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ON FIELDS OF DEFINITION OF ARITHMETIC KLEINIAN REFLECTION GROUPS

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Abstract. We show that the degrees of the real fields of definition of arithmetic Kleinian reflection groups are bounded by 35.

1. Introduction

In two recent articles [7], [8], Nikulin gave explicit upper bounds for the degrees of the fields of definition of arithmetic hyperbolic reflection groups in dimensions $n \geq 4$ and $n = 2$. He pointed out that only the case $n = 3$ remained open. Later, in [9], Nikulin extended his method to this case as well and thus completed in general the solution of the problem. Still the explicit bounds obtained in [7, 8, 9] are far from being sharp.

In this paper we explore an alternative approach to the problem. As a result we show that the fields of definition of arithmetic Kleinian reflection groups have degrees less than 70. This implies that real fields of definition of corresponding orthogonal groups have degrees bounded by 35 (compare with the 10000-bound in [9], which was improved to 909 in the latest version of the preprint). We also give general bounds for discriminants of the fields of definition and good upper bounds for the discriminants which correspond to non-cocompact Kleinian groups.

A theorem of Nikulin [7, Th. 4.8] states that the degrees of the real fields of definition in all dimensions are bounded by a maximum of 56, and the degrees in dimensions 2 and 3. In dimension 2, following the previous work [5], Nikulin gave an upper bound of 44 for the degrees (see [7, Sec. 4.5]). Combining this with the result of the present paper, we get a universal upper bound of 56 for all dimensions, which, in particular, improves on the bounds for the dimensions 4 and 5 obtained by Nikulin in [8].

Our method is based on the work of Agol [1] combined with Borel's volume formula [2] and some number-theoretic results of Chinburg and Friedman [3, 4].

2. Preliminaries

Discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ are called *Kleinian groups*. As $\mathrm{PSL}(2, \mathbb{C})$ is isomorphic to the group of orientation preserving isometries of the hyperbolic 3-space $\mathbb{H}^3$, Kleinian groups act isometrically on $\mathbb{H}^3$. If a Kleinian group Γ acts as

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an orientation preserving subgroup of a discrete group generated by reflections in hyperbolic hyperplanes, it is called a *reflection group*. In this case the volume of the hyperbolic polyhedron $\mathcal{P}$ bounded by the reflection hyperplanes is half of the covolume of Γ. If Γ is an arithmetic subgroup of $\mathrm{PSL}(2,\mathbb{C})$ (we refer to [6, Sec. 8] for the definition and basic properties of arithmetic Kleinian groups), its covolume can be estimated using its arithmetic invariants. A volume formula of Borel [2] allows us to write down the estimate in an explicit form. On the other hand, the covolume of Γ can be estimated from the geometry of the polyhedron $\mathcal{P}$. One of the main ideas of [1] is that the interplay between these two estimates leads to the finiteness result for the number of conjugacy classes of arithmetic Kleinian reflection groups. Our purpose is to make this relation explicit and then apply it to get quantitative bounds for the finiteness theorem.

3. Results

Theorem 1. *Let k be a field of definition of an arithmetic Kleinian reflection group. Then its degree $n_k \leq 70$ and the absolute value of the discriminant $\mathcal{D}_k < 4.4\times 10^{273}$. Moreover, if the group is non-cocompact, then $n_k = 2$ and $\mathcal{D}_k \leq 9240$.*

Remark 1. It is well-known due to Vinberg that arithmetic reflection groups are defined by quadratic forms [10, Lemma 7]. Corresponding forms are defined over totally real fields, and the fields of definition of arithmetic Kleinian reflection groups are quadratic extensions of these fields (see [6, Sec. 10.2]). Therefore, the degrees of the fields of definition of arithmetic Kleinian reflection groups are necessarily even, and the corresponding real fields of definition of orthogonal groups have degrees half that. Theorem 1 implies that the degrees of the real fields of definition are bounded by $70/2 = 35$.

Remark 2. The non-cocompact arithmetic Kleinian groups are also known as Bianchi groups and have a long history in the mathematical literature. While our general bound for discriminants of the fields of definition is more an existence bound, the bound for the Bianchi groups is much better (compare with 1.02×10^8-bound in [1, Sec. 7]). Moreover, in the proof of the theorem we obtained a list of 330 imaginary quadratic fields which contains all possible candidates for the fields of definition of Bianchi reflection groups. The key ingredients for this enumeration are Gauss' theorem on the 2-class number of a quadratic field and Vinberg's result on class groups of fields of definition of Bianchi reflection groups (see part 3 of the proof for details). No proper analogue of any of these two results is currently known in a more general setting.

