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ON SPACES OF OPERATORS ON $C(Q)$ SPACES (Q COUNTABLE METRIC SPACE)

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Abstract. In this paper we study spaces of nuclear operators $\mathcal{N}(C(Q))$ and spaces of compact operators $\mathcal{K}(C(Q))$ on spaces of continuous functions $C(Q)$, where Q is a countable compact metric space, in connection with the C. Bessaga and A. Pełczyński isomorphic classification of these spaces.

We show that the spaces $\mathcal{K}(C(Q))$ [resp. $\mathcal{N}(C(Q))$] and $\mathcal{K}(C(Q'))$ [resp. $\mathcal{N}(C(Q'))$] are isomorphic if, and only if, $C(Q)$ and $C(Q')$ are isomorphic. We show also that $\mathcal{N}(C(Q))$ is not isomorphic to a subspace of $\mathcal{K}(C(Q))$.

1. Notation and terminology

Throughout this paper, the symbols $E, F, X, Y, \dots$ denote Banach spaces. B_E denotes the closed unit ball of E. "Subspace" means closed linear subspace. "Operator" means "bounded linear operator".

In the sequel we denote $\alpha, \beta, \gamma, \dots$ ordinal numbers, ω denotes the first infinite ordinal and ω_1 the first uncountable ordinal. Let $\alpha < \beta$ be ordinals; then $\langle \alpha, \beta \rangle$ denotes the interval $\{ \gamma\,;\, \alpha \le \gamma \le \beta \}$ and $\langle \alpha, \beta)$ denotes the interval $\{ \gamma\,;\, \alpha \le \gamma < \beta \}$ endowed with the order topology. It is a well-known theorem of Mazurkiewicz and Sierpiński, cf. [5], that every countable compact metric space is homeomorphic to an interval $\langle 1, \alpha \rangle$ with $\omega \le \alpha < \omega_1$.

$C(\alpha)$ denotes the space of all continuous scalar-valued functions defined on $\langle 1, \alpha \rangle$ with the norm $\|x\| = \sup\limits_{\gamma \in \langle 1, \alpha \rangle} |x(\gamma)|$ and $C_0(\alpha)$ is the subspace $\{ x \in C(\alpha)\,;\, x(\alpha) = 0 \}$ of $C(\alpha)$. The space of all continuous E-valued functions defined on $\langle 1, \alpha \rangle$ with the norm $\|x\| = \sup\limits_{\gamma \in \langle 1, \alpha \rangle} \|x(\gamma)\|$ is denoted $C(\alpha, E)$ and $C_0(\alpha, E)$ is the subspace $\{ x \in C(\alpha, E)\,;\, x(\alpha) = 0 \}$. The spaces $C(\alpha, E)$ and $C_0(\alpha, E)$ are isomorphic for all $\alpha \in \langle \omega, \omega_1 \rangle$; cf. [1].

The injective norm on $E \otimes F$, denoted by $\| \ \|_\varepsilon$, is given by

$$\left\| \sum_{i=1}^{n} x_i \otimes y_i \right\|_\varepsilon = \sup \left\{ \left| \sum_{i=1}^{n} x^*(x_i) y^*(y_i) \right| \,;\, x^* \in B_{E^*}, y^* \in B_{F^*} \right\}.$$

The completion of $(E \otimes F, \| \ \|_\varepsilon)$ is denoted by $E \widehat{\widehat{\otimes}} F$.

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The projective norm on $E \otimes F$, denoted by $\| \; \|_\pi$, is given by

$$\|u\|_\pi = \sup \{ |\varphi(u)| \,;\, \varphi \text{ bounded bilinear form on } E \times F,\ \|\varphi\| \le 1 \} .$$

The completion of $(E \otimes F, \| \; \|_\pi)$ is denoted by $E \widehat{\otimes} F$.

It is well known that (cf. [2], [3]):

i) given $(\Omega, \mathcal{S}, \mu)$ a measure space, the space $L^1(\mu)\widehat{\otimes} E$ may be isometrically identified with the space $L^1(\mu, E)$ of Bochner-integrable "functions" from Ω to E,

ii) given K a topological compact space, the space $C(K)\widehat{\widehat{\otimes}} E$ is isometrically isomorphic to the space $C(K, E)$ of all continuous E-valued functions defined on K.

