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WINNING TACTICS IN A GEOMETRICAL GAME

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Abstract. A winning tactic for the point-closed slice game in a closed bounded convex set K with Radon-Nikodým property (RNP) is constructed. Consequently a Banach space X has the RNP if and only if there exists a winning tactic in the point-closed slice game played in the unit ball of X. By contrast, there is no winning tactic in the point-open slice game in K. Finally, a more subtle analysis of the properties of the winning tactics leads to a characterization of superreflexive spaces.

1. Introduction

Let K be a set in a real Banach space X. Let $\mathcal{A}$ be a collection of subsets of K such that for every $x \in K$ there exists $A \in \mathcal{A}$ which has $x \in A$ (we will be working with open slices of K ($\mathcal{A} = \mathcal{S}_o$), closed slices of K ($\mathcal{A} = \mathcal{S}_c$), and hyperplane sections of K ($\mathcal{A} = \mathcal{H}$)). We define a game $\mathbf{G}(K, \mathcal{A})$ as follows. There are two players, Player I and Player II. Player I starts the game by choosing an arbitrary point $x_1 \in K$. Player II then chooses a set $A_1 \in \mathcal{A}$ so that $x_1 \in A_1$; then Player I chooses a point $x_2 \in A_1$, and Player II chooses a set $A_2 \in \mathcal{A}$ so that $x_2 \in A_2$; and so on. Summing up, the rules are:

- Player I starts by playing $x_1 \in K$ arbitrarily;
- after x_n has been played, Player II must choose A_n so that $A_n \in \mathcal{A}$ and $x_n \in A_n$;
- after A_n has been played, Player I must play x_{n+1} so that $x_{n+1} \in A_n$.

Formally

$$\mathbf{G}(K, \mathcal{A}) = \{(x, A) \in K^{\mathbb{N}} \times \mathcal{A}^{\mathbb{N}} : x = (x_n),\ A = (A_n) \text{ and } x_n \in A_n \ni x_{n+1} \text{ for all } n \in \mathbb{N}\},$$

and each element of this set is called a *run* of the game. Player II wins the run (x, A) if the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy. Otherwise Player I wins.

J. Malý and M. Zelený [6], who were the first to mention the game, have established that Player II has a *winning strategy* in the *point-line game* $\mathbf{G}(B_{\mathbb{R}^2}, \{\text{lines}\})$. Later R. Deville and É. Matheron [4] have shown that Player II has a winning

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strategy in $\mathbf{G}(B_X, \mathcal{S}_o)$ iff he has a winning strategy in $\mathbf{G}(B_X, \mathcal{S}_c)$ iff he has a winning strategy in $\mathbf{G}(B_X, \mathcal{H})$ iff X has the Radon-Nikodým property (RNP). They also proved that in the case that X is a superreflexive space, Player II has a *winning tactic* (WT) in $\mathbf{G}(B_X, \mathcal{S}_c)$. Zelený [9] even constructed a continuous WT for Player II in the game $\mathbf{G}(B_{\mathbb{R}^n}, \mathcal{H})$. The existence of a winning strategy or tactic for Player II translates easily to a convenient sufficient condition for convergence of bounded sequences in X that has proved useful in applications (see [6, 4, 9, 3]).

In everyday life, there is hardly any difference between the words 'tactic' and 'strategy'. Mathematically, they stand for different concepts. If Player II plays according to a tactic (see Definition 2.1), he decides his next move A_n only taking into account the last move x_n of Player I. If Player II plays according to a strategy, he considers the whole history of Player I's moves $(x_i)_{i=1}^n$ before playing A_n.

In this article, we continue the ideas of [4] and prove that actually the RNP of the space X is sufficient for Player II to have a WT in the games $\mathbf{G}(B_X, \mathcal{S}_c)$ and $\mathbf{G}(B_X, \mathcal{H})$ (Theorem 2.3). Thus we answer positively the question [4, 3.4(1)]. On the other hand, we prove that Player II does not have any WT in the game $\mathbf{G}(B_X, \mathcal{S}_o)$ (Theorem 2.6).

Finally, we study the question for how many steps can Player I keep $\|x_i - x_{i+1}\| \geq \varepsilon$ for some fixed $\varepsilon > 0$. It turns out that each tactic of Player II in $\mathbf{G}(B_X, \mathcal{S}_c)$ allows Player I to do an arbitrary number of such steps if and only if X is not superreflexive (Theorem 2.9).

The main results are collected in Section 2. The construction of the WT and the proof of Theorem 2.9 are done in Section 3.

2. Results

Definition 2.1. We say that a function $t : K \to \mathcal{A}$ is a *tactic for Player II* if $x \in t(x)$ for all $x \in K$. We say that a tactic $t : K \to \mathcal{A}$ for Player II is *winning* if any sequence $(x_n) \subset K$ which satisfies $x_{n+1} \in t(x_n)$ for all $n \in \mathbb{N}$ is necessarily Cauchy.

