ON SEMILOCAL RINGS

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Abstract. In this paper, semilocal rings are characterized in different ways; in particular, it is proved that a ring $R$ is semilocal if and only if every descending chain of principal right ideals of $R$, $a_0R \supseteq a_1R \supseteq a_2R \supseteq \cdots \supseteq a_nR \supseteq \cdots$ with $a_{i+1} = a_i - a_i b_i a_i$ eventually terminates. Then modules with semilocal endomorphism rings are characterized by chain conditions.

1. Introduction

All rings in this paper are associative with identity; modules are unital right $R$-modules.

A finite set $A_1, \cdots, A_n$ of proper submodules of $M$ is said to be coindependent if for each $i$, $1 \leq i \leq n$, $A_i + \bigcap_{j \neq i} A_j = M$, and a family of submodules of $M$ is said to be coindependent if each of its finite subfamilies is coindependent. The module $M$ is said to have finite hollow dimension (or finite dual Goldie dimension) if every coindependent family of submodules of $M$ is finite. It can be shown that, in this case, there is a maximal coindependent family of submodules of $M$. If this set is finite, then its cardinality (denoted by $h.dim(M)$ or $codim(M)$) is uniquely determined and is called the hollow dimension of $M$ (or dual Goldie dimension of $M$). A module $M$ with hollow dimension 1 is said to be hollow, and a cyclic hollow module is said to be local.

A ring $R$ with Jacobson radical $J(R)$ is said to be semilocal if $R/J(R)$ is a semisimple ring. Semilocal rings are characterized as those rings with finite hollow dimension (see [2], Proposition 2.43). For a semilocal ring $R$,

$$h.dim(R) = \text{composition length of the right } R\text{-module } R/J(R).$$

It is well known that semilocal rings have stable range one, and so any modules $M$ with semilocal endomorphism rings can cancel from the direct sum; i.e. $M \oplus B \cong M \oplus C$ implies $B \cong C$. Facchini, Herbera, Levy and Vámos [3] proved that if $M$ has a semilocal endomorphism ring, then $M$ has the “$n$-th root property”; i.e., $M^n \cong N^n$ implies $M \cong N$.

In 1993, Camps and Dicks [1] obtained a number of new characterizations of semilocal rings. Facchini stated two of their many characterizations in [2], Theorem 4.2 as follows.
Theorem 1 (Camps and Dicks). The following conditions are equivalent for a ring $R$.

(a) $R$ is semilocal.

(b) There exists an integer $n \geq 0$ and a function $d : R \to \{0, 1, \cdots, n\}$ such that

(i) for every $a, b \in R$, $d(a - aba) = d(a) + d(1 - ab)$;

(ii) if $a \in R$ and $d(a) = 0$, then $a \in U(R)$.

(c) There exists a partial order $\leq$ on the set $R$ such that

(iii) $(R, \leq)$ is an Artinian partial order set;

(iv) if $a, b \in R$ and $1 - ab \notin U(R)$, then $a - aba < a$.

In this paper, inspired by Camps and Dicks’s Theorem 1, a new notion “hollow length” is introduced (Definition 2). Then some properties of hollow length are presented in Proposition 6, Corollary 7 and Proposition 8. We give different characterizations of semilocal rings in Theorem 9 and Corollary 10. Modules with hollow length are characterized in Theorem 11. Theorem 3 (1), (2) of [4] and Theorem 5 of [1] are extended in Corollaries 12 and 13.

Remark. “Left hollow chain” and “left hollow length” can be defined similarly. For simplicity, “right hollow chain” and “hollow length”, and “r.h.length(a)” is written as “h.length(a)”.

Lemma 3. Let $R$ be a ring, $a, x \in R$. The following conditions are equivalent:

\begin{enumerate}
  \item $aR = (a - axa)R$.
  \item $a \in (1 - ax)R$.
  \item $1 - ax \in r.U(R)$.
  \item $\overline{aR} = (a - axa)\overline{R}$.
\end{enumerate}

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are trivial.

(2) $\Rightarrow$ (3) Set $a = (1 - ax)t$. Then $1 = (1 - ax) + ax = (1 - ax) + (1 - ax)tx = (1 - ax)(1 + tx)$, so $1 - ax \in r.U(R)$.