An operator $T : E \to F$ is called nuclear if it can be written as the sum of an absolutely converging series of rank-one operators, i.e., if there are sequences $(x_n^*)_n$ in E^* and $(y_n)_n$ in F with $\sum_n \|x_n^*\| \, \|y_n\| < +\infty$ and such that $T(x) = \sum_n x_n^*(x) y_n$ for all $x \in E$. Such a series $\sum_n x_n^* \otimes y_n$ is called a nuclear representation of T ; the infimum

$$\|T\|_{\mathcal{N}} = \inf \left\{ \sum_n \|x_n^*\| \, \|y_n\| \,;\, \sum_n x_n^* \otimes y_n \ \text{ nuclear representation of } T \right\}$$

is called the nuclear norm of T. We denote by $\mathcal{N}(E, F)$ the space of nuclear operators from E to F, $\mathcal{N}(E) = \mathcal{N}(E, E)$. The space $(\mathcal{N}(E, F), \| \; \|_{\mathcal{N}})$ is a Banach space. Also, $\mathcal{L}(X, Y)$ [resp. $\mathcal{K}(X, Y)$] denotes the space of operators [resp. of compact operators] from X to Y, $\mathcal{L}(X) = \mathcal{L}(X, X)$, $\mathcal{K}(X) = \mathcal{K}(X, X)$.

There exists a closed connection between compact or nuclear operators and tensor products. If E^* or F has the approximation property, then (cf. [2], [3]):

i) the canonical map from $E^* \widehat{\otimes} F$ into $\mathcal{L}(E, F)$ is an isomorphism onto $\mathcal{N}(E, F)$,

ii) the canonical map from $E^* \widehat{\widehat{\otimes}} F$ into $\mathcal{L}(E, F)$ is an isomorphism onto $\mathcal{K}(E, F)$.

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2. The spaces $\mathcal{N}(C(Q))$ and $\mathcal{K}(C(Q'))$

W.B. Johnson has shown in [4] that if either X or Y has a local unconditional structure, then $\mathcal{N}(X, Y)$ is a proper subset of $\mathcal{K}(X, Y)$. In this section we shall show that no subspace of $\mathcal{K}(C(Q'))$ is isomorphic to $\mathcal{N}(C(Q))$ whatever the countable metric compact spaces Q and Q' are.

Lemma 2.1. *Let α be a countable ordinal ($\omega \le \alpha < \omega_1$) and let E be a Banach space which does not contain a subspace isomorphic to c_0 and such that $E \times E$ is isomorphic to a subspace of E. Then every subspace of $C(\alpha, E)$ which does not contain a subspace isomorphic to c_0 is isomorphic to a subspace of E.*

Proof. We prove the lemma by a transfinite induction on α. The proof of the initial case $\alpha = \omega$ is essentially the same as the proof of the case $\omega < \alpha$ where α is a limit ordinal. Suppose that α is a limit ordinal which satisfies $\omega < \alpha < \omega_1$ and that, for every $\beta \in \langle \omega, \alpha)$, every subspace of $C(\beta, E)$ which does not contain a subspace isomorphic to c_0 is isomorphic to a subspace of E.

Consider a subspace X of $C_0(\alpha, E)$ which does not contain a subspace isomorphic to c_0.

Let $\omega \leq \alpha_1 < \alpha_2 < \ldots < \alpha_k < \ldots < \alpha$ be a strictly increasing sequence of ordinal numbers converging to α. If there exists an integer k_0 such that X is isomorphic to a subspace of $C(\alpha_{k_0}, E)$, the conclusion will then follow from the induction hypothesis.

Assume on the contrary that, for every integer k, X is not isomorphic to a subspace of $C(\alpha_k, E)$. Let $\varepsilon \in]0,1[$. By a standard procedure using the operators

$$\pi_k : C_0(\alpha, E) \to C(\alpha_k, E)$$

defined, for every integer k, by $\pi_k(f) = f_{|\langle 1,\alpha_k\rangle}$, we can find a strictly increasing sequence of integers $(k_l)_{l\geq 1}$ and a normalized sequence $(f_l)_{l\geq 1}$ of X such that, for every integer $l \geq 2$, we have $\|\pi_{k_{l-1}}(f_l)\| \leq \frac{\varepsilon}{2^l}$ and, for every integer $l \geq 1$ and for every ordinal $\gamma > \alpha_{k_l}$, we have $\|f_l(\gamma)\| \leq \frac{\varepsilon}{2^l}$.

