UNIFORMLY CONVEX FUNCTIONS ON BANACH SPACES

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ABSTRACT. Given a Banach space $(X, \|\cdot\|)$, we study the connection between uniformly convex functions $f : X \to \mathbb{R}$ bounded above by $\|\cdot\|^p$ and the existence of norms on X with moduli of convexity of power type. In particular, we show that there exists a uniformly convex function $f : X \to \mathbb{R}$ bounded above by $\|\cdot\|^2$ if and only if X admits an equivalent norm with modulus of convexity of power type 2.

1. INTRODUCTION

Uniformly convex functions on Banach spaces were introduced by Levitin and Poljak in [10]. Their properties were studied in depth by Zălinescu [13], and then later Azé and Penot [1] studied their duality with uniformly smooth convex functions. The monograph [14] provides a systematic development of these topics. Additionally, related properties of convex functions and their applications have been studied in papers such as [2, 3, 4, 5]. In particular, [3] examines various properties of $\|\cdot\|^r$ when $\|\cdot\|$ is a uniformly convex norm. In this note, we will present a related result that determines when functions of the form $f = \|\cdot\|^r$ are uniformly convex. We also examine a more general converse problem: if $f : X \to \mathbb{R}$ is uniformly convex and bounded above by $\|\cdot\|^r$, does X admit a norm with a modulus of convexity of power type related to r?

We work with a real Banach space $(X, \|\cdot\|)$ with dual X^*, and let B_X and S_X denote the closed unit ball and unit sphere respectively. The *modulus of convexity* of a norm $\|\cdot\|$ on X is defined for $\varepsilon \in [0, 2]$ by

$$\delta_{\|\cdot\|}(\varepsilon) = \inf\left\{ 1 - \frac{1}{2}\|x + y\| : \quad x, y \in S_X, \|x - y\| \geq \varepsilon \right\}.$$

The norm $\|\cdot\|$ is called *uniformly convex* if $\delta_{\|\cdot\|}(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$; additionally, we say that $\|\cdot\|$ has *modulus of convexity of power type* p if there exists $C > 0$ so that $\delta_{\|\cdot\|}(\varepsilon) \geq C\varepsilon^p$ for $\varepsilon \in [0, 2]$. Note that from [11] it follows that $p \geq 2$. The

modulus of smoothness of the norm $\| \cdot \|$ is defined for $\tau > 0$ by

$$\rho_{\|\cdot\|}(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \quad \|x\| = \|y\| = 1 \right\}.$$

The norm $\| \cdot \|$ is called *uniformly smooth* if $\lim_{\tau \to 0^+} \rho_{\|\cdot\|}(\tau)/\tau = 0$; additionally, we say that $\|\cdot\|$ has *modulus of smoothness of power type* q if there exists $C > 0$ such that $\rho_{\|\cdot\|}(\tau) \leq C\tau^q$ for $\tau > 0$. It follows also from [11] that $q \leq 2$. See [6, Chapter IV] for more information on these notions.

We now introduce the like-named concepts for convex functions whose definitions are different from—but motivated by—the norm cases. Given a convex function $f : X \to (-\infty, +\infty]$ we define its *modulus of convexity* as the function $\delta_f : (0, +\infty) \to [0, +\infty]$ given by

$$\delta_f(t) := \inf \left\{ \frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) : \quad \|x - y\| = t, \ x, y \in \mathrm{dom}\, f \right\}, \quad t > 0,$$

where the infimum over the empty set is $+\infty$. We say that f is *uniformly convex* when $\delta_f(t) > 0$ for all $t > 0$; additionally, we say that f has a *modulus of convexity of power type* p if there exists $C > 0$ so that $\delta_f(t) \geq Ct^p$ for all $t > 0$.

Similarly we consider the *modulus of smoothness* of the convex function $f : X \to (-\infty, +\infty)$ as the function $\rho_f : (0, +\infty) \to [0, +\infty]$ defined by

$$\rho_f(t) := \sup \left\{ \frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) : \quad \|x - y\| = t \right\}, \quad t > 0.$$

We will say f is *uniformly smooth* if $\lim_{t \to 0^+} \rho_f(t)/t = 0$; additionally we say that f has a *modulus of smoothness of power type* q if there is a constant $C > 0$, so that $\rho_f(t) \leq Ct^q$ for all $t > 0$.

This terminology may cause some confusion, because, for example, $f = \|\cdot\|$ is never uniformly convex as a function, even when $\|\cdot\|$ is a uniformly convex norm. Therefore, it is important to note the context in which the terms are used. Moreover, the concepts of uniform smoothness and uniform convexity for functions are sometimes defined using the *gage of uniform convexity* and *gage of uniform smoothness* respectively as found in [14]. It is important to note that these alternate definitions using the respective gages are equivalent to those just given; cf. [13, Remark 2.1] and [14, p. 205].