(3) $\Rightarrow$ (1) Set $v = (1 - ax)v = 1$. Then $v = 1 + axv$, and so $a = [(1 - ax)v]a = (1 - ax)(1 + axv)a = (a - axa)(1 + xva) \in (a - axa)R$.

(4) $\Rightarrow$ (3) $\overline{aR} = (a - axa)\overline{R}$ shows that there exists $t \in R$ such that $a - (a - axa)t \in J(R)$, so $(1 - ax)(1 + ax) = 1 - [a - (a - axa)t]x \in U(R)$, and so $1 - ax \in r.U(R)$. 

2. Main results

Definition 2. Let $R$ be a ring, $a \in R$. A right hollow chain of $a$ is a strictly descending chain

$$a_0R \supseteq a_1R \supseteq a_2R \supseteq a_3R \supseteq \cdots,$$

with $a_0 = a$ and for all $n \geq 0$, $a_{n+1} = a_n - a_nb_na_n$ for some $b_n \in R$.

Set $r = \sup \{n \in \mathbb{Z} | a_0R \supseteq a_1R \supseteq a_2R \supseteq \cdots \supseteq a_{n-1}R \supseteq a_nR$ is a hollow chain of $a_0 = a \}$. $r$ is called the right hollow length of $a$, denoted as $r.h.length(a) = r$.

Remark. “Left hollow chain” and “left hollow length” can be defined similarly. For simplicity, “right hollow chain” and “hollow length” are called in this paper respectively “hollow chain” and “hollow length”, and “r.h.length(a)” is written as “h.length(a)”.

Theorem 11. Let $R$ be a ring, $a, x \in R$. The following conditions are equivalent:

\begin{enumerate}
  \item $aR = (a - axa)R$.
  \item $a \in (1 - ax)R$.
  \item $1 - ax \in r.U(R)$.
  \item $\overline{aR} = (a - axa)\overline{R}$.
\end{enumerate}

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are trivial.

(2) $\Rightarrow$ (3) Set $a = (1 - ax)t$. Then $1 = (1 - ax) + ax = (1 - ax) + (1 - ax)tx = (1 - ax)(1 + tx)$, so $1 - ax \in r.U(R)$.

(3) $\Rightarrow$ (1) Set $v = (1 - ax)v = 1$. Then $v = 1 + axv$, and so $a = [(1 - ax)v]a = (1 - ax)(1 + axv)a = (a - axa)(1 + xva) \in (a - axa)R$.

(4) $\Rightarrow$ (3) $\overline{aR} = (a - axa)\overline{R}$ shows that there exists $t \in R$ such that $a - (a - axa)t \in J(R)$, so $(1 - ax)(1 + ax) = 1 - [a - (a - axa)t]x \in U(R)$, and so $1 - ax \in r.U(R)$. 


Remark 4. Let $M$ be a module and $\text{End}(M)$ its endomorphism ring. For all $f, g \in \text{End}(M)$, $(f - fgf)(M) = f(M)$ implies that $1 - fg$ is surjective. In fact, $f(M) = (f - fgf)(M)$ shows $f(M) = (1 - fg)f(M) \subseteq (1 - fg)(M)$. Thus $\forall x \in M$, $fg(x) = (1 - fg)(y)$ for some $y \in M$, so $x = (1 - fg)(x) + fg(x) = (1 - fg)(x) + (1 - fg)(y) = (1 - fg)(x + y) \subseteq (1 - fg)(M)$; i.e., $1 - fg$ is surjective.

Now we present some properties of hollow length.

**Lemma 5.** Let $R$ be a ring, $a \in R$.

1. $h.length(a) = 0$ if and only if $a \in J(R)$.
2. If $h.length(a) = 1$, then $\overline{aR}$ is a simple $\overline{R}$-module.

**Proof.** (1) $h.length(a) = 0$ if and only if for all $x \in R, aR = (a - axa)R$. Lemma 3 shows that this means $1 - ax \in r.U(R)$ for all $x \in R$, i.e., $a \in J(R)$.