It is easy to show that for every integer l_0 and for every finite scalar sequence $a_1, \ldots, a_{l_0}$ we have

$$(1-\varepsilon) \max_{1\leq l\leq l_0} |a_l| \leq \left\| \sum_{l=1}^{l_0} a_l f_l \right\| \leq (1+\varepsilon) \max_{1\leq l\leq l_0} |a_l|.$$

So $(f_l)_{l\geq 1}$ is a normalized basic sequence of X equivalent to the canonical basis of c_0. The assumption that, for every integer k, X is not isomorphic to a subspace of $C(\alpha_k, E)$ leads to a contradiction.

In the case $\alpha = \omega$ we have to observe that a subspace X of $C_0(\omega, E)$ which does not contain a subspace isomorphic to c_0 is isomorphic to a subspace of E^n for some integer n, so X is isomorphic to a subspace of E. □

The case where α is a successor ordinal is obvious.

The proof of the following lemma is straightforward.

Lemma 2.2. *Every infinite dimensional subspace of* $(\ell_\infty^1 \oplus \ldots \oplus \ell_\infty^n \oplus \ldots)_{\ell_1}$ *contains a subspace isomorphic to* ℓ_1.

Theorem 2.3. *Let* α, β *be two countable ordinals; then* $\mathcal{N}(C(\alpha))$ *is not isomorphic to a subspace of* $\mathcal{K}(C(\beta))$.

Proof. Suppose that there exists an isomorphism T from $\mathcal{N}(C_0(\alpha))$ onto a subspace of $\mathcal{K}(C_0(\beta))$. We know that $\mathcal{N}(C_0(\alpha))$ is isometrically isomorphic to $\ell_1 \widehat{\otimes} C_0(\alpha) = \ell_1(\mathbb{N}, C_0(\alpha))$ and $\mathcal{K}(C_0(\beta))$ is isometrically isomorphic to $\ell_1 \widehat{\widehat{\otimes}} C_0(\beta) = C_0(\beta, \ell_1)$. Let, for every integer n, X_n be a subspace of $C_0(\alpha)$ isometric to ℓ_∞^n. By Lemma 2.2, the subspace $X = (X_1 \oplus \cdots \oplus X_n \oplus \cdots)_{\ell_1}$ of $\mathcal{N}(C_0(\alpha))$ does not contain a subspace isomorphic to c_0, so, by Lemma 2.1 with $E = \ell_1$, the subspace $T(X)$ of $C_0(\beta, \ell_1)$ is isomorphic to a subspace of ℓ_1. This means that c_0 is finitely representable in ℓ_1, which is, of course, false and ends the proof. □

Using the same arguments we can prove

Theorem 2.4. *Let* Q_1, Q_2, Q_3 *and* Q_4 *be four countable compact metric spaces. Then* $\mathcal{N}(C(Q_1), C(Q_2))$ *is not isomorphic to a subspace of* $\mathcal{K}(C(Q_3), C(Q_4))$.

3. Spaces of compact operators

The isomorphic classification of $C(Q)$ spaces with Q a countable compact metric space is due to C. Bessaga and A. Pełczyński [1]. Let $\omega \leq \alpha \leq \beta < \omega_1$ be countable ordinal numbers. Then the spaces $C(\alpha)$ and $C(\beta)$ are isomorphic if, and only if, $\beta < \alpha^\omega$. Thus, under the conditions $\omega \leq \alpha \leq \beta < \alpha^\omega < \omega_1$, the spaces $\mathcal{K}(C(\alpha))$ and $\mathcal{K}(C(\beta))$ are isomorphic. Conversely we shall show that, if $\mathcal{K}(C(\alpha))$ and $\mathcal{K}(C(\beta))$ are isomorphic, then the spaces $C(\alpha)$ and $C(\beta)$ are also isomorphic.