In contrast to the situation for norms, neither the restriction $p \geq 2$ for the modulus of convexity of a convex function nor $q \leq 2$ for the modulus of smoothness of a convex function are necessary. To see this, consider an indicator function of a single point and a constant function respectively.

It is also instructive to consider the function $f = |\cdot|^r$ on the real line where $r > 1$ is fixed. For $1 < r \leq 2$, one can check that f' satisfies an $(r-1)$-Hölder condition. Then using the Mean Value Theorem, it follows that f has a modulus of smoothness of power type r. On the other hand, for $t > 0$, when taking $x = \frac{t}{2}$, $y = -\frac{t}{2}$, we have $\rho_f(t) \geq \frac{1}{2}\left(\frac{t}{2}\right)^r + \frac{1}{2}\left(\frac{t}{2}\right)^r = \frac{1}{2^r}t^r$. Hence f does not have a modulus of smoothness of power type q where $q < r$. In the case $r > 2$, it is straightforward to check that f is not uniformly smooth. Dually, but certainly not trivially, when $r \geq 2$, f is uniformly convex with modulus of convexity of power type r. The next section will elaborate on results of this nature in a more general setting.

Finally, the *Fenchel conjugate* of $f : X \to (-\infty, +\infty]$ is the function $f^* : X^* \to [-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x) : \quad x \in X\}, \quad x^* \in X^*.$$

It is through this concept that duality between uniform convexity and uniform smoothness is studied in the context of convex functions; see [1, 14].

2. UNIFORM CONVEXITY OF FUNCTIONS AND NORMS

This section will demonstrate for $2 \leq p < \infty$ that $f = \| \cdot \|^p$ is uniformly convex if and only if the norm $\| \cdot \|$ has modulus of convexity of power type p.

Lemma 2.1. *Let* $0 < r \leq 1$. *Then* $|t^r - s^r| \leq |t - s|^r$ *for all* $s, t \in [0, \infty)$.

Proof. For $x \geq 0$ we have $(1 + x)^r \leq 1 + x^r$ (see [12, Example 4.20]). Setting $x = (t - s)/s$ with $t \geq s > 0$, and then multiplying by s^r, we get $t^r \leq s^r + (t - s)^r$. The conclusion follows from this. □

Theorem 2.2. *For* $1 < q \leq 2$, *the following are equivalent in a Banach space* $(X, \|\cdot\|)$.
(a) *The norm* $\|\cdot\|$ *has modulus of smoothness of power type* q.
(b) *The derivative of* $f = \|\cdot\|^q$ *exists everywhere and satisfies a* $(q - 1)$-*Hölder condition.*
(c) *The function* $f = \|\cdot\|^q$ *has modulus of smoothness of power type* q.
(d) *The function* $f = \|\cdot\|^q$ *is uniformly smooth.*

Proof. (a) $\Rightarrow$ (b): Assume that $\|\cdot\|$ has modulus of smoothness of power type q. According to [6, Lemma IV.5.1], the norm $\| \cdot \|$ has (Fréchet) derivative at each point $x \in X \setminus \{0\}$—call it ϕ_x—and moreover it satisfies on S_X a Hölder condition; that is, there is $C > 0$ so that

$$(2.1) \qquad \|\phi_x - \phi_y\| \leq C \|x - y\|^{q-1} \quad \text{for all } x, y \in S_X.$$

Let $f = \|\cdot\|^q$. Then $f'(0) = 0$, and $f'(x) = q \|x\|^{q-1} \phi_x$ for $x \neq 0$. Thus if $x = 0$ or $y = 0$, then $\|f'(x) - f'(y)\| \leq q \|x - y\|^{q-1}$. Let $x, y \in X \setminus \{0\}$. Then

$$
\begin{aligned}
f'(x) - f'(y) &= q \|x\|^{q-1} \phi_x - q \|y\|^{q-1} \phi_y \\
(2.2) \qquad &= q \|x\|^{q-1} (\phi_x - \phi_y) + \left(q \|x\|^{q-1} - q \|y\|^{q-1}\right) \phi_y.
\end{aligned}
$$

Using Lemma 2.1 we also compute

$$(2.3) \qquad \left| q \|x\|^{q-1} - q \|y\|^{q-1} \right| \leq q \left| \|x\| - \|y\| \right|^{q-1} \leq q \|x - y\|^{q-1}.$$