A *strategy for Player II* is a sequence $(t_n)_{n\in\mathbb{N}}$ where $t_n : D_n \to \mathcal{A}$. The domains D_n are defined inductively by $D_1 = K$ and

$$D_{n+1} = \left\{(x_i)_{i=1}^{n+1} : (x_i)_{i=1}^n \in D_n, x_{n+1} \in t_n(x_1, \dots, x_n)\right\}.$$

Each t_n must satisfy $x_n \in t_n(x_1, \dots, x_n)$ for all $(x_i)_{i=1}^n \in D_n$. A strategy (t_n) for Player II is *winning* if every $(x_n) \subset K$ is Cauchy whenever it satisfies $x_{n+1} \in t_n(x_1, \dots, x_n)$ for all $n \in \mathbb{N}$.

Winning tactics for Player II in $\mathbf{G}(K, \mathcal{A})$ are obviously a subset of winning strategies for Player II in $\mathbf{G}(K, \mathcal{A})$.

We will only deal with the tactics and winning tactics for Player II, so for economic reasons we will not usually mention it.

2.1. Winning tactics and the Radon-Nikodým property.

We will use X for a real Banach space. For $z \in X$ and $r > 0$ we denote $B_X(z, r) = \{x \in X : \|x - z\| \leq r\}$ and $B_X^O(z, r) = \{x \in X : \|x - z\| < r\}$. We abbreviate $B_X = B_X(0, 1)$.

For $f \in X^* \setminus \{0\}$, $a \in \mathbb{R}$ we define an *open halfspace* $H(f, a) = \{x \in X : f(x) > a\}$ and a *closed halfspace* $\overline{H(f, a)} = \{x \in X : f(x) \geq a\}$. If the intersection $K \cap H(f, a)$ is nonempty, we refer to it as the *open slice* of K given by $f \in X^*$ and $a \in \mathbb{R}$. Similarly, the nonempty intersection $K \cap \overline{H(f, a)}$ is the *closed slice* of K given by

f and a. For the sake of completeness, we recall that a *hyperplane section* h of K is a set of the form $h = K \cap (\overline{H(f,a)} \setminus H(f,a))$.

We recall that a set $K \subset X$ has the *Radon-Nikodým property* (RNP) if every subset L of K has nonempty open slices of arbitrarily small diameter. A Banach space X has the RNP if B_X does. Examples of spaces which enjoy this property include reflexive spaces or separable dual spaces (see e.g. [8]).

Let $\mathcal{S}_c$ and $\mathcal{S}_o$ be all the closed and open slices of K, respectively. Let $\mathcal{H}$ be all the hyperplane sections of K. We will consider the games $\mathbf{G}(K, \mathcal{S}_c)$ (*point-closed slice game*), $\mathbf{G}(K, \mathcal{S}_o)$ (*point-open slice game*) and $\mathbf{G}(K, \mathcal{H})$ (*point-hyperplane game*). The following observation is inspired by [4] and offers a different view of the game and tactic business.

Proposition 2.2. *Let K be a subset of X. Then the following are equivalent:*

(1) *Player II has a winning tactic in the game $\mathbf{G}(K, \mathcal{S}_c)$.*
(2) *There exists a function $F : K \to X^*$ such that whenever a sequence $(x_n) \subset K$ satisfies $\langle F(x_n), x_{n+1}\rangle \geq \langle F(x_n), x_n\rangle$ for all $n \in \mathbb{N}$, then (x_n) is Cauchy.*

Proof. The condition (2) is obviously equivalent to $t'(x) := K \cap \overline{H(F(x), F(x)x)}$ being a WT.

On the other hand, a general WT t in $\mathbf{G}(K, \mathcal{S}_c)$ is determined by functions $F : K \to X^*$ and $a : K \to \mathbb{R}$ by means of the equality $t(x) = K \cap \overline{H(F(x), a(x))}$ for all $x \in K$. The function F then satisfies (2). Indeed, we may define $t'(x) := K \cap \overline{H(F(x), F(x)x)}$. Then $x \in t'(x) \subset t(x)$ for all $x \in K$, and thus t' is a WT in $\mathbf{G}(K, \mathcal{S}_c)$. □

Later we will be using tactics of the general form, but because of the above proposition we will never be interested in the function a as much as in the function F.

Our main theorem is the following.

Theorem 2.3. *Let K be a closed convex bounded subset of X and let K have the RNP. Then there exists a winning tactic $t : K \to \mathcal{S}_c$ for Player II in the game $\mathbf{G}(K, \mathcal{S}_c)$.*

Moreover, our particular construction yields a tactic of the form $t(x) = K \cap \overline{H(F(x), F(x)x)}$ where $F : K \to X^*$ is a Baire 1 function. Moreover, given $0 \neq f \in X^*$ and $\varepsilon > 0$, the tactic t may be constructed in such a way that $F(K) \subset B_{X^*}(f, \varepsilon)$.

Corollary 2.4. *If the Banach space X has the RNP, then, for any bounded set $K \subset X$, Player II has a winning tactic in the point-closed slice game $\mathbf{G}(K, \mathcal{S}_c)$.*

Proof. We may suppose that $K \subset B_X$, while Theorem 2.3 provides a WT t for $\mathbf{G}(B_X, \mathcal{S}_c)$. Clearly the restriction $t_{|K}$ (more precisely $x \in K \mapsto t(x) \cap K$) is a WT for $\mathbf{G}(K, \mathcal{S}_c)$. □

Since Deville and Matheron have proved that, for $\Omega \subset X$ with $\operatorname{int} \Omega \neq \emptyset$, the existence of a winning strategy for Player II in the point-hyperplane game in Ω implies the RNP for Ω, we may restate their result (see [4, Theorem 3.4]).