Lemma 3.1. *Let α be a countable ordinal ($\omega \leq \alpha < \omega_1$) and let E be a Banach space. If, for every ordinal $\gamma < \alpha$, the space $C_0(\alpha)$ is not isomorphic to a subspace of $C_0(\gamma, E)$, then the space $C_0(\alpha^\omega)$ is not isomorphic to a subspace of $C_0(\alpha, E)$.*

Proof. We use ideas of the proof of Lemma 3 in [1]. Suppose that α is a countable ordinal such that for every ordinal $\gamma < \alpha$, the space $C_0(\alpha)$ is not isomorphic to a subspace of $C_0(\gamma, E)$ and $C_0(\alpha^\omega)$ is isomorphic to a subspace of $C_0(\alpha, E)$; we shall show that this assumption leads to a contradiction. Let T be an operator from $C_0(\alpha^\omega)$ into $C_0(\alpha, E)$ and b be a real number such that, for every $f \in C_0(\alpha^\omega)$, we have

$$\|f\| \leq \|T(f)\| \leq b\|f\|\,.$$

Now we fix an integer N and $\varepsilon \in]0,1[$ such that $b < N$ and $\dfrac{N}{N+1} \leq \dfrac{1-\varepsilon}{1+\varepsilon}$.

Let f_0 be the function identically equal to 1 on $\langle 1, \alpha^N\rangle$ and equal to 0 on $\langle \alpha^N + 1, \alpha^\omega\rangle$. There exists an ordinal $\gamma_1 < \alpha$ such that $\|T(f_0)(\gamma)\| \leq \varepsilon$ for every $\gamma_1 < \gamma \leq \alpha$. Let $\Delta^1_\beta = \langle \alpha^{N-1}\beta + 1, \alpha^{N-1}(\beta+1)\rangle$ for $0 \leq \beta < \alpha$ and

$$E_1 = \left\{ f \in C_0(\alpha^\omega)\,;\, \forall\, \gamma > \alpha^N,\ f(\gamma) = 0 \text{ and } \forall\, 0 \leq \beta < \alpha,\ f \text{ is constant on } \Delta^1_\beta \right\}.$$

It is obvious that E_1 is isomorphic to $C_0(\alpha)$. There is no subspace of $C_0(\gamma_1, \ell_1)$ isomorphic to $C_0(\alpha)$, so there exists $f_1 \in E_1$ such that $\|f_1\| = 1$ and, for every $\gamma \leq \gamma_1$, we have $\|T(f_1)(\gamma)\| \leq \varepsilon$. We may change f_1 to $-f_1$ and suppose that there exists $\beta_1 \in \langle 0, \alpha)$ such that, for every $\lambda \in \Delta^1_{\beta_1}$, we have $1 - \varepsilon \leq f_1(\lambda) \leq 1$. We have $T(f_1) \in C_0(\alpha, \ell_1)$, so we can find $\gamma_2 \in (\gamma_1, \alpha)$ such that $\|T(f_1)(\gamma)\| \leq \varepsilon$ for every $\gamma_2 < \gamma < \alpha$. This ends the first step of the proof. □

For the second step, let

$$\Delta^2_\beta = \langle \alpha^{N-1}\beta_1 + \alpha^{N-2}\beta + 1, \alpha^{N-1}\beta_1 + \alpha^{N-2}(\beta+1)\rangle$$

for $0 \leq \beta < \alpha$ and let

$$E_2 = \left\{ f \in C_0(\alpha^\omega)\,;\, \forall\, \gamma \notin \Delta^1_{\beta_1},\ f(\gamma) = 0 \text{ and } \forall\, 0 \leq \beta < \alpha,\ f \text{ is constant on } \Delta^2_\beta \right\}.$$

The subspace E_2 of $C_0(\alpha^\omega)$ is isomorphic to $C_0(\alpha)$. In the same way as in the previous case, we can find $f_2 \in E_2, \|f_2\| = 1$ and two ordinal numbers $0 \leq \beta_2 < \alpha$, $\gamma_2 < \gamma_3 < \alpha$ such that $\|T(f_2)(\gamma)\| \leq \varepsilon$ for all $\gamma \leq \gamma_2$, $\|T(f_2)(\gamma)\| \leq \varepsilon$ for all $\gamma > \gamma_3$ and $1 - \varepsilon \leq f_2(\gamma) \leq 1$ for every $\gamma \in \Delta^2_{\beta_2}$. Repeating this procedure N times we shall find $f_0, \ldots, f_N \in C_0(\alpha^\omega)$ and $\gamma_1 < \gamma_2 < \cdots < \gamma_N < \alpha$ such that :