We now work on an estimate for $q \|x\|^{q-1} \|\phi_x - \phi_y\|$. We may and do assume that $0 < \|y\| \leq \|x\|$. If $\|y\| \leq \|x\|/2$, then

$$(2.4) \qquad q \|x\|^{q-1} \|\phi_x - \phi_y\| \leq 2q \|x\|^{q-1} \leq q 2^q \|x - y\|^{q-1}.$$

Further, assume that $\|y\| \geq \|x\|/2$. Consider $x' = \lambda x$ where $\lambda = \|y\| / \|x\|$, so that $\|x'\| = \|y\|$. Then

$$(2.5) \qquad \|x' - y\| \leq \|x' - x\| + \|x - y\| = \|x\| - \|y\| + \|x - y\| \leq 2 \|x - y\|.$$

Now let $\alpha = \|y\|$. Observe that ϕ_x and ϕ_y are also support functionals for $\alpha^{-1}x'$ and $\alpha^{-1}y$ respectively. Applying (2.1), the fact that $\|x\| \leq 2\alpha$, and (2.5), we obtain

$$
\begin{aligned}
\|\phi_x - \phi_y\| &\leq C\|\alpha^{-1}x' - \alpha^{-1}y\|^{q-1} \leq \frac{C}{\alpha^{q-1}}\|x' - y\|^{q-1} \\
&\leq \frac{C2^{q-1}}{\|x\|^{q-1}}(2\|x - y\|)^{q-1} = \frac{C4^{q-1}}{\|x\|^{q-1}}\|x - y\|^{q-1}.
\end{aligned}
$$

Consequently, $q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \leq C4^{q-1}q\,\|x - y\|^{q-1}$. This inequality and (2.4) show there exists $K > 0$ such that

$$(2.6) \qquad q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \leq K\,\|x - y\|^{q-1} \quad \text{for all } x, y \in X \setminus \{0\}.$$

Combining (2.2), (2.3) and (2.6) shows that f' satisfies a $(q-1)$-Hölder condition.

(b) $\Rightarrow$ (c) follows from [14, Corollary 3.5.7] (see also [6, Lemma V.3.5]) and (c) $\Rightarrow$ (d) is trivial, so we prove (d) $\Rightarrow$ (a). Suppose $\|\cdot\|$ does not have modulus of smoothness of power type q. According to [6, Lemma IV.5.1] there exist x_n, $y_n \in X$ and $\varepsilon > 0$ such that $x_n \neq y_n$, $\max\{\|x_n\|, \|y_n\|\} \geq \varepsilon$ and

$$\|\phi_{x_n} - \phi_{y_n}\| \geq n\,\|x_n - y_n\|^{q-1} \quad \text{for all } n \in \mathbb{N}.$$

This implies $\|x_n - y_n\| \to 0$. Consequently, there exists $N \in \mathbb{N}$ so that

$$\min\{\|x_n\|, \|y_n\|\} \geq \varepsilon/2 \quad \text{for all } \ n \geq N.$$

Now put $\tilde{x}_n = x_n/\|x_n\|$ and $\tilde{y}_n = y_n/\|y_n\|$ for $n \geq N$. Then, for $n \geq N$,

$$\|\tilde{x}_n - \tilde{y}_n\| \leq \left\|\frac{x_n}{\|x_n\|} - \frac{y_n}{\|x_n\|}\right\| + \left\|\frac{y_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\right\| \leq \frac{4}{\varepsilon}\|x_n - y_n\|.$$

Let $\delta_n = \|x_n - y_n\|$ and define $u_n = \frac{1}{\delta_n\sqrt{n}}\tilde{x}_n$ and $v_n = \frac{1}{\delta_n\sqrt{n}}\tilde{y}_n$ for $n \geq N$. Then $\|u_n - v_n\| \leq \frac{4}{\varepsilon\sqrt{n}} \to 0$. However

$$
\begin{aligned}
\|f'(u_n) - f'(v_n)\| &= \left\|q\,\|u_n\|^{q-1}\,\phi_{u_n} - q\,\|v_n\|^{q-1}\,\phi_{v_n}\right\| \\
&= \left\|q\,\|u_n\|^{q-1}\,\phi_{x_n} - q\,\|v_n\|^{q-1}\,\phi_{y_n}\right\| \\
&= \frac{q}{\delta_n^{q-1}n^{\frac{q-1}{2}}}\,\|\phi_{x_n} - \phi_{y_n}\| \\
&\geq \frac{q}{\delta_n^{q-1}n^{\frac{q-1}{2}}}\left(n\delta_n^{q-1}\right) = qn^{\frac{3-q}{2}} \to \infty.
\end{aligned}
$$

Consequently, f' is not uniformly continuous, and so [14, Theorem 3.5.6] (see also [6, Lemma V.3.5]) shows that $f = \|\cdot\|^q$ is not a uniformly smooth function. □

The results in [1] enable us to derive the dual version of Theorem 2.2 for uniformly convex functions.