Corollary 2.5. *Let Ω be a bounded subset of X with nonempty interior. Then the following are equivalent:*

(1) *X has the Radon-Nikodým property;*

(2) *Player II has a winning tactic in the point-closed slice game for* Ω*;*
(3) *Player II has a winning tactic in the point-hyperplane game for* Ω*.*

It is neither a coincidence nor a limitation of the method that the tactic is constructed for the game $\mathbf{G}(K, \mathcal{S}_c)$ but not for the game $\mathbf{G}(K, \mathcal{S}_o)$. Indeed, take a look at the next more general result.

Theorem 2.6. *Let* (E, d) *be a nonscattered complete metric space. Let* $\mathcal{A} \subset \{$*open sets*$\}$*. Then there is no winning tactic for Player II in the game* $\mathbf{G}(E, \mathcal{A})$*.*

Remark 2.7. In particular, Player II has no winning tactic in the game $\mathbf{G}(K, \mathcal{S}_o)$ whenever K is nonscattered. This contrasts with the result of Deville and Matheron that Player II has a winning strategy in $\mathbf{G}(K, \mathcal{S}_o)$ provided K has the RNP.

Proof. Since E is nonscattered, it has nonempty perfect part $F \subset E$. We continue by contradiction. Let $t : E \to \mathcal{A}$ be a WT. For $n \in \mathbb{N}$, we denote

$$D_n = \{x \in F : B_E(x, 1/n) \subset t(x)\}.$$

Then $\bigcup D_n = F$ and so, by the Baire category theorem, for some index n the relative (with respect to F) interior of $\overline{D_n}$ is nonempty. Hence there is a relatively open set $G \subset F$ such that $G \cap D_n$ is dense in G and $\operatorname{diam} G < 1/n$. For any $x \in G \cap D_n$, one has $G \subset t(x)$ by the definition of D_n. Also, $G \cap D_n$ is in fact infinite since F is perfect. Player I is therefore recommended to stay in the set $G \cap D_n$, switching there merely between two different points to produce a divergent sequence and the contradiction. □

2.2. **Winning tactics and superreflexivity.** Let t be a winning tactic in $\mathbf{G}(B_X, \mathcal{S}_c)$ and let $\varepsilon > 0$. Since t is winning, there clearly does not exist any infinite sequence $(x_i) \subset B_X$ satisfying

$$x_{i+1} \in t(x_i) \text{ and } \|x_i - x_{i+1}\| \geq \varepsilon \tag{2.1}$$

for all $i \in \mathbb{N}$, but one may ask whether there exists some uniform bound on the length of the sequences that satisfy the above condition. One result in this direction was obtained by Zelený [9] who constructed a WT t in $\mathbf{G}(B_{\mathbb{R}^N}, \mathcal{H})$ with the property that for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that no sequence $(x_i)_{i=1}^m \subset B_{\mathbb{R}^N}$ satisfies (2.1) for all $i < m$.

Definition 2.8. Let $t : B_X \to \mathcal{S}_c$ be a tactic (winning or not) in the game $\mathbf{G}(B_X, \mathcal{S}_c)$. Let $\varepsilon > 0$. We say that t has *uniformly short* ε*-separated runs* if the following holds: there exists $m \in \mathbb{N}$ such that whenever $(x_i)_{i=1}^n \subset B_X$ satisfies (2.1) for all $i < n$, then $n < m$.

Zelený's result therefore reads: there is a winning tactic t in $\mathbf{G}(B_{\mathbb{R}^N}, \mathcal{H})$ which has uniformly short ε-separated runs for every $\varepsilon > 0$.

Our next theorem shows, in particular, that this is not possible in spaces that are not superreflexive. (Superreflexive spaces are those that have an equivalent uniformly convex norm.)

Theorem 2.9. *The following are equivalent:*

(1) X *is superreflexive;*
(2) *for every* $0 < \varepsilon < 1$ *there exists a winning tactic* t_ε *for Player II in* $\mathbf{G}(B_X, \mathcal{S}_c)$ *which has uniformly short* ε*-separated runs.*

3. Construction of winning tactic

In the rest of the article, we will be proving Theorems 2.3 and 2.9. From now on, K will always be a closed convex bounded subset of X that has the RNP even though much of the following would make sense also in more generality.

3.1. **ε-slicings, ε-tactics.** For $\varepsilon > 0$, we will consider the game ε-$\mathbf{G}(K, \mathcal{S}_o)$ with the objective of making the sequence (x_n) ε-Cauchy (i.e. $\|x_n - x_m\| < \varepsilon$ for n, m large enough). A WT in this game will be simply an ε-winning tactic (ε-WT). The WT in $\mathbf{G}(K, \mathcal{S}_c)$ from Theorem 2.3 will be constructed as a limit of a sequence of 2^{-n}-winning tactics.

Definition 3.1. A *slicing* $\mathcal{Z}$ of a convex bounded $L \subset X$ given by the halfspaces $(H(f_\xi, a_\xi))_{\xi \le \eta}$ is a family $(Z_\xi)_{\xi \le \eta}$ of relatively closed convex subsets of L, where η is an ordinal, satisfying:

(a) $Z_{\xi+1} = Z_\xi \setminus H(f_\xi, a_\xi)$;
(b) for each limit ordinal $\lambda \le \eta$, $Z_\lambda = \bigcap_{\xi<\lambda} Z_\xi$;
(c) $Z_0 = L$ and $Z_\eta = \emptyset$.