(2) Suppose that $h.length(a) = 1$. We need to prove that for all $x \in R$, if $0 \neq \overline{axR}$, then $\overline{axR} = \overline{aR}$. In fact, $0 \neq \overline{axR}$ gives $ax \notin J(R)$, so there exists $y \in R$ such that $1 - axy \notin r.U(R)$. Lemma 3 shows $aR \supseteq (a - axy)a$. Since $h.length(a) = 1$, we have $h.length(a - axy) = 0$, so $a - axy \in J(R)$, i.e., $\overline{a - axy} = \overline{a} \in \overline{R}$. Thus $\overline{axR} = \overline{aR}$, as desired. \qed

Remark. Essentially, the proof of Lemma 5(2) has its origin in Camps and Dicks’s proof of Theorem 1.

We can characterize hollow length as follows.

**Proposition 6.** Let $R$ be a ring, $a \in R$ and let $n \geq 0$ be an integer. The following conditions are equivalent:

1. $h.length(a) = n$.
2. $\overline{aR}$ is a semisimple $\overline{R}$-module with $\dim(\overline{aR}) = n$.

**Proof.** For $n = 0$, this is Lemma 5(1). We now prove the equivalence for $n \geq 1$.

(1) $\Rightarrow$ (2) We use induction on $n$. The case $n = 1$ is Lemma 5(2). Let $n > 1$ and suppose that our implication holds for all $x \in R$ with $h.length(x) \leq n - 1$. Let $aR \supseteq a_1R \supseteq \cdots \supseteq a_{n-1}R \supseteq a_nR$ be a hollow chain of $a$, where $a_1 = a - aba$ and $a_{i+1} = a_i - a_ib_ia$ for all $i \geq 1$. Then $h.length(a_i) = n - i$ for all $i \in \{0, 1, \cdots, n\}$.

Since $h.length(a_n) = 0$, Lemma 5(1) gives $a_n \in J(R)$. Therefore $\overline{a_n} = \overline{a_{n-1}} - \overline{a_{n-1}b_{n-1}a_{n-1}} = 0$, i.e., $\overline{a_{n-1}} - \overline{a_{n-1}b_{n-1}a_{n-1}} = 0$. Thus $\overline{a_{n-1}b_{n-1}}$ is idempotent and $\overline{R} = (1 - \overline{a_{n-1}b_{n-1}})\overline{aR} \oplus \overline{a_{n-1}b_{n-1}aR}$, and so $\overline{aR} = (1 - \overline{a_{n-1}b_{n-1}})\overline{aR} \oplus \overline{a_{n-1}b_{n-1}aR}$. Noting that $a_{n-1} \in \overline{aR}$, we have $\overline{a_{n-1}R} = \overline{a_{n-1}b_{n-1}aR}$, so $\overline{a_{n-1}R} = (1 - \overline{a_{n-1}b_{n-1}})\overline{aR} \oplus \overline{a_{n-1}b_{n-1}aR}$. Therefore we have

$$\overline{aR} = (1 - \overline{a_{n-1}b_{n-1}})\overline{aR} \oplus \overline{a_{n-1}b_{n-1}aR} \oplus \overline{a_{n-1}b_{n-1}aR}$$

Note that $h.length(a_{n-1}) = 1$. Lemma 5(2) shows that $\overline{a_{n-1}b_{n-1}aR}$ is a simple $\overline{R}$-module.

Since $a_{n-1} \supseteq a_nR = (a_{n-1} - a_{n-1}b_{n-1})a_{n-1}R$, we know from Lemma 3 that $(1 - a_{n-1}b_{n-1}) \notin r.U(R)$, and so Lemma 3 gives $aR \supseteq (a - a_{n-1}b_{n-1})aR$. Thus we have $h.length(a - a_{n-1}b_{n-1}a) < h.length(a) = n$. By induction we know $(a - a_{n-1}b_{n-1}a)\overline{R}$ is a semisimple module with $\dim((a - a_{n-1}b_{n-1}a)\overline{R}) \leq n - 1$. Therefore equation $(*)$ yields $\overline{aR}$ is a semisimple module with $\dim(\overline{aR}) \leq n$.