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a Banach space, and let $2 \leq p < \infty$. Then the following are equivalent.*
(a) The norm $\|\cdot\|$ on X has modulus of convexity of power type p.
(b) The function $f = \|\cdot\|^p$ has modulus of convexity of power type p.
(c) The function $f = \|\cdot\|^p$ is uniformly convex.

Proof. (a) $\Rightarrow$ (b): Let us assume that $\|\cdot\|$ has modulus of convexity of power type p. Then the modulus of smoothness of the dual norm on X^*, which we denote in this proof as $\|\cdot\|_*$, is of power type q, where $\frac{1}{p} + \frac{1}{q} = 1$; see [6, Proposition IV.1.12]. By Theorem 2.2 the function $g = \frac{1}{q} \|\cdot\|_*^q$ has modulus of smoothness of power type q. The Fenchel conjugate of g is $g^* = \frac{1}{p} \|\cdot\|^p$; see [1, 14]. Now g^*—and hence $\|\cdot\|^p$—has a modulus of convexity of power type p according to [1] (see also [14, Corollary 3.5.11]).

(b) $\Rightarrow$ (c) is trivial, so we prove (c) $\Rightarrow$ (a). Assume that $f = \|\cdot\|^p$ is a uniformly convex function. Then [1] shows that the Fenchel conjugate f^* (and hence $\|\cdot\|_*^q$) is a uniformly smooth function. According to Theorem 2.2, $\|\cdot\|_*$ has modulus of smoothness of power type q; therefore $\|\cdot\|$ has modulus of convexity of power type p; see [6, Proposition IV.1.12]. $\qquad\square$

We conclude this section by showing that the spaces with nontrivial uniformly convex functions are those that admit equivalent uniformly convex norms.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a Banach space. Then the following are equivalent.*
(a) *There exists a continuous uniformly convex function f defined on B_X.*
(b) *X admits an equivalent uniformly convex norm.*
(c) *There exist $p \geq 2$ and an equivalent norm $|\cdot|$ on X so that the function $f = |\cdot|^p$ is uniformly convex.*

Proof. (a) $\Rightarrow$ (b): By replacing f with the function $x \mapsto \frac{f(x) + f(-x)}{2}$ we may and do assume that f is symmetric, and a suitable shift guarantees that $f(0) = 0$. It then follows that $f(x) \geq 0$ for all $x \in X$. Because f is convex and continuous on B_X, we can fix $\varepsilon \in (0, 1)$ such that f is Lipschitz on εB_X (see [14, Corollary 2.2.13]). Let us consider the norm $\|\cdot\|$ whose unit ball is $B = \{x \in B_X : \quad f(x) \leq \delta_f(\varepsilon)\}$. The continuity of f at 0 implies $0 \in \mathrm{int} B$. Moreover, $B \subset \varepsilon B_X$. This follows because

$$f(x) \geq 2 \left[\frac{1}{2} f(x) + \frac{1}{2} f(0) - \frac{1}{2} f\left(\frac{x}{2}\right) \right] \geq 2\delta_f(\varepsilon) > \delta_f(\varepsilon) \ \text{ if } \|x\| > \varepsilon.$$

Thus, $\|\cdot\|$ is an equivalent norm on X.

Consider $x_n, \ y_n \in B_X$ such that $\|x_n\| = \|y_n\| = 1$ and $\|x_n + y_n\| \to 2$. Because f is Lipschitz on B, we have that $f\left(\frac{x_n + y_n}{2}\right) \to \delta_f(\varepsilon)$. Consequently $\frac{1}{2} f(x_n) + \frac{1}{2} f(y_n) - f\left(\frac{x_n + y_n}{2}\right) \to 0$. Thus, the uniform convexity of f ensures that $\|x_n - y_n\| \to 0$ and hence $\|x_n - y_n\| \to 0$.

(b) $\Rightarrow$ (c): From (b) it follows that X is superreflexive (see for instance [6, Theorem IV.4.1]). Then, there exist $p \geq 2$ and an equivalent norm $|\cdot|$ whose modulus of convexity is of power type p (see [6, Theorem IV.4.8]). Consequently, Theorem 2.3 ensures that the function $f = |\cdot|^p$ is uniformly convex.

(c) $\Rightarrow$ (a): This is trivial. $\qquad\square$

3. Growth rates of uniformly convex functions and renorming

In this section we will construct a uniformly convex norm whose modulus of convexity is related to the growth rate of a given uniformly convex function on the Banach space. We begin with some preliminary results.