For $x \in L$, let $\Gamma_{\mathcal{Z}}(x)$ be the unique ordinal $\gamma < \eta$ such that $x \in Z_\gamma \setminus Z_{\gamma+1}$. Also notice that if $\alpha \le \beta$, then $Z_\alpha \supset Z_\beta$.

If moreover $\mathcal{Z}$ has small difference sets, i.e. it satisfies $\operatorname{diam} Z_\xi \setminus Z_{\xi+1} < \varepsilon$ for some $\varepsilon > 0$ and all $\xi < \eta$, we shall call it *ε-slicing*.

The following proposition shows that there is a canonical way of defining an ε-WT once we have an ε-slicing.

Proposition 3.2. *Let $\mathcal{Z}$ be an ε-slicing of K given by the halfspaces $(H(f_\xi, a_\xi))_{\xi \le \eta}$. Then $t_{\mathcal{Z}} : K \to \mathcal{S}_o$ defined as $t_{\mathcal{Z}}(x) := K \cap H(f_{\Gamma_{\mathcal{Z}}(x)}, a_{\Gamma_{\mathcal{Z}}(x)})$ is an ε-winning tactic in ε-$\mathbf{G}(K, \mathcal{S}_o)$.*

Proof. Let x_n be the last move of Player I. Then $H(f_{\Gamma_{\mathcal{Z}}(x_n)}, a_{\Gamma_{\mathcal{Z}}(x_n)}) \cap Z_\beta = \emptyset$ for any $\beta > \Gamma_{\mathcal{Z}}(x_n)$, which shows that $(\Gamma_{\mathcal{Z}}(x_n))_{n=1}^\infty$ is a nonincreasing sequence of ordinals if Player II sticks to the tactic $t_{\mathcal{Z}}$. Hence $(\Gamma_{\mathcal{Z}}(x_n))_{n=1}^\infty$ must be eventually constant or, equivalently, x_n stays eventually in $Z_\xi \setminus Z_{\xi+1}$ for some particular $\xi < \eta$. This difference set has diameter smaller than ε as $\mathcal{Z}$ is an ε-slicing. Thus (x_n) is ε-Cauchy. $\square$

It is useful to notice that if $t_{\mathcal{Z}}(x) = K \cap H(F(x), a(x))$ is a tactic obtained from a slicing $\mathcal{Z}$ as in the previous proposition, then $F : K \to X^*$ is constant on difference sets $Z_\xi \setminus Z_{\xi+1}$ (we say it is a *slice constant mapping*).

3.2. **Refining ε-slicings.** Let us treat the space of all mappings from K to X^* as the power X^{*K}. We recall that the *box topology* [7, page 114] on the product X^{*K} is the one generated by the basis of open sets of the form $\prod_{x \in K} B^O_{X^*}(f_x, r_x)$, where $f_x \in X^*$ and $r_x > 0$ for all $x \in K$. We call these basis sets *open boxes*. The closure of an open box admits the representation $\overline{\prod_{x \in K} U_x} = \prod_{x \in K} \overline{U_x}$.

For the purposes of this article, we will be interested in boxes of a rather special kind.

Definition 3.3. Let $\mathcal{Z} = (Z_\xi)_{\xi \le \eta}$ be a slicing of K and let $t_{\mathcal{Z}}(x) = K \cap H(F(x), a(x))$ be the canonically corresponding tactic. We say that a box U is a *box around* $\mathcal{Z}$ if

$$U = \prod_{x \in K} B^O_{X^*}(F(x), r(x))$$

and if $r : K \to (0, +\infty)$ is constant on the difference sets $Z_\xi \setminus Z_{\xi+1}$; i.e. there exists some transfinite sequence $(r_\xi)_{\xi\le\eta}$ of positive numbers such that $r(x) = r_{\Gamma_{\mathcal{Z}}(x)}$ for every $x \in K$. Remember that, by definition, $F(x) = f_{\Gamma_{\mathcal{Z}}(x)}$, so we may view the box U around $\mathcal{Z}$ as a set-valued mapping that is constant on difference sets $Z_\xi \setminus Z_{\xi+1}$. We will use the term *selection* known from this context.

Definition 3.4. Let $\mathcal{Z} = (Z_\xi)_{\xi\le\eta}$ and $\mathcal{Y} = (Y_\mu)_{\mu\le\lambda}$ be slicings of K. If $\mathcal{Z} \subset \mathcal{Y}$, then we say that $\mathcal{Y}$ is a *refinement* of $\mathcal{Z}$. Further if $\mathcal{Z}$ is a slicing of K and U is a box around $\mathcal{Z}$, then we say that $\mathcal{Y}$ is a *U-refinement* of $\mathcal{Z}$ if $\mathcal{Y}$ is a refinement of $\mathcal{Z}$ and $t_{\mathcal{Y}}(x) = K \cap H(G(x), b(x))$ satisfies $G \in U$.

One can build up refinements in the following manner.