4. Proof of Theorem 1

4.1. Let Γ be a maximal arithmetic Kleinian reflection group and let $\mu(\Gamma)$ denote its covolume. By the proof of Theorem 6.1 in [1], we have

$$\mu(\Gamma) \leq 2\times 64\pi^2. \tag{4.1}$$

We now recall a non-trivial corollary of Borel's volume formula which was obtained in [4, Lemma 4.3]. It implies

$$\mu(\Gamma) > 0.69\exp\left(0.37 n_k - \frac{19.08}{h(k,2,B)}\right), \tag{4.2}$$

where n_k is the degree of the field of definition of Γ and $h(k,2,B)$ is the order of a certain subgroup of the 2-class group of k (see [4, Sec. 2] for a precise definition of $h(k,2,B)$).

From (4.1) and (4.2) we obtain

$$\begin{aligned} n_k &< \frac{\log(128\pi^2/0.69) + 19.08/h(k,2,B)}{0.37} \\ &\le \frac{\log(128\pi^2/0.69) + 19.08}{0.37} \\ &< 71.88. \end{aligned}$$

It was already pointed out in Remark 1 that the degree of k is necessarily an even integer, which implies $n_k \le 70$. As any arithmetic reflection group is contained in some maximal arithmetic reflection group it follows that the same bound holds for the fields of definition of arbitrary arithmetic Kleinian reflection groups.

4.2. Using the Brauer-Siegel theorem and Zimmert's bound for the regulator, we can show that the class number

$$h_k \le 10^2 \left(\frac{\pi}{12}\right)^{n_k} \mathcal{D}_k \tag{4.3}$$

(see [4, Sec. 3] for more details).

The bound (4.3) together with the upper bound for n_k gives

$$h(k,2,B) \le h_k \le c_1 \mathcal{D}_k, \quad c_1 = 10^2 \left(\frac{\pi}{12}\right)^{70}.$$

Now the volume formula [2] implies (here again we use the notation from [4]):

$$\begin{aligned} \mu(\Gamma) &\ge \frac{\mathcal{D}_k^{3/2}\zeta_k(2)2^{t-t'}}{2^{2(n_k-2)+4+t-1}\pi^{2(n_k-2)+2}[K(B):k]} \prod_{\substack{v\in R_f \\ 2\ne Nv}} \frac{(Nv-1)}{2} \\ &\ge \frac{\mathcal{D}_k^{3/2}}{2^{t'}2^{2n_k-1}\pi^{2n_k-2}2^{2(n_k-2)}h(k,2,B)} \\ &\ge \frac{\mathcal{D}_k^{1/2}}{c_2 c_1}, \quad c_2 = 2^{70}2^{4\times 70-5}\pi^{2\times 70-2}. \end{aligned} \tag{4.4}$$

Combined with (4.1) this gives

$$\mathcal{D}_k \le (128\pi^2 c_1 c_2)^2 < 4.4 \times 10^{273}.$$

4.3. Now let Γ be a non-cocompact arithmetic Kleinian group. It follows that k is necessarily an imaginary quadratic field and the set of places of k at which the quaternion division algebra associated to Γ is ramified is empty (see [6, Th. 8.2.3]). In the notation of [4] the latter implies $t' = 0$. Thus the volume estimate (4.4) applied to this case gives

$$\mu(\Gamma) \ge \frac{\mathcal{D}_k^{3/2}\zeta_k(2)}{2^3\pi^2 h(k,2,B)}. \tag{4.5}$$

We have $h(k,2,B) \le h_2(k)$, where $h_2(k)$ is the 2-class number of k. By Gauss' theorem, $h_2(k) \le 2^{t_k-1}$, where t_k is the number of distinct prime divisors of the discriminant $\mathcal{D}_k$. Well-known number-theoretic estimates imply $t_k \le 1.5\log\mathcal{D}_k/\log\log\mathcal{D}_k$.

Therefore,

$$\mu(\Gamma) \geq \frac{\mathcal{D}_k^{3/2}\zeta_k(2)}{2^{1.5\log\mathcal{D}_k/\log\log\mathcal{D}_k - 1}8\pi^2}$$
$$\geq \frac{\mathcal{D}_k^{3/2}}{2^{1.5\log\mathcal{D}_k/\log\log\mathcal{D}_k}16\pi^2}.$$

If $\mathcal{D}_k \geq 10^5$, then $\mu(\Gamma) > 1492.9$ and inequality (4.1) fails. Considering the remaining fields, a simple program for the GP PARI calculator allows us to compute the lower bound (4.5) for $\mu(\Gamma)$ using precise values of $h_2(k)$ and $\zeta_k(2)$. This way we obtain that there are in total 882 fields which satisfy the criteria and that the largest $\mathcal{D}_k = 9240$. The list of the admissible fields can be further improved using a theorem of Vinberg [11], which implies that the class numbers of the fields under consideration should be powers of 2. As a result we obtain that there are only 330 such fields.

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