Lemma 3.1. *Let $\|\cdot\|$ be a norm on a Banach space X. Consider $x, y \in X$ such that $\|x\| = \|y\|$, and $\|x - y\| \geq \delta$ where $0 < \delta \leq 2 \|x\|$. Then $\inf_{t \geq 0} \|x - ty\| \geq \delta/2$.*

Proof. Assume that $\|x - t_0 y\| < \delta/2$ for some $t_0 \geq 0$. Then $|1 - t_0| \|y\| < \delta/2$ and so

$$\|x - y\| \leq \|x - t_0 y\| + |1 - t_0| \|y\| < \delta,$$

which is a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The next lemma will be used later to estimate the modulus of convexity of a norm constructed by using level sets of a symmetric uniformly convex function.

Lemma 3.2. *Let $N \in \mathbb{N}$ and consider norms $\|\cdot\|_N$, $\|\cdot\|_{N+1}, \ldots$ on $(X, \|\cdot\|)$ satisfying*

(3.1) $$\frac{1}{2^{n+1}} \|\cdot\| \leq \|\cdot\|_n \leq \frac{1}{2^n} \|\cdot\| \quad for \ \ n \geq N.$$

For each $n \geq N$, suppose there exists $d_n > 0$ so that

$$\left\| \frac{x+y}{2} \right\|_n \leq 1 - d_n, \quad whenever \ x, y \in X, \|x\|_n = \|y\|_n = 1 \ and \ \|x - y\| \geq 1.$$

Then there exist an equivalent norm $|\cdot|$ on X and $M \in \mathbb{N}$ so that

$$\delta_{|\cdot|}(t) \geq \frac{d_n}{n^2} \quad whenever \ \ 2^{M+1-n} \leq t \leq 2 \ and \ n \geq M.$$

Proof. Put $M = \max\{4, N\}$, and define $|\cdot|$ by

$$|\cdot| = \sum_{m=M}^{\infty} \frac{2^{m+1}}{m^2} \|\cdot\|_m.$$

Observe that $|\cdot| \leq \sum_{m=M}^{\infty} \frac{2^{m+1}}{m^2 2^m} \|\cdot\| \leq \sum_{m=4}^{\infty} \frac{2}{m^2} \|\cdot\| \leq \|\cdot\|$; then

(3.2) $$\frac{1}{2^M} \|\cdot\| \leq \frac{1}{M^2} \|\cdot\| \leq \frac{2^{M+1}}{M^2} \|\cdot\|_M \leq |\cdot| \leq \|\cdot\|.$$

Fix $n \geq M$ and $2^{M+1-n} \leq t \leq 2$. Consider $x, y \in X$ such that $|x| = |y| = 1$ and $|x - y| = t$. Because $|x| = |y| = 1$, it follows from (3.2) that

(3.3) $$1 \leq \|x\| \leq 2^M \qquad \text{and} \qquad 1 \leq \|y\| \leq 2^M.$$

We assume, without loss of generality, $\|x\|_n \leq \|y\|_n$. Now let us denote $a = \|x\|_n^{-1}$ and $b = \|y\|_n^{-1}$. It follows from (3.1) and (3.3) that $2^{n-M} \leq b \leq a \leq 2^{n+1}$, which in turn implies $|ax - ay| \geq 2$.

According to Lemma 3.1, $|ax - by| \geq 1$, and hence $\|ax - by\| \geq 1$. Thus we can estimate

$$\begin{aligned} \left\| \frac{ax + ay}{2} \right\|_n &\leq \left\| \frac{ax + by}{2} \right\|_n + \frac{1}{2}(a - b) \|y\|_n \\ &\leq \frac{1}{2} \|ax\|_n + \frac{1}{2} \|by\|_n + \frac{1}{2}(a - b) \|y\|_n - d_n \\ &= \frac{a}{2}(\|x\|_n + \|y\|_n) - d_n. \end{aligned}$$

This inequality implies

(3.4) $$\left\| \frac{x+y}{2} \right\|_n \leq \frac{1}{2} \|x\|_n + \frac{1}{2} \|y\|_n - \frac{d_n}{a}.$$

Thus, using (3.4) and the triangle inequality for $\|\cdot\|_j$ when $j \neq n$, and then that $a \leq 2^{n+1}$, we obtain

$$\left| \frac{x+y}{2} \right| \leq \sum_{j=M}^{\infty} \frac{2^{j+1}}{2j^2} \|x\|_j + \sum_{j=M}^{\infty} \frac{2^{j+1}}{2j^2} \|y\|_j - \frac{2^{n+1}d_n}{n^2 a} \leq 1 - \frac{d_n}{n^2},$$

which finishes the proof. $\qquad\square$

Lemma 3.3. *Let $(X, \|\cdot\|)$ be a Banach space and let $f : X \to (-\infty, +\infty)$ be a symmetric, continuous, and uniformly convex function, with $f(0) < 0$. Then $f(x) > 0$ whenever $x \in X$ and $\|x\| > \max\{-f(0)/(2\delta_f(1)), 1\}$.*