Lemma 3.5. *Let $\mathcal{Z} = (Z_\xi)_{\xi\le\eta}$ be a slicing of K and for every $\xi < \eta$ let $\mathcal{Y}_\xi = (Y_{(\xi,\mu)})_{\mu\le\eta_\xi}$ be a slicing of the difference set $Z_\xi \setminus Z_{\xi+1}$ given by the halfspaces $(H(f_{(\xi,\mu)}, a_{(\xi,\mu)}))_{\mu\le\eta_\xi}$. If*

$$H(f_{(\xi,\mu)}, a_{(\xi,\mu)}) \cap Z_{\xi+1} = \emptyset \text{ for all } \xi < \eta \text{ and } \mu \le \eta_\xi, \tag{3.1}$$

then $\mathcal{Y} = (Y_{(\xi,\mu)} \cup Z_{\xi+1})_{(\xi,\mu)}$ with the lexicographical order on the doubles (ξ, μ) is a refinement of $\mathcal{Z}$.

Proof. This is a straightforward verification of the definition of slicing. It is exactly condition (3.1) that makes it possible to verify property (a) of Definition 3.1. □

We will use refinements in order to achieve two things. The first of them is to make the ε of an ε-slicing smaller. This is the moment when we start making use of the RNP of the set K.

Proposition 3.6. *Let $\varepsilon > 0$. Let $\mathcal{Z}$ be an ε-slicing of K and let U be a box around $\mathcal{Z}$. Then for any $0 < \delta < \varepsilon$ there is a U-refinement $\mathcal{Y}$ of $\mathcal{Z}$ which is a δ-slicing.*

In the proof we will need the following fact.

Fact 3.7. Let $L \subset X$ be a closed convex bounded set with RNP and $L \cap H(f, a)$ a nonempty open slice of it. For any $\delta > 0$, $r > 0$ there is a nonempty slice $L \cap H(g, b) \subset L \cap H(f, a)$ with diameter less than δ and $\|g - f\| < r$.

Proof of the fact. The Bourgain-Phelps theorem [8, Theorem 5.20] claims that for a set L of given properties the set of strongly exposing functionals is dense. So it is enough to choose some strongly exposing g sufficiently close to f and b accordingly. □

Proof of Proposition 3.6. Suppose that $\mathcal{Z} = (Z_\xi)_{\xi\le\eta}$ is given by halfspaces $(H(f_\xi, a_\xi))_{\xi\le\eta}$ and suppose that the box U is given by positive numbers $(r_\xi)_{\xi\le\eta}$. For $\xi < \eta$ fixed, the slicing $(Y_{(\xi,\mu)})_\mu)_{\mu\le\eta_\xi}$ of $Z_\xi \setminus Z_{\xi+1}$ and the corresponding $(H(g_{(\xi,\mu)}, b_{(\xi,\mu)}))_{\mu\le\eta_\xi}$ with $\left\|g_{(\xi,\mu)} - f_\xi\right\| < r_\xi$ are obtained by iterated use of Fact 3.7 in an obvious way. The proof is completed using Lemma 3.5. □

3.3. **Stability.** The second reason for our interest in refinements is that they provide a way to get an additional stability property of ε-WT's. Roughly speaking, the next defined *stable ε-winning tactic* is such an ε-WT whose suitable perturbations are again ε-WT's.

Definition 3.8. An ε-winning tactic $t : K \to \mathcal{S}_o$, $t(x) = K \cap H(F(x), a(x))$ is *stable* if the mapping $F : K \to X^*$ is an interior point of the set

$$W = \{G : K \to X^* \mid x \mapsto K \cap H(G(x), b(x)) \text{ is an } \varepsilon\text{-WT for some } b : K \to \mathbb{R}\}$$

in the box topology on the product ${X^*}^K$.

Let $\mathcal{Z}$ be an ε-slicing of K and let $t_{\mathcal{Z}}$ be the corresponding ε-WT. If $t_{\mathcal{Z}}$ is stable and there exists a box U around $\mathcal{Z}$ such that $F \in U \subset \overline{U} \subset W$, we say that $t_{\mathcal{Z}}$ is *U-stable* and $\mathcal{Z}$ is a *U-stable ε-slicing*. In this case we also call $\overline{U}$ a *stability box* of $t_{\mathcal{Z}}$. This terminology is motivated by the important fact that any selection G from $\overline{U}$ then gives rise to an ε-WT.

Clearly, if $U' \subset U$ are boxes around $\mathcal{Z}$ and $\mathcal{Z}$ is a U-stable ε-slicing, then it is also U'-stable.

We observe that the ε-winning tactics that arise from ε-slicings are close to being stable. In fact, to any ε-slicing there exists a stable refinement.

Proposition 3.9. *Let $\mathcal{Z}$ be an ε-slicing of K. Then there exists an ε-slicing $\mathcal{Y}$ of K which is a V-refinement of $\mathcal{Z}$ for every box V around $\mathcal{Z}$ (i.e. we use the same functionals). Moreover, there exists a box U around $\mathcal{Y}$ such that $\mathcal{Y}$ is U-stable.*

We will lean on the following geometrical fact.