Now we prove $\dim(\overline{(aR)}) \geq n$. Noting that $\overline{aR} \subseteq \overline{aR}$ and $\overline{aR}$ is semisimple, we have $\overline{aR} = \overline{aR} \oplus D$ for some $D \subseteq \overline{aR}$. Since $h.length(a_i) = n - i$, by induction, $\overline{a_iR}$ is a semisimple module with $\dim(\overline{a_iR}) = n - i$. If we can prove $D \neq 0$, then
we have $\dim(\overline{aR}) = \dim(\overline{aR}) + \dim(D) \geq n$. In fact, if $D = 0$, then $\overline{aR} = \overline{aR}$. Lemma 3 gives $aR = a_1R$, a contradiction.

(2) ⇒ (1) Suppose that $\overline{aR}$ is a semisimple module with $\dim(\overline{aR}) = n$.

First assume that $\text{h.length}(a) = \infty$. Then there is a finite descending chain

$$a_0R \supseteq a_1R \supseteq a_2R \supseteq \cdots \supseteq a_mR$$

with $a_0 = a$, $a_{i+1} = a_i - a_i b_i a_i$ ($0 \leq i \leq m - 1$) and $m > n$. By Lemma 3, there is a finite descending chain

$$\overline{aR} \supseteq \overline{a_1R} \supseteq \overline{a_2R} \supseteq \cdots \supseteq \overline{a_mR}.$$

Thus $\dim(\overline{aR}) \geq m > n$, a contradiction.

Suppose $\text{h.length}(a) = m < \infty$. Then by (i) ⇒ (ii), we know that $\overline{aR}$ is a semisimple module with $\dim(\overline{aR}) = m$. Therefore $m = n$. $\square$

**Corollary 7.** Let $R$ be a ring. If $a, b \in R$ and $aR \subseteq bR$, then $\text{h.length}(a) \leq \text{h.length}(b)$. In particular, for all $a \in R$, $\text{h.length}(a) \leq \text{h.length}(1_R)$.

**Proof.** Suppose that $\text{h.length}(b) = n < \infty$. By Proposition 6, $\overline{aR}$ is a semisimple $\overline{R}$-module with $\dim(\overline{aR}) = n$, so $\overline{aR}$ is a semisimple $\overline{R}$-module with $\dim(\overline{aR}) \leq n$. Therefore $\text{h.length}(a) = \dim(\overline{aR}) \leq n = \text{h.length}(b)$. $\square$

**Proposition 8.** Let $R$ be a semilocal ring, $a, b \in R$. Then

1. $\text{h.length}(a) = \dim(\overline{R}) - \dim(\overline{aR})$.
2. $\text{h.length}(a) + \text{h.length}(1 - ab) - \text{h.length}(a - aba) = \text{h.dim}(R)$.
3. $\text{h.length}(a) = \text{h.length}(a - aba) + 1$ if and only if $aR/(a - aba)R$ is local.

**Proof.** (1) Since $(aR + J(R))/aR$ is small in $R/aR$, applying [2], Proposition 2.42 (d) to the exact sequence

$$(aR + J(R))/aR \hookrightarrow R/aR \rightarrow (R/aR)/(aR + J(R))/aR \cong R/(aR + J(R)),$$

we have

$$\dim(R/aR) = \dim(R/(aR + J(R))) = \dim(\overline{R}/\overline{aR}) = \dim(\overline{R}) - \dim(\overline{aR}) = \dim(R) - \dim(\overline{aR}).$$

By Proposition 6, we get $\dim(R/aR) = \dim(R) - \text{h.length}(a)$.

(2) This result comes directly from (1) and the fact that $\dim(\overline{R}/(a - aba)R) = \dim(R/aR \oplus R/(1 - ab)R) = \dim(R/aR) + \dim(R/(1 - ab)R)$.

(3) By (2), $\text{h.length}(a) = \text{h.length}(a - aba) + 1$ if and only if $\text{h.length}(1 - ab) = \dim(R) - 1$, i.e., $\dim(R) - \dim(R/(1 - ab)R) = \dim(R) - 1$; this means that $\dim(R/(1 - ab)R) = 1$. By [2], Proposition 2.42 (b), this means that $R/(1 - ab)R$ is hollow. Since $R/(1 - ab)R$ is cyclic, $[5, 1.1.5 (b)]$ shows that $R/(1 - ab)R = (aR + (1 - ab)R)/(1 - ab)R \cong aR/(a - aba)R$ is local. $\square$

We now give our characterizations of semilocal rings.