Proof. The symmetry of f implies that $f \geq f(0)$. Also, the continuity and uniform convexity of f guarantees that f is also convex. Now, take any $x \in X$ as in the conclusion of the lemma. Then

$$\delta_f(1) \leq \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{x}{\|x\|}\right) - f\left(\frac{x}{2\|x\|}\right) \leq \frac{1}{2}f(0) + \frac{1}{2}f\left(\frac{x}{\|x\|}\right) - f(0)$$

$$\leq -\frac{1}{2}f(0) + \frac{1}{2}\left[\left(1 - \frac{1}{\|x\|}\right)f(0) + \frac{1}{\|x\|}f(x)\right] = -\frac{1}{2\|x\|}f(0) + \frac{1}{\|x\|}f(x),$$

and hence, $f(x) \geq \|x\| \delta_f(1) + \frac{1}{2}f(0) > 0$. $\qquad\square$

Proposition 3.4. *Let $(X, \|\cdot\|)$ be a Banach space and let $F : [0, +\infty) \to [0, +\infty)$ be a continuous convex function satisfying $F(0) = 0$. Suppose $f : X \to \mathbb{R}$ is a continuous symmetric uniformly convex function satisfying $f(x) \leq F(\|x\|)$ for all $x \in X$. Then there exists an equivalent norm $|\cdot|$ on X such that given any $\gamma > 0$, there are constants $\alpha > 0$ and $\beta > 0$ so that*

$$\delta_{|\cdot|}(t) \geq \frac{\alpha}{F(\beta t^{-1})} t^{\gamma} \qquad \text{for } 0 < t \leq 2.$$

Proof. Because $F(0) = 0$, the convexity of F ensures that $F(\lambda t) \leq \lambda F(t)$ for $0 \leq \lambda \leq 1$; hence F is nondecreasing on $[0, +\infty)$ because it is nonnegative there. Consequently, $F \circ \|\cdot\|$ is a convex function. According to Lemma 3.3, there exists $N \in \mathbb{N}$ so that $f(x) \geq 0$ if $\|x\| \geq N$. Now replace f with $[f + F \circ \|\cdot\|]/2$. Obviously, this new f will still be uniformly convex. We have

$$(3.5) \qquad F\left(\frac{\|x\|}{2}\right) \leq \frac{1}{2}F(\|x\|) \leq f(x) \leq F(\|x\|) \text{ whenever } \|x\| \geq N.$$

For $n \geq N$, let $\|\cdot\|_n$ be the norm whose unit ball is $B_n = \{x \in X : \quad f(x) \leq F(2^n)\}$. It follows from (3.5) that if $\|x\|_n = 1$, then $2^n \leq \|x\| \leq 2^{n+1}$. Consequently,

$$\frac{1}{2^{n+1}} \|\cdot\| \leq \|\cdot\|_n \leq \frac{1}{2^n} \|\cdot\|.$$

Let

$$M_n = \sup\{f'_+(u, v) : \quad u, v \in X, \quad \|u\|_n = \|v\|_n = 1\}.$$

If u, v are such, we have $\|u\| \geq 2^n \|u\|_n = 2^n \geq 2^N > N$ and hence

$$f'_+(u, v) \leq f(u+v) - f(u) \leq f(u+v) \leq F(\|u+v\|)$$

$$(3.6) \qquad\qquad \leq F\left(2^{n+1} \|u+v\|_n\right) \leq F\left(2^{n+2}\right).$$

Therefore, $M_n \leq F\left(2^{n+2}\right)$.

Now consider $x, y \in X$ such that $\|x\|_n = \|y\|_n = 1$ and $\|x - y\| \geq 1$. We will verify the hypothesis of Lemma 3.2. Put $z = \frac{x+y}{2}$ and $z' = \frac{x+y}{\|x+y\|_n}$. We observe

that $f(x) = f(y) = f(z') = F(2^n)$. Then, using the definition of $\delta_f(1)$ and the convexity of f, we have

$$\delta_f(1) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) = f(z') - f(z)$$

(3.7)
$$\leq f'_+(z', z' - z) \leq M_n \|z' - z\|_n,$$

and hence $\|z' - z\|_n \geq \delta_f(1)/M_n$. Therefore

(3.8)
$$\left\|\frac{x+y}{2}\right\|_n = 1 - \|z - z'\|_n \leq 1 - \frac{\delta_f(1)}{M_n} \leq 1 - \frac{\delta_f(1)}{F(2^{n+2})}.$$