Fact 3.10. Let L be a closed convex bounded subset of X. Let L be sliced by two disjoint parallel hyperplanes into three nonempty disjoint parts. More exactly, let $a > b$ and let $L_1 = L \cap H(f, a) \neq \emptyset$, $L_2 = L \cap H(f, b) \setminus H(f, a)$ and $L_3 = L \setminus H(f, b) \neq \emptyset$. Then there exists $r > 0$ with the property that for every $g \in B_{X^*}(f, r)$ there exists $\alpha \in \mathbb{R}$ such that $L_1 \subset L \cap H(g, \alpha) \subset L_1 \cup L_2$.

Proof of the fact. We may push the scene (i.e. $x \mapsto x - y$ for some y) in order to have $|a| = |b|$ and $0 \in L_2$. Also, since L is bounded, we may suppose without loss of generality that $L_2 \subset B_X$. Let $M = \{x \in B_X : |f(x)| = |a|\}$ (by our assumptions $M \neq \emptyset$) and let $\|f - g\| < |a|$. Then $M \cap \ker g = \emptyset$. Indeed, let $x \in M \cap \ker g$. Then $|a| = |f(x)| = |(f - g)(x)| \leq \|f - g\| < |a|$. We see that $\{g = 0\}$ separates L_1 from L_3. So we may set $r := |a|/2$. Finally, we push the scene back so $\alpha := g(y)$. $\square$

Proof of Proposition 3.9. Suppose that $\mathcal{Z} = (Z_\xi)_{\xi \leq \eta}$ is given by halfspaces $(H(f_\xi, a_\xi))_{\xi \leq \eta}$. Let $\xi < \eta$ be fixed. We will slice up the difference set $Z_\xi \setminus Z_{\xi+1}$ by countably many hyperplanes parallel to $\{f_\xi = a_\xi\}$.

Denote $A = \sup\{f_\xi(x) : x \in Z_\xi\}$ and define a slicing $(Y_{(\xi,n)})_n$ of $Z_\xi \setminus Z_{\xi+1}$ by

$$Y_{(\xi,n)} := Z_\xi \setminus H(f_\xi, \frac{1}{n}A + (1 - \frac{1}{n})a_\xi).$$

So $g_{(\xi,n)} = f_\xi$ and obviously $H(g_{(\xi,n)}, a_{(\xi,n)}) \cap Z_{\xi+1} = \emptyset$. This tells us (using Lemma 3.5) that $(Y_{(\xi,n)})$ with the lexicographical order on the doubles (ξ, n) is an ε-slicing of K. It is of course a V-refinement of $\mathcal{Z}$ for every box V around $\mathcal{Z}$ since $g_{(\xi,n)} = f_\xi$ for all $\xi < \eta$ and $n \in \mathbb{N}$.

In order to prove the stability claim we will show that it is possible to perturb the ε-WT $t_{\mathcal{Y}}$ corresponding to $\mathcal{Y}$ and still get an ε-WT. We start by defining $r_{(\xi,n)} > 0$ using Fact 3.10 with $L = Y_{(\xi,n)}$, $L_1 = L \setminus Y_{(\xi,n+1)}$, $L_3 = Y_{(\xi,n+2)}$, and $L_2 = L \setminus (L_1 \cup L_3)$. We claim that $\mathcal{Y}$ is U-stable with

$$U = \prod_{x \in K} B^O_{X^*}(t_{\mathcal{Y}}(x), r_{\Gamma_{\mathcal{Y}}(x)}).$$

Indeed, suppose that $F : K \to X^*$ is any selection from $\overline{U}$ and consider $x \in K$ such that it is in the difference set $Y_{(\xi,i)} \setminus Y_{(\xi,i+1)}$. Fact 3.10 insures existence of $\alpha(x)$ such that the hyperplane $\{y : F(x)y = \alpha(x)\}$ separates $Y_{(\xi,i)} \setminus Y_{(\xi,i+1)}$ from $Y_{(\xi,i+2)}$. Thus, more importantly, $x \in H(F(x), \alpha(x))$ and $H(F(x), \alpha(x)) \cap Z_{\xi+1} = \emptyset$. We may therefore define $t(x) := K \cap H(F(x), \alpha(x))$ and it will satisfy $x \in t(x) \subset t_{\mathcal{Z}}(x)$. This of course implies that t is an ε-WT since $t_{\mathcal{Z}}$ was. That means that $\mathcal{Y}$ is U-stable. □

3.4. **Induction.** The proof of Theorem 2.3 has an inductive character. Let us isolate the main ingredient of the induction step in the following corollary.

Corollary 3.11. *Let $\mathcal{Z}_1$ be a U_1-stable ε-slicing of K for some box U_1 around $\mathcal{Z}_1$ and for some $\varepsilon > 0$. Then there exists an $\frac{\varepsilon}{2}$-slicing $\mathcal{Z}_2$ of K which is a U_1-refinement of $\mathcal{Z}_1$. Moreover, $\mathcal{Z}_2$ is U_2-stable for some box U_2 around $\mathcal{Z}_2$ and $U_2 \subset U_1$. (See Figure 1.)*