- $\|f_0\| = \ldots = \|f_N\| = 1$,
- $\emptyset \neq f_N^{-1}([1-\varepsilon, 1]) \subset f_{N-1}^{-1}([1-\varepsilon, 1]) \subset \cdots \subset f_1^{-1}([1-\varepsilon, 1]) \subset \langle 1, \alpha^N\rangle = f_0^{-1}(1)$,
- $\forall\, 1 \leq k \leq N,\ \forall\, \gamma \leq \gamma_k$ we have $\|T(f_k)(\gamma)\| \leq \varepsilon$,
- $\forall\, 1 \leq k \leq N,\ \forall\, \gamma > \gamma_k$ we have $\|T(f_{k-1})(\gamma)\| \leq \varepsilon$.

It is obvious that $(N+1)(1-\varepsilon) \leq \|f_0 + \cdots + f_N\|$ and $\|T(f_0) + \cdots + T(f_N)\| \leq b + N\varepsilon$; hence $(N+1)(1-\varepsilon) \leq b + N\varepsilon$, and so $N \leq b$. We obtain a contradiction with $b < N$, and so the conclusion of the lemma holds.

Theorem 3.2. *Let $\omega \leq \alpha \leq \beta < \omega_1$ be two countable infinite ordinal numbers and let E be a Banach space which does not contain a subspace isomorphic to c_0. Then $C(\beta)$ is isomorphic to a subspace of $C(\alpha, E)$ if, and only if, $C(\beta)$ is isomorphic to a subspace of $C(\alpha)$.*

Proof. It is obvious that if $C(\beta)$ is isomorphic to a subspace of $C(\alpha)$, then $C(\beta)$ is isomorphic to a subspace of $C(\alpha, E)$.

For the converse we introduce two sets of ordinal numbers:

$$I_1 = \{\, \alpha \in \langle \omega, \omega_1) \,;\, \forall\, \gamma < \alpha,\ C(\alpha) \ \text{ is not isomorphic to a subspace of } \ C(\gamma) \,\},$$

$$I_2 = \{\, \alpha \in \langle \omega, \omega_1) \,;\, \forall\, \gamma < \alpha,\ C(\alpha) \ \text{ is not isomorphic to a subspace of } \ C(\gamma, E) \,\}.$$

We shall prove that $I_1 = I_2$. It is obvious that $I_2 \subset I_1$. The space E does not contain a subspace isomorphic to c_0, so, for every integer n, the space E^n does not contain a subspace isomorphic to c_0; therefore $\omega \in I_2$. Now, suppose that I_2 is a proper subset of I_1. Let α_1 be the least element of $I_1 \setminus I_2$. We have $\omega < \alpha_1$ and for every $\beta \in I_1$, $\beta < \alpha_1$ implies $\beta \in I_2$. The ordinal $\alpha_1 \notin I_2$, so there exists an ordinal $\gamma_1 < \alpha_1$ be such that $C(\alpha_1)$ is isomorphic to a subspace of $C(\gamma_1, E)$. Let

$$\alpha_2 = \min \{\, \gamma < \alpha_1 \,;\, C(\alpha_1) \ \text{ is isomorphic to a subspace of } \ C(\gamma, E) \,\}.$$

We have $\alpha_2 \leq \gamma_1$ and $C(\alpha_1)$ is isomorphic to a subspace of $C(\alpha_2, E)$. Let us show that $\alpha_2 \in I_1$. If this is not the case, there exists an ordinal $\gamma_2 < \alpha_2$ such that $C(\alpha_2)$ is isomorphic to a subspace of $C(\gamma_2)$. Therefore $C(\alpha_2, E)$ is isomorphic to a subspace of $C(\gamma_2, E)$ and so $C(\alpha_1)$ is isomorphic to a subspace of $C(\gamma_2, E)$ with $\gamma_2 < \alpha_2$, in contradiction with the definition of α_2. We have $\alpha_2 \in I_1$ and $\alpha_2 < \alpha_1$, so $\alpha_2 \in I_2$. By Lemma 3.1, the space $C(\alpha_2^\omega)$ is not isomorphic to a subspace of $C(\alpha_2, E)$. We have $\alpha_1 \in I_1$ and $\alpha_2 < \alpha_1$, so, by the result of C. Bessaga and A. Pełczyński [1], the space $C(\alpha_2^\omega)$ is isomorphic to a subspace of $C(\alpha_1)$. Therefore $C(\alpha_1)$ is not isomorphic to a subspace of $C(\alpha_2, E)$ in contradiction with the definition of α_2. Hence $I_1 = I_2$ and the converse is proved. □

The following theorem is a straightforward consequence of the previous one.