Applying Lemma 3.2, we find an equivalent norm $|\cdot|$ and $M \geq N$ such that

$$\delta_{|\cdot|}(t) \geq \frac{\delta_f(1)}{F(2^{n+2})} \cdot \frac{1}{n^2}, \quad \text{whenever } n \geq M \text{ and } 2^{M+1-n} \leq t \leq 2.$$

Fix any $\gamma > 0$. Find $n_0 > M$ so large that $n^{-2} \geq (2^{-n})^\gamma$ for all $n \geq n_0$. Then

$$\delta_{|\cdot|}(t) \geq \frac{\delta_f(1)}{F(2^{n+2})} \cdot \left(\frac{1}{2^n}\right)^\gamma \quad \text{whenever } n \geq n_0 \text{ and } 2^{M+1-n} \leq t \leq 2.$$

Let $\alpha = \delta_f(1)\left(\frac{1}{2^{n_0+1}}\right)^\gamma$ and $\beta = 2^{n_0+3}$. The previous inequality, along with the fact that F is nondecreasing, ensures that for $2^{M+1-n_0} \leq t \leq 2$ we have

$$\delta_{|\cdot|}(t) \geq \frac{\delta_f(1)}{F(2^{n_0+2})}\left(\frac{1}{2^{n_0}}\right)^\gamma \geq \frac{\delta_f(1)}{F(2^{n_0+3}t^{-1})}\left(\frac{t}{2^{n_0+1}}\right)^\gamma = \frac{\alpha t^\gamma}{F(\beta t^{-1})},$$

and for $2^{M+1-n} \leq t \leq 2^{M+2-n}$ where $n \geq n_0 + 1$, we have

$$\delta_{|\cdot|}(t) \geq \frac{\delta_f(1)}{F(2^{n+2})}\left(\frac{1}{2^n}\right)^\gamma \geq \frac{\delta_f(1)}{F(2^{M+4}t^{-1})}\left(\frac{t}{2^{M+2}}\right)^\gamma \geq \frac{\alpha t^\gamma}{F(\beta t^{-1})}.$$

Altogether, $\delta_{|\cdot|}(t) \geq \frac{\alpha}{F(\beta t^{-1})}t^\gamma$ for all $0 < t \leq 2$, as desired. $\qquad\square$

Theorem 3.5. *Let $(X, \|\cdot\|)$ be a Banach space, and let $p \geq 2$. Suppose $f : X \to \mathbb{R}$ is a continuous uniformly convex function such that $f(x) \leq \|x\|^p$ for all $x \in X$. Then for any $r > p$, X admits an equivalent norm with modulus of convexity of power type r.*

Proof. Apply Proposition 3.4 with $F(t) = t^p$. $\qquad\square$

Remark 3.6. It should be noted that $p < 2$ cannot be in Theorem 3.5, since by [14, Proposition 3.5.8], $\liminf_{\|x\|\to\infty} f(x)/\|x\|^2 > 0$.

In the case $p = 2$ we will prove the following sharp result.

Theorem 3.7. *Let $(X, \|\cdot\|)$ be a Banach space. Then there is a continuous uniformly convex funtion $f : X \to \mathbb{R}$ satisfying $f \leq \|\cdot\|^2$ if and only if X admits an equivalent norm with modulus of convexity of power type 2.*

Before proving this theorem, we will present a preliminary lemma, and we also refer the reader to [7] for some related information about this case.

Lemma 3.8. *Let $(X, \|\cdot\|)$ be a Banach space and $N \in \mathbb{N}$. Suppose that $|\cdot|_N$, $|\cdot|_{N+1}, \ldots$ are norms on X such that*

$$(3.9) \qquad\qquad K \|\cdot\| \leq |\cdot|_n \leq \|\cdot\|,$$

for some $K > 0$ and all $n \geq N$. Then there exists an equivalent norm $|\cdot|$ such that

$$\delta_{|\cdot|}(t) \geq \liminf_{n \to \infty} \delta_{|\cdot|_n}(t), \quad for \quad 0 < t < 2.$$

Proof. Let us consider a free (i.e., nonprincipal) ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then $\lim_{\mathcal{U}} |x|_n$ exists for each $x \in X$, where $\lim_{\mathcal{U}} |x|_n = L$ means that for each $\varepsilon > 0$, there exists $A \in \mathcal{U}$ such that $|\,|x|_n - L| < \varepsilon$ for all $n \in A$. Now define $|\cdot| : X \to [0, +\infty)$ by

$$|x| = \lim_{\mathcal{U}} |x|_n, \quad \text{for all} \quad x \in X.$$

The definition of $|\cdot|$ together with (3.9) ensures that $|\cdot|$ is an equivalent norm on X.