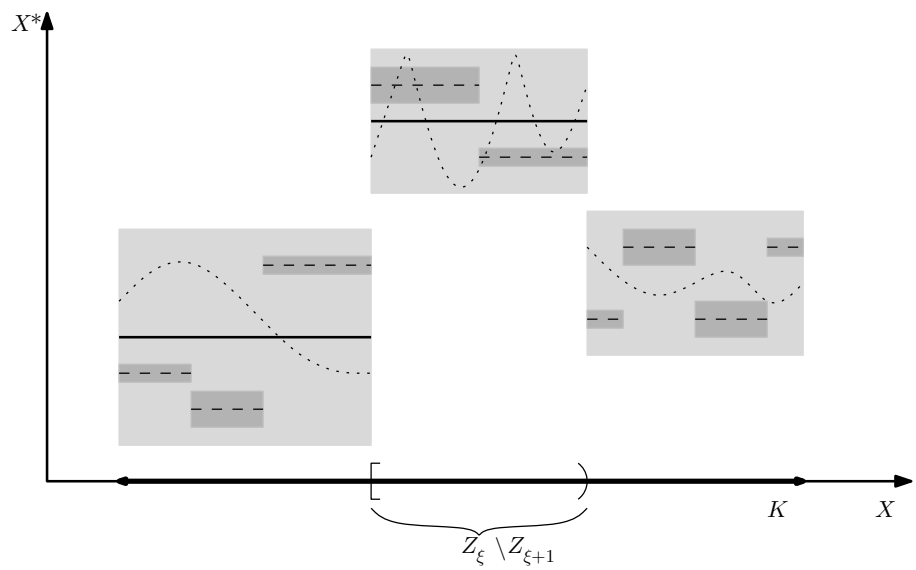

FIGURE 1. The solid line is $t_{\mathcal{Z}_1}$ and the light grey region is its stability box U_1. Any selection (dotted line) from U_1 would correspond to an ε-winning tactic. The dashed line is $t_{\mathcal{Z}_2}$ from Corollary 3.11 with its stability box U_2 drawn in dark grey.

Proof. We may apply Proposition 3.6 to get a U_1-refinement $\mathcal{Y}$ of $\mathcal{Z}_1$ which is an $\frac{\varepsilon}{2}$-slicing of K. Then we refine $\mathcal{Y}$ (using Proposition 3.9) in order to get $\mathcal{Z}_2$, which is a U_2-stable $\frac{\varepsilon}{2}$-slicing of K for some box U_2 around $\mathcal{Z}_2$. Since $\mathcal{Z}_2$ is a V-refinement of $\mathcal{Y}$ for every box V around $\mathcal{Y}$ (says Proposition 3.9), it is a U_1-refinement of the original $\mathcal{Z}_1$. Of course, U_2 may be chosen to satisfy $U_2 \subset U_1$. □

We are now ready to complete the proof of Theorem 2.3.

Proof of Theorem 2.3. Remember that $0 \neq f \in X^*$ and $\varepsilon > 0$ are given. Let us suppose without loss of generality that $K \subset \frac{1}{2}B_X$. Let $\frac{1}{2}B_X \subset H(f, a)$ for some

$a \in \mathbb{R}$. Our induction begins with a trivial observation that $\mathcal{Y} = (K, \emptyset)$ is a 1-slicing of K given by the halfspace $H(f, a)$. We use Proposition 3.9 to get a refinement $\mathcal{Z}_0$ of $\mathcal{Y}$ which is 1-slicing and for which there exists a box $U_0 = \prod_{x \in K} U_0(x)$ around $\mathcal{Z}_0$ such that $\mathcal{Z}_0$ is U_0-stable. We may suppose that $\operatorname{diam} U_0(x) < \varepsilon$ for all $x \in K$.

Let $\mathcal{Z}_n$ be a U_n-stable 2^{-n}-slicing with $\operatorname{diam} U_n(x) < \varepsilon/2^n$ for all $x \in K$. Then we may apply Corollary 3.11 to get a U_n-refinement $\mathcal{Z}_{n+1}$ of $\mathcal{Z}_n$ which is a U_{n+1}-stable 2^{-n-1}-slicing for which the box U_{n+1} satisfies $U_{n+1} \subset U_n$. Moreover, we may suppose that $\operatorname{diam} U_{n+1}(x) < \varepsilon/2^{n+1}$ for all $x \in K$. It is therefore possible to define $t(x) = K \cap \overline{H(F(x), a(x))}$ where $F(x)$ is the unique member of the intersection $\bigcap_{n=1}^{\infty} \overline{U_n(x)}$ and $a(x) = F(x)x$. Now for every $n \in \mathbb{N}$, $F : K \to X^*$ is a selection from the stability box $\overline{U_n}$ of the 2^{-n}-WT $t_{\mathcal{Z}_n}$; thus the mapping $t : x \mapsto K \cap \overline{H(F(x), a(x))}$ is a 2^{-n}-WT itself (in 2^{-n}-$\mathbf{G}(K, \mathcal{S}_c)$). That obviously implies that t is a winning tactic in $\mathbf{G}(K, \mathcal{S}_c)$.

For every $n \in \mathbb{N} \cup \{0\}$, let $t_{\mathcal{Z}_n} = K \cap H(F_n, a_n)$ be the 2^{-n}-WT canonically corresponding to the slicing $\mathcal{Z}_n$. Since $F \in \overline{U_n}$, one has $\sup_{x \in K} \|F(x) - F_n(x)\| \leq \varepsilon/2^n$ for every $n \geq 0$. Thus F is a uniform limit of *slice constant* mappings. By [2, Proposition I.4.5] these mappings are Baire 1, so F is a Baire 1 mapping, too. Since $F_0(x) = f$ for all $x \in K$, we also get that $F(K) \subset B_{X^*}(f, \varepsilon)$. □

Remark 3.12. It is not difficult to observe that our tactic t is continuous with respect to the game; i.e. if (x_n) satisfies $x_{n+1} \in t(x_n)$ and $t(x_n) = K \cap \overline{H(f_n, a_n)}$, then we have both (x_n) and (f_n) convergent.