Theorem 3.3. *Let $\omega \leq \alpha \leq \beta < \omega_1$ be two countable infinite ordinal numbers. Then $\mathcal{K}(C(\alpha))$ is isomorphic to $\mathcal{K}(C(\beta))$ if, and only if, $\beta < \alpha^\omega$.*

4. Spaces of nuclear operators

Under the conditions $\omega \leq \alpha \leq \beta < \alpha^\omega < \omega_1$ the spaces $\mathcal{N}(C(\alpha))$ and $\mathcal{N}(C(\beta))$ are isomorphic. We shall show conversely that if $\mathcal{N}(C(\alpha))$ and $\mathcal{N}(C(\beta))$ are isomorphic, then $C(\alpha)$ and $C(\beta)$ are also isomorphic.

Lemma 4.1. *Let $\alpha \in \langle \omega, \omega_1)$ and let E be a subspace of $\ell_1(\mathbb{N}, C(\alpha))$ which is not isomorphic to a subspace of $C(\alpha)$. Then there is a subspace of E which is isomorphic to ℓ_1.*

Proof. For every integer k, let π_k be the operator on $\ell_1(\mathbb{N}, C(\alpha))$ defined by $\pi_k(x) = (x_1, \ldots, x_k, 0, 0 \ldots)$ for every $x = (x_i)_{i \geq 1} \in \ell_1(\mathbb{N}, C(\alpha))$. Let $\varepsilon \in \,]0, 1/2[$. Using a standard trick we can find a normalized sequence $(X_n)_{n \geq 1}$ of E and a strictly increasing sequence $(k_n)_{n \geq 1}$ of integers such that $\|X_1 - \pi_{k_1}(X_1)\| \leq \varepsilon$ and

$\|X_n - (\pi_{k_n} - \pi_{k_{n-1}})(X_n)\| \leq \varepsilon$ for every integer $n \geq 2$. Let $Y_1 = \pi_{k_1}(X_1)$ and $Y_n = (\pi_{k_n} - \pi_{k_{n-1}})(X_n)$ for $n = 2, 3, \ldots$. It is obvious that for every integer $n \geq 1$ and for every finite sequence $(a_i)_{1 \leq i \leq n}$ of real numbers we have

$$(1-\varepsilon)\sum_{i=1}^{n}|a_i| \leq \left\|\sum_{i=1}^{n} a_i Y_i\right\| \leq \sum_{i=1}^{n}|a_i|$$

and

$$\left\|\sum_{i=1}^{n} a_i Y_i - \sum_{i=1}^{n} a_i X_i\right\| \leq \varepsilon \sum_{i=1}^{n}|a_i|,$$

so

$$(1-2\varepsilon)\sum_{i=1}^{n}|a_i| \leq \left\|\sum_{i=1}^{n} a_i X_i\right\| \leq (1+\varepsilon)\sum_{i=1}^{n}|a_i|\,.$$

Therefore, the sequence $(X_n)_{n\geq 1}$ is equivalent to the canonical basis of ℓ_1. $\square$

Theorem 4.2. *Let $\omega \leq \alpha \leq \beta < \omega_1$ be two countable infinite ordinal numbers. Then $\mathcal{N}(C(\alpha))$ is isomorphic to $\mathcal{N}(C(\beta))$ if, and only if, $\beta < \alpha^\omega$.*

Proof. For $\alpha \leq \beta < \alpha^\omega$ the spaces $C(\alpha)$ and $C(\beta)$ are isomorphic and so $\mathcal{N}(C(\alpha))$ and $\mathcal{N}(C(\beta))$ are also isomorphic. For the converse, suppose that the spaces $\mathcal{N}(C(\alpha))$ and $\mathcal{N}(C(\beta))$ are isomorphic. Then, the space $C(\beta)$ is isomorphic to a subspace of $\mathcal{N}(C(\alpha)) = \ell_1(\mathbb{N}, C(\alpha))$ and so, by Lemma 4.1, $C(\beta)$ is isomorphic to a subspace of $C(\alpha)$. This implies that $\beta < \alpha^\omega$. $\square$

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