We will proceed by reductio ad absurdum. Assume there is $t \in (0, 2)$ such that $\delta_{|\cdot|}(t) < \liminf \delta_{|\cdot|_n}(t)$. Since $\delta_{|\cdot|}$ is continuous—see [9]—there exists $t' \in (t, 2)$ such that $\delta_{|\cdot|}(t') < \liminf \delta_{|\cdot|_n}(t)$. Then, there exist $x, y \in X$ and a constant $a > 0$ such that $|x| = |y| = 1$, $|x - y| \geq t'$ and $1 - |(x+y)/2| < a < \liminf \delta_{|\cdot|_n}(t)$. For $n \in \mathbb{N}$, let $x_n = x/|x|_n$ and $y_n = y/|y|_n$. By the definition of $|\cdot|$, there exists $A \in \mathcal{U}$ such that $|x_m - y_m|_m > t$ and $1 - |(x_m + y_m)/2|_m < a$ for all $m \in A$. Therefore $\delta_{|\cdot|_m}(t) < a < \liminf \delta_{|\cdot|_n}(t)$ for all $m \in A$, which yields a contradiction, since $\mathcal{U}$ is free and then A is infinite. $\qquad\square$

Proof (*of Theorem* 3.7). Suppose $(X, \|\cdot\|)$ admits an equivalent norm $|\cdot|$ that has modulus of convexity of power type 2. We may and do assume that $|\cdot|$ has modulus of convexity of power type 2 and satisfies $|\cdot| \leq \|\cdot\|$. According to Theorem 2.3, $f = |\cdot|^2$ is uniformly convex as desired.

Conversely, suppose $f : X \to \mathbb{R}$ is a uniformly convex function such that $f \leq \|\cdot\|^2$. Proceeding as in Theorem 3.4 when $F(t) = t^2$ we obtain norms $\{\|\cdot\|_n\}_{n \geq N}$ satisfying $\frac{1}{2^{n+1}} \|\cdot\| \leq \|\cdot\|_n \leq \frac{1}{2^n} \|\cdot\|$, and then (3.8) becomes

$$\left\| \frac{x+y}{2} \right\|_n \leq 1 - \frac{\delta_f(1)}{16} \left(\frac{1}{2^n} \right)^2, \text{ whenever } \|x\|_n = \|y\|_n = 1 \text{ and } \|x - y\|_n \geq \frac{1}{2^n}.$$

The previous inequality implies

$$\delta_{\|\cdot\|_n}(2^{-n}) \geq \frac{\delta_f(1)}{16}(2^{-n})^2.$$

According to [8, Corollary 11] there is a universal constant $L > 0$ such that

$$\frac{\delta_{\|\cdot\|_n}(2^{-n})}{(2^{-n})^2} \leq 4L \frac{\delta_{\|\cdot\|_n}(\eta)}{\eta^2} \quad \text{for } 2^{-n} \leq \eta \leq 2.$$

Let $R = \frac{\delta_f(1)}{64L}$; then the previous two inequalities imply

$$(3.10) \qquad\qquad \delta_{\|\cdot\|_n}(t) \geq Rt^2 \text{ for } 2^{-n} \leq t \leq 2.$$

For each $n \geq N$, let us consider the new norm $|\cdot|_n = 2^n \|\cdot\|_n$. These new norms satisfy $\frac{1}{2} \|\cdot\| \leq |\cdot|_n \leq \|\cdot\|$ and $\delta_{|\cdot|_n}(\cdot) = \delta_{\|\cdot\|_n}(\cdot)$. Let $|\cdot|$ be the norm found for our $|\cdot|_N, |\cdot|_{N+1}, \ldots$ in Lemma 3.8. Then (3.10) yields that

$$\delta_{|\cdot|}(t) \geq \liminf_{n \to \infty} \delta_{|\cdot|_n}(t) = \liminf_{n \to \infty} \delta_{\|\cdot\|_n}(t) \geq Rt^2 \text{ for } 0 < t \leq 2,$$

which finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It does not appear as though the proof of Theorem 3.7 can be adapted to work in the case $p > 2$. This is because the analogue of Figiel's result ([8, Corollary 11]) is not valid when $p > 2$. Indeed, if $\delta_{\|\cdot\|}$ has modulus of power type less than p for some norm $\|\cdot\|$, then $\lim \delta_{\|\cdot\|}(\varepsilon)/\varepsilon^p \to \infty$ as $\varepsilon \to 0$; hence a Figiel constant does not exist. However we have not checked whether such a constant exists for the family of norms $\|\cdot\|_n$ used in Proposition 3.4.

Acknowledgments

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