequality, the positivity of the kernel of G, and D of Lemma 3.1 give us

$$\begin{aligned}\left|\iint_{\Gamma\backslash\mathbb{D}} G(\nu_\alpha\bar{\nu}_\beta)\nu_\alpha\bar{\nu}_\beta\rho d^2z\right| &\leq \iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha||\nu_\beta|)|\nu_\alpha||\nu_\beta|\rho d^2z\\ &\leq \iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)^{1/2}G(|\nu_\beta|^2)^{1/2}|\nu_\alpha||\nu_\beta|\rho d^2z\\ &\leq \left\{\iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2\rho d^2z\right\}^{1/2}\left\{\iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\beta|^2)|\nu_\alpha|^2\rho d^2z\right\}^{1/2}\\ &= \iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2\rho d^2z.\end{aligned}$$

Similarly,

$$\left|\iint_{\Gamma\backslash\mathbb{D}} G(\nu_\alpha\bar{\nu}_\beta)\bar{\nu}_\alpha\nu_\beta\rho d^2z\right| \leq \iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2\rho d^2z. \tag{3.8}$$

Therefore,

$$K_{\alpha,\beta} \geq -2\iint_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2\rho d^2z.$$

The same reasoning as in the case of holomorphic sectional curvature shows that

$$K_{\alpha,\beta} \geq -2C(r_X)^2.$$

□

If we naively use the approach above to find the lower bounds for the Ricci and scalar curvatures, we will find that the Ricci and scalar curvatures are bounded below by $-2(3g-3)C(r_X)^2$ and $-2(3g-3)^2C(r_X)^2$ respectively. In fact, these bounds are obtained in [4]. However, by doing slightly more work, we can greatly improve the bounds. Observe that given an orthonormal basis $\{\nu_1, \ldots, \nu_{3g-3}\}$ of $\Omega^{-1,1}(\mathbb{D},\Gamma)$, the kernel

$$P(z,w) = \rho(z)\rho(w)\sum_{\beta=1}^{3g-3}\overline{\nu_\beta(z)}\nu_\beta(w) \tag{3.9}$$

is the kernel of the projection operator mapping bounded quadratic differentials to holomorphic quadratic differentials. Namely, for any bounded quadratic differential q of X,

$$(Pq)(z) := \iint_{\Gamma\backslash\mathbb{D}} P(z,w)q(w)\rho(w)^{-1}d^2w$$

is a holomorphic quadratic differential, and $Pq = q$ if and only if q is holomorphic. Let

$$\Lambda = \sup_{z\in\mathbb{D}}\sum_{\beta=1}^{3g-3}|\nu_\beta(z)|^2 = \sup_{z\in\mathbb{D}}\rho(z)^{-2}P(z,z). \tag{3.10}$$

Obviously,

$$\Lambda \leq \sup_{z\in\mathbb{D}}\sup_{w\in\mathbb{D}}\left|\sum_{\beta=1}^{3g-3}\overline{\nu_\beta(z)}\nu_\beta(w)\right|. \tag{3.11}$$

For fixed z, $\sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)}\nu_\beta(w) \in \Omega^{-1,1}(\mathbb{D},\Gamma)$. By (3.9), its Weil–Petersson norm is

$$\left\| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)}\nu_\beta(w) \right\|_{WP}^2 = \rho(z)^{-2} \iint_{\Gamma\backslash\mathbb{D}} P(z,w)\overline{P(z,w)}\rho(w)^{-1} d^2w.$$

Since as a function of w, $\overline{P(z,w)} = P(w,z)$ is a holomorphic quadratic differential, the projection property of P implies that

$$\iint_{\Gamma\backslash\mathbb{D}} P(z,w)\overline{P(z,w)}\rho(w)^{-1} d^2w = \overline{P(z,z)} = P(z,z).$$

Therefore,

$$\left\| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)}\nu_\beta(w) \right\|_{WP}^2 = \rho(z)^{-2}P(z,z).$$

Using Proposition 3.1, this gives

$$\sup_{w\in\mathbb{D}} \left| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)}\nu_\beta(w) \right| \leq C(r_X)\sqrt{\rho(z)^{-2}P(z,z)}.$$

Consequently, (3.11) and (3.10) imply that

$$\Lambda \leq \sup_{z\in\mathbb{D}} \sup_{w\in\mathbb{D}} \left| \sum_{\beta=1}^{3g-3} \overline{\nu_\beta(z)}\nu_\beta(w) \right| \leq C(r_X) \sup_{z\in\mathbb{D}} \sqrt{\rho(z)^{-2}P(z,z)} = C(r_X)\Lambda^{1/2}.$$

In other words,

$$\Lambda = \sup_{z\in\mathbb{D}} \sum_{\beta=1}^{3g-3} |\nu_\beta(z)|^2 \leq C(r_X)^2. \tag{3.12}$$

Notice that we greatly improve the naive bound $\Lambda \leq (3g-3)C(r_X)^2$ to $\Lambda \leq C(r_X)^2$. Now we can prove the following lower bounds for Ricci and scalar curvatures.

Proposition 3.3.