**Theorem 9.** The following conditions are equivalent for a ring $R$ and an integer $n \geq 0$:

1. $R$ is a semilocal ring and $\dim(R/J(R)) = n$.
2. $\text{h.length}(1_R) = n$.
3. There exists a descending chain of principal right ideals of $R$

$$R = a_0R \supseteq a_1R \supseteq a_2R \supseteq \cdots \supseteq a_nR$$

with $a_{i+1} = a_i - a_i b_i a_i$.
such that

(i) \( a_n \in J(R) \);

(ii) \( a_i R/a_{i+1} R \) is local for all \( 0 \leq i \leq n - 1 \).

(4) There exist \( d, d_1, \ldots, d_n \in R \) with \( d R = \bigcap_{i=1}^{n} d_i R \subseteq J(R) \) such that \( R/d_i R \) is local for all \( i \in \{1, \ldots, n\} \) and \( R/d R \cong \bigoplus_{i=1}^{n} R/d_i R \).

(5) There exists \( d \in J(R) \) such that \( R/d R \cong \bigoplus_{i=1}^{n} A_i \), where \( \{A_i\}_{i \in \{1, \ldots, n\}} \) are local modules.

\((2^*), (3^*), (4^*), (5^*)\). The left-right duals of (2), (3), (4) and (5).

Proof. (1) \( \iff \) (2) comes directly from Proposition 6 by setting \( a = 1_R \).

(2) \( \Rightarrow \) (3) Let \( R = a_0 R \supseteq a_1 R \supseteq a_2 R \supseteq \cdots \supseteq a_n R \) with \( a_{i+1} = a_i - a_ib_i a_i \) a hollow chain of \( a_0 = 1_R \). Then (i) and (ii) come respectively from Lemma 5(1) and Proposition 8(3).

(3) \( \Rightarrow \) (4) Assume that (3) holds. Let \( d = a_n, d_1 = a_1 \) and \( d_{i+1} = (1 - a_i b_i) \) for all \( i \in \{1, \ldots, n - 1\} \). Then \( d R = \bigcap_{i=1}^{n} d_i R \subseteq J(R) \) and \( R/d_1 R \) is local. Obviously, for all \( i \in \{2, \ldots, n\} \), \( R/d_i R \cong a_{i-1} R/a_i R \) is also local. Moreover,

\[
R/d R = R/a_0 R = R/(a_{n-1} - a_{n-1} b_{n-1} a_{n-1}) R \cong R/a_{n-1} R \oplus R/(1 - a_{n-1} b_{n-1}) R
\]

\[
= R/(a_{n-2} - a_{n-2} b_{n-2} a_{n-2}) R \oplus R/(1 - a_{n-2} b_{n-2}) R \cong \cdots
\]

\[
= R/a_1 R \oplus R/(1 - a_1 b_1) R \oplus R/(1 - a_{n-2} b_{n-2}) R \oplus R/(1 - a_{n-1} b_{n-1}) R
\]

\[
= R/d_1 R \oplus R/d_2 R \oplus \cdots \oplus R/d_{n-1} R \oplus R/d_n R.
\]

(4) \( \Rightarrow \) (5) is trivial.

(5) \( \Rightarrow \) (1) \( d \in J(R) \) shows that \( d R \) is small in \( R \), and so

\[
h.dim(R) = h.dim(R/d R) = \sum_{i=1}^{n} h.dim(A_i)
\]

\[
= \sum_{i=1}^{n} 1 = n.
\]

\( \square \)

Corollary 10. The following conditions are equivalent for a ring \( R \):

(1) \( R \) is a semilocal ring.

(2) Every right hollow chain

\[
a_0 R \supseteq a_1 R \supseteq a_2 R \supseteq \cdots \supseteq a_n R \supseteq \cdots \text{ with } a_{i+1} = a_i - a_i b_i a_i
\]

eventually terminates.