Remark 3.13. Let X^* be a dual space with the RNP. Let K be a w*-compact convex set in X^* and let $w^*\mathcal{S}_c$ be the closed slices of K given by functionals from X. Then we claim that Player II has a winning tactic in the game $\mathbf{G}(K, w^*\mathcal{S}_c)$. Indeed, the whole proof may be easily rephrased in terms of functionals from X. More attention must be paid only in the proof of Fact 3.7. Still, it is enough to realize that the RNP is equivalent to the w*-dentability in the dual space X^* [8]. Then the w*-version of the Bourgain-Phelps theorem [1, Theorem 3.5.4 (w*)] may be used.

3.5. **Superreflexivity.** Recall that a Banach space X has the *finite tree property* [5], [1, page 34] if for each $0 < \varepsilon < 1$ and each $n \in \mathbb{N}$ there is a bounded *ε-tree of height n*, i.e. a finite sequence $(x_i)_{i=1}^{2^n-1} \subset B_X$ such that

$$x_i = \frac{x_{2i} + x_{2i+1}}{2} \text{ and } \|x_{2i} - x_i\| \geq \varepsilon$$

for $i = 1, \ldots, 2^{n-1} - 1$.

Proof of Theorem 2.9. Let us suppose first that X is not superreflexive, and let t be any (winning or not winning) tactic of Player II. By a result of James [5, Lemma C], X has the finite tree property. So for arbitrary $0 < \varepsilon < 1$ and $n \in \mathbb{N}$, let $T \subset B_X$ be an ε-tree of height n. Clearly, at least one branch of T satisfies the game which Player II played according to his tactic t; i.e. there is a sequence $(y_i)_{i=1}^n$ such that it is a branch of T and $y_{i+1} \in t(y_i)$ for all $i < n$. Being a branch of T, the sequence $(y_i)_{i=1}^n$ also satisfies $\|y_i - y_{i+1}\| \geq \varepsilon$ for all $i < n$. Thus, since ε and n have been arbitrary, t fails (2).

For the converse, fix $\varepsilon > 0$. Let us suppose that $\mathcal{Z}$ is a slicing of B_X such that $t_{\mathcal{Z}}$ has uniformly short ε-separated runs. The limiting process described in the proof of Theorem 2.3 then yields a winning tactic t_ε which satisfies $t_\varepsilon(x) \subset t_{\mathcal{Z}}(x)$ for every $x \in B_X$. Thus t_ε has uniformly short ε-separated runs, too. Therefore, it is enough to show that, for X superreflexive, one may always find such a slicing $\mathcal{Z}$. Indeed, let us suppose without loss of generality that B_X is uniformly convex with the modulus of convexity $\delta(t) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X,\ \|x-y\| \geq t\right\}$. Any slice S of B_X which does not intersect $(1-\delta(\varepsilon))B_X$ has diameter smaller than ε. Similarly, for $n \in \mathbb{N}$,

$$\operatorname{diam}\left((1-\delta(\varepsilon))^n B_X \cap S\right) < \varepsilon$$

whenever $(1-\delta(\varepsilon))^{n+1}B_X \cap S = \emptyset$. On the other hand, there exists $m \in \mathbb{N}$ such that $(1-\delta(\varepsilon))^m < \varepsilon/2$, so all slices of $(1-\delta(\varepsilon))^m B_X$ have automatically diameter smaller than ε. With the help of the separation theorem, there exist a slicing $\mathcal{Z} = (Z_\xi)_{\xi \leq \eta}$ and ordinals $\xi_1 \leq \ldots \leq \xi_m \leq \eta$ such that, for every $n \leq m$,

$$\bigcap_{\lambda \leq \xi_n} Z_\lambda = (1-\delta(\varepsilon))^n B_X$$

is satisfied. Now let Player II play according to the tactic $t_{\mathcal{Z}}$ and let x_i, x_{i+1} be consecutive moves of Player I such that $\|x_i - x_{i+1}\| \geq \varepsilon$. If $x_i \in (1-\delta(\varepsilon))^m B_X$, then clearly $x_{i+1} \notin (1-\delta(\varepsilon))^m B_X$. If $x_i \in (1-\delta(\varepsilon))^n B_X \setminus (1-\delta(\varepsilon))^{n+1} B_X$ for some $n < m$, then $t_{\mathcal{Z}}(x_i) \cap (1-\delta(\varepsilon))^{n+1} B_X = \emptyset$. Since $x_{i+1} \in t_{\mathcal{Z}}(x_i)$ and $\operatorname{diam}\left((1-\delta(\varepsilon))^n B_X \cap t_{\mathcal{Z}}(x_i)\right) < \varepsilon$, we conclude that $x_{i+1} \notin (1-\delta(\varepsilon))^n B_X$. One can see that ε-separated runs of the game cannot be longer than $m+1$ steps. $\square$

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