A. *The Ricci curvature of the Weil–Petersson metric is bounded below by* $-2C(r_X)^2$.

B. *The scalar curvature of the Weil–Petersson metric is bounded below by* $-2(3g-3)C(r_X)^2$.

Proof. Using (3.2), (3.3) and (3.8), we have

$$\mathcal{R}_{\alpha\bar{\alpha}} \geq -2 \sum_{\beta=1}^{3g-3} \int_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2 \rho d^2z,$$

$$S \geq -2 \sum_{\alpha=1}^{3g-3}\sum_{\beta=1}^{3g-3} \int_{\Gamma\backslash\mathbb{D}} G(|\nu_\alpha|^2)|\nu_\beta|^2 \rho d^2z.$$

Equation (3.12) and the same method used in the proof of Proposition 3.2 then give us immediately

$$\mathcal{R}_{\alpha\bar{\alpha}} \geq -2C(r_X)^2, \qquad S \geq -2(3g-3)C(r_X)^2.$$

□

Notice that we have established that the Ricci curvature of the Weil–Petersson metric is bounded below by a constant depending only on the injectivity radius of the corresponding Riemann surface, but independent of the genus. This substantially improves the result of [4].

We would also like to remark that tending to the boundary of the moduli spaces, the injectivity radius $r_X = inj(X)$ decreases to zero. The results of Propositions 3.2 and 3.3 and the estimate (3.7) show that when $r_X \to 0$, the holomorphic sectional, sectional and Ricci curvatures of the Weil–Petersson metric are all bounded below by a constant of order $1/r_X^2$. It is interesting to compare this with the asymptotics of the curvatures obtained in [11].

4. Bounds of curvatures on the universal Teichmüller space

In [12], we have shown that the holomorphic sectional and sectional curvatures of the Weil–Petersson metric on the Hilbert manifold $T_H(1)$ are negative. We also showed that $T_H(1)$ is a Kähler–Einstein manifold with constant Ricci curvature $-\frac{13}{12\pi}$. In this section, we show that the holomorphic sectional and sectional curvatures are bounded below by a universal constant. We also show that these curvatures do not have negative upper bounds.

An analog of Proposition 3.1 for the universal Teichmüller space $T_H(1)$ is

Lemma 4.1. *Let $\nu \in \Omega^{-1,1}(\mathbb{D})$ be a harmonic Beltrami differential on $\mathbb{D}$. Then*

$$\|\nu\|_\infty \leq \sqrt{\frac{3}{4\pi}}\|\nu\|_{WP}. \tag{4.1}$$

This can be considered as the limiting case of Proposition 3.1 when $r_X \to \infty$. In fact, in the present situation, the corresponding Riemann surface is isomorphic to the disc which has infinite hyperbolic radius. Another proof of (4.1) is given in the proof of Lemma 2.1 in [12].

Using Lemma 4.1, one obtains immediately as in Proposition 3.2 that

Proposition 4.1. *On the universal Teichmüller space $T_H(1)$, the holomorphic sectional and sectional curvatures are bounded below by $-\frac{3}{2\pi}$.*

To prove the statements about upper bounds, we define for $n \geq 2$,

$$\nu_n = \rho(z)^{-1}\sqrt{\frac{2(n^3-n)}{\pi}}\bar{z}^{n-2}.$$

It is easy to show that $\{\nu_2, \nu_3, \dots\}$ is an orthonormal basis of $H^{-1,1}(\mathbb{D})$. It is elementary to find the sup-norm of ν_n explicitly:

Lemma 4.2. *For $n \geq 2$,*

$$\|\nu_n\|_\infty = \sqrt{\frac{2(n^3-n)}{\pi}}\frac{4}{(n+2)^2}\left(\frac{n-2}{n+2}\right)^{\frac{n-2}{2}}.$$

Proof. For $n \geq 2$, define

$$h_n(r) = (1-r^2)^2 r^{n-2}, \qquad r \in [0,1].$$

Then h_n is a nonnegative function and

$$h_n'(r) = r^{n-3}(1-r^2)((n-2)-(n+2)r^2).$$

This implies that $h_n(r)$ has a maximum at $r = \sqrt{(n-2)/(n+2)}$ and its maximum value is

$$\max_{r\in[0,1]} h_n(r) = \frac{16}{(n+2)^2}\left(\frac{n-2}{n+2}\right)^{\frac{n-2}{2}}.$$

The assertion follows. □

Notice that $\|\nu_2\|_\infty = \sqrt{3/(4\pi)}$. This shows that the result of Lemma 4.1 is sharp. On the other hand, it is easy to see that

$$\|\nu_n\|_\infty \leq \sqrt{\frac{32}{\pi(n+2)}} \to 0 \qquad \text{as } n \to \infty. \tag{4.2}$$

Using this, we can prove that

Proposition 4.2. *On the universal Teichmüller space $T_H(1)$, the holomorphic sectional and sectional curvatures do not have negative upper bounds.*

Proof. For the holomorphic sectional curvature, we obtain as in the proof of Proposition 3.2 that

$$|s_n| = 2\iint_{\mathbb{D}} G(|\nu_n|^2)|\nu_n|^2\rho d^2z \leq 2\|\nu_n\|_\infty^2.$$

On the other hand, the proof of Proposition 3.2 shows that the sectional curvature $K_{m,n}$ (3.4) is bounded by

$$|K_{m,n}| \leq 2\iint_{\mathbb{D}} G(|\nu_m|^2)|\nu_n|^2\rho d^2z \leq 2\|\nu_n\|_\infty^2.$$

Since by (4.2), $\|\nu_n\|_\infty \to 0$ as $n \to \infty$, we conclude that the holomorphic sectional and sectional curvatures do not have negative upper bounds. □

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