(3) There exists a partial order \( \leq \) on the set \( R \) such that

(i) \( (R, \leq) \) is an Artinian partial order set;

(ii) if \( a, b \in R \) and \( 1 - ab \notin R.U(R) \), then \( a - aba < a \).

(4) Every descending chain of principal right ideals of \( R \)

\[
a_0 R \supseteq a_1 R \supseteq a_2 R \supseteq \cdots \supseteq a_n R \supseteq \cdots \text{ with } a_{i+1} = a_i - a_i b_i a_i
\]

eventually terminates.

\((2^*), (3^*), (4^*)\). The left-right duals of (2), (3) and (4).
Proof. (4) ⇒ (2) is trivial.

(1) ⇒ (2) comes from Theorem 9(2) and Corollary 7.

(2) ⇒ (3) If (2) holds, define an order ≤ on R via

\[ b ≤ a \text{ if and only if } aR ≥ bR \text{ and } b = a - ada \text{ for some } d ∈ R. \]

If we can prove that ≤ is a partial order, then (3) is easily verified. We need only to prove the transitivity of “≤”. In fact, suppose that \( a_3 < a_2 \) and \( a_2 < a_1 \).

Set \( a_{i+1} = a_i - a_ib_ia_i \) (i = 1, 2), where \( 1 - ab_i ∉ r.U(R) \). Write \( d = b_1 + (1 - b_1a_1)b_2(1 - b_1b_1) \). Then

\[
a_3 = a_2 - a_2b_2a_2 = (a_1 - a_1b_1a_1) - (1 - a_1b_1a_1)b_2(1 - a_1b_1a_1) = a_1 - a_1[1 + (1 - b_1a_1)b_2(1 - a_1b_1)]a_1 = a_1 - a_1a_1. \]

Noting that \( 1 - a_1d = 1 - a_1b_1 + (1 - b_1a_1)b_2(1 - a_1b_1) = (1 - a_1b_1)(1 - a_2b_1(1 - a_1b_1)) \) and \( 1 - a_1b_1 ∉ r.U(R) \), we get \( 1 - a_1d ∉ r.U(R) \), and so \( a_3 < a_1 \), as desired.

(3) ⇒ (1) Note that \( ∀a ∈ R, a ∈ J(R) \) if and only if \( 1 - ab ∉ r.U(R) \) for all \( b ∈ R \). Then using the same proof of Camps and Dicks’ Theorem 1 (see (c) ⇒ (a) of [2] Theorem 4.2) or (f) ⇒ (a) of [1] Theorem 1, we can prove (1).

(1), (2) ⇒ (4) We prove first the following result.

Claim. For all descending chains

\[ a_1R ≥ a_2R ≥ a_3R ≥ \cdots ≥ a_nR ≥ \cdots \] with \( a_{n+1} = a_n - a_nb_na_n \).

If \( a_1R = a_2R \), then we have a descending chain

\[ a_1R ≥ a_2R ≥ \cdots ≥ a_nR ≥ \cdots \] with \( a_n' = a_1 - a_1da_1, a_{n+1}' = a_n' - a_n'a_n \);

for some \( d, n \in R \) (\( n ≥ 3 \)), and \( a_n'R = a_{n+1}R \) if and only if \( a_nR = a_{n+1}R \).

If \( a_1R = a_2R \) with \( a_2 = a_1 - a_1b_1a_1 \) implies that \( 1 - a_1b_1 ∈ r.U(R) = U(R) \), i.e., \( 1 - b_1a_1 ∈ U(R) \).

For \( n ≥ 3 \), set \( a_n' = a_n(1 - b_1a_1)^{-1} \). Then \( a_n'R = a_nR \), so \( a_n'R = a_{n+1}'R \) if and only if \( a_nR = a_{n+1}R \). In addition, \( a_{n+1}' = (a_n - a_nb_na_n)(1 - b_1a_1)^{-1} = a_n' - a_n'(1 - b_1a_1)b_1a_n \).

\[ a_3 = a_3(1 - b_1a_1)^{-1} = (a_2 - a_2b_2a_2)(1 - b_1a_1)^{-1} = [a_1 - a_1(1 - b_1a_1)b_2a_1], \]

and \( a_3R = a_1R \) means that \( a_3(1 - b_1a_1)^{-1}R = a_1R \), that is, \( a_3R = a_1R = a_2R \).

Assume that there exists a descending chain

\[ a_0R ≥ a_1R ≥ a_2R ≥ \cdots ≥ a_nR ≥ \cdots \] with \( a_{i+1} = a_i - a_ib_ia_i \)

which never terminates. Then, by the claim, we can obtain an infinite hollow chain

\[ d_0R ≥ d_1R ≥ d_2R ≥ \cdots ≥ d_nR ≥ \cdots \] with \( d_{i+1} = d_i - d_ic_id_i \),

a contradiction. \( \square \)

As an application of Corollary 10, we can characterize modules with a semilocal endomorphism ring as follows.

**Theorem 11.** The following conditions are equivalent for a module M:

1. The ring \( \text{End}(M) \) is semilocal.
2. For every \( f_0 ∈ \text{End}(M) \) and every sequence \( g_0, g_1, g_2, \cdots \) of elements of \( \text{End}(M) \), if we set \( f_{n+1} = f_0 - f_n g_n f_n \) for every \( n ≥ 0 \), then the chains
   
   (i) \( f_0(M) ≥ f_1(M) ≥ \cdots ≥ f_n(M) ≥ \cdots \)
   
   (ii) \( \ker f_0 ≤ \ker f_1 ≤ \cdots ≤ \ker f_n ≤ \cdots \)
of submodules of $M$ both eventually terminate.

**Proof.** (1) $\Rightarrow$ (2) Set $E = \text{End}(M)$. Corollary 10 shows that

$$f_0E \supseteq f_1E \supseteq f_2E \supseteq \cdots$$

of $E$ eventually terminates, so there exists $m \in \mathbb{N}$ such that $\forall n > m$, $f_{n+1}E = (f_n - f_ng_n)E$, i.e., $1 - f_ng_n \in r.U(E) = U(E)$. Thus $1 - g_nf_n \in U(E)$. So $\forall n > m$, $f_{n+1}(M) = (f_n - f_ng_n)(M) = f_n(1 - g_nf_n)(M) = f_n(M)$, and $\ker f_{n+1} = \ker f_n \oplus \ker(1 - g_nf_n) = \ker f_n \oplus 0 = \ker f_n$. Therefore the two chains (i) and (ii) eventually terminate.

(2) $\Rightarrow$ (1) Suppose that $\forall n > m$, $\ker f_{n+1} = \ker f_n \oplus \ker(1 - g_nf_n) = \ker f_n$ and $f_{n+1}(M) = (f_n - f_ng_n)(M) = f_n(M)$.

Then $\ker(1 - g_nf_n) = 0$. By [2] Lemma 4.1 (a), $\ker(1 - f_n) = 0$. So $1 - f_ng_n$ is injective.

By Remark 4, $f_{n+1}(M) = (f_n - f_ng_n) = f_n(M)$ implies that $1 - f_ng_n$ is surjective.

Therefore $\forall n > m$, $1 - f_ng_n$ is bijective. Lemma 3 shows that the descending chain of $E$

$$E = f_0E \supseteq f_1E \supseteq f_2E \supseteq \cdots$$

with $f_{n+1} = f_n - f_ng_n$ eventually terminates. Corollary 10 yields that $\text{End}(M)$ is semilocal.

We extend Theorem 3 (2) of [4] as follows.

**Corollary 12.** The following conditions are equivalent for a module $M$ for which every epimorphism $M \to M$ splits:

(i) The ring $\text{End}(M)$ is semilocal.

(ii) For every $f_0 \in \text{End}(M)$ and every sequence $g_0, g_1, g_2, \cdots$ of elements of $\text{End}(M)$, if we set $f_{n+1} = f_n - f_ng_nf_n$ for every $n \geq 0$, then the chain $f_0(M) \supseteq f_1(M) \supseteq \cdots \supseteq f_n(M) \supseteq \cdots$ of submodules of $M$ eventually terminates.

Moreover, $\text{h.dim(End(M))} \leq \text{h.dim}(M)$.

**Proof.** By Theorem 11, we need only to prove (ii) $\Rightarrow$ (i). Set $E = \text{End}(M)$. Given a descending chain $f_0E \supseteq f_1E \supseteq \cdots \supseteq f_nE \supseteq \cdots$ with $f_{n+1} = f_n - f_ng_nf_n$, by (ii), there exists $m \in \mathbb{N}$ such that $\forall n > m$, $f_{n+1}(M) = (f_n - f_ng_nf_n)(M) = f_n(M)$. By Remark 4, $1 - f_ng_n$ is surjective, so $1 - f_ng_n \in r.U(E)$. Lemma 3 and Corollary 10 yield that $\text{End}(M)$ is semilocal.

Assume that $\text{h.dim}(M) = m < \infty$ and $\text{h.dim(End(M))} > m$. Then by Theorem 9 there exists a hollow chain in $E$,

$$E = f_0E \supseteq f_1E \supseteq \cdots \supseteq f_mE \supseteq f_{m+1}E,$$

with $f_{i+1} = f_i - f_ig_i, f_i \in E$ for all $i \in \{0, 1, \cdots, m\}$. Then for all $i \in \{0, 1, \cdots, m\}$, $1 - f_ig_i \notin r.U(E)$, and so $1 - f_ig_i$ is not surjective. By Remark 4, $f_0(M) \supseteq f_1(M) \supseteq \cdots \supseteq f_n(M) \supseteq f_{n+1}(M)$ is a strictly descending chain. Write $N_i = f_i(M) \subseteq M$ and $N_{i+1} = (1_M - f_ig_i)(M) \subseteq M$ for all $i \in \{1, \cdots, m\}$. Then for all $i \geq 1$, $\bigcap N_i = f_i(M) + (1_M - f_ig_i)(M) = M$, and so $\{N_i|i = 1, \cdots, m + 1\}$ is a coindependent set of proper submodules of $M$. Thus $\text{h.dim}(M) > m$, a contradiction.

Similarly, we extend Theorem 5 of [1] and Theorem 3 (1) of [4] as follows.
Corollary 13. The following conditions are equivalent for a module $M$ for which every monomorphism $M \to M$ splits:

(i) The ring $\text{End}(M)$ is semilocal.

(ii) For every $f_0 \in \text{End}(M)$ and every sequence $g_0, g_1, g_2, \cdots$ of elements of $\text{End}(M)$, if we set $f_{n+1} = f_n - f_ng_nf_n$ for every $n \geq 0$, then the chain $\ker f_0 \subseteq \ker f_1 \subseteq \cdots \subseteq \ker f_n \subseteq \cdots$ of submodules of $M$ eventually terminates.

Moreover, $\text{h.dim}(\text{End}(M)) \leq \dim(M)$.

Proof. Set $E = \text{End}(M)$. We need only to prove (ii) $\Rightarrow$ (i). For this, it is enough to show that every descending chain $Ef_0 \supseteq Ef_1 \supseteq \cdots \supseteq Ef_n \supseteq \cdots$ of left ideals of $E$ with $f_{n+1} = f_n - f_ng_nf_n$ eventually terminates. Given such a chain, by (ii) there exists $m \in \mathbb{N}$ such that $\forall n > m$, $\ker f_{n+1} = \ker(f_n - f_ng_nf_n) = \ker f_n$, i.e., $\ker f_n \oplus \ker(1-g_nf_n) = \ker f_n$, so $\ker(1-g_nf_n) = 0$, i.e., $1-g_nf_n$ is injective, and so $1-g_nf_n \in l.U(E)$ for all $n > m$. It follows that $Ef_0 \subseteq Ef_1 \subseteq \cdots \subseteq Ef_n \subseteq \cdots$ eventually terminates, so $\text{End}(M)$ is semilocal. $\text{h.dim}(\text{End}(M)) \leq \dim(M)$ comes from Theorem 9(2) and the fact that $\forall i \geq 1$, $\ker f_{i+1} = \ker(f_i - f_ig_if_i) = \ker f_i \oplus \ker(1-g_if_i)$. \hfill \Box

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References


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