PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 3, March 2009, Pages 1073–1080
S 0002-9939(08)09722-0
Article electronically published on October 22, 2008

AHS–STRUCTURES AND AFFINE HOLONOMIES

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(Communicated by Jon G. Wolfson)

Abstract. We show that a large class of non-metric, non-symplectic affine holonomies can be realized, uniformly and without case by case considerations, by Weyl connections associated to the natural AHS–structures on certain generalized flag manifolds.

1. Introduction

The classification of the possible irreducible holonomies of non-locally symmetric torsion-free affine connections is a cornerstone in differential geometry. A list of possible holonomy Lie algebras was compiled by M. Berger (see [1]), and later a few small corrections and several extensions to this list were found. It took quite a long time until the existence of all these holonomies was proved, often case by case and uncovering interesting relations to several areas of differential geometry and related fields. The program was finally completed by S. Merkulov and L. Schwachhöfer in [7].

Apart from the case of metric holonomies (i.e. holonomies of connections admitting a parallel pseudo-Riemannian metric), the classification of possible affine holonomies has surprisingly close relations to the classification of certain types of symmetric spaces or, equivalently, of certain types of parabolic subalgebras in simple Lie algebras, respectively certain generalized flag manifolds. In the original existence proofs for holonomies, these relations were not exploited systematically. For the case of symplectic holonomies (i.e. holonomies of connections admitting a parallel symplectic form) this was done later by M. Cahen and L. Schwachhöfer in [2]. There the authors construct special symplectic connections starting from certain generalized flag manifolds. This not only provides examples of all symplectic holonomies, but locally produces all connections with such holonomies.

The aim of this article is to give a conceptual proof of existence of affine holonomies, which exploits the relation to parabolic subalgebras, in an easier case, namely for the holonomies related to the classification of Hermitian symmetric spaces. Here the simplest description is in terms of so-called $|1|$–graded simple Lie algebras. For a $|1|$–grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of a simple Lie algebra $\mathfrak{g}$, the subspace $\mathfrak{g}_0$ turns

Received by the editors April 22, 2008.
2000 *Mathematics Subject Classification.* Primary 53C29, 53C10; Secondary 53C15, 53C30, 53B15.
The author was supported by project P 19500–N13 of the "Fonds zur Förderung de wissenschaftlichen Forschung" (FWF).

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out to be a reductive subalgebra with a one-dimensional center. The adjoint action defines a representation of $\mathfrak{g}_0$ on the vector space $\mathfrak{g}_{-1}$. Our main result is:

Theorem 1. *For any $|1|$–graded simple Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the adjoint representations of $\mathfrak{g}_0$ and of its semisimple part on $\mathfrak{g}_{-1}$ can be realized as holonomy Lie algebras of torsion free, non-locally symmetric affine connections on a compact manifold.*

The classification of $|1|$–gradings on simple Lie algebras and hence of the holonomies covered by this construction is well known and can be found in Table 1 below. Note that in contrast to the results of [2] for symplectic holonomies, which are local in nature, we obtain global connections on compact manifolds, indeed on generalized flag manifolds.

A Lie group with $|1|$–graded simple Lie algebra determines a geometric structure, namely a first order structure with a certain structure group G_0, which can be canonically prolonged to a normal Cartan geometry. These are the so-called AHS–structures, with conformal structures providing the prototypical example. Any structure of this type (as well as the more general parabolic geometries) comes with a class of distinguished affine connections, called Weyl connections. The theory of Weyl structures (which are equivalent descriptions of the Weyl connections) as developed in [4] provides the main technical input for our results.

2. AHS–structures and Weyl connections

A $|1|$–grading on a simple Lie algebra $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ which defines a grading of $\mathfrak{g}$, i.e. is such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ where $\mathfrak{g}_k = 0$ for $k \notin \{-1, 0, 1\}$. We will always assume that the rank of $\mathfrak{g}$ is greater than one, so $\dim(\mathfrak{g}_{-1}) > 1$. It turns out that the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is always reductive with a one-dimensional center. The semisimple part of $\mathfrak{g}_0$ will be denoted by $\mathfrak{g}_0^{ss}$. By the grading property, the adjoint representation of $\mathfrak{g}$ restricts to representations of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$. These representations are always irreducible and dual to each other via the Killing form of $\mathfrak{g}$. Finally, $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a maximal parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{g}_1$.

$|1|$–gradings on simple Lie algebras are closely related to Hermitian and para–Hermitian symmetric spaces, as well as to questions of Lie algebras with non-trivial prolongations. The full classification is well known; see e.g. [6].

Proposition 2. *Table 1 lists all real and complex simple $|1|$–graded Lie algebras $\mathfrak{g}$ of rank greater than one, together with the subalgebras $\mathfrak{g}_0$ and the representations on $\mathfrak{g}_{-1}$, with $\mathbb{K}$ denoting $\mathbb{R}$ or $\mathbb{C}$.*

The algebras $\mathfrak{g}_0$ in Table 1 and their semisimple parts exhaust all of Berger's original list of non-metric holonomies (Table 2 of [7]) except the full symplectic algebras, as well as some exotic holonomies (Table 3 of [7]).

Suppose that we have given a $|1|$–graded simple Lie algebra $\mathfrak{g}$ and a Lie group G with Lie algebra $\mathfrak{g}$. Then there are natural subgroups $G_0 \subset P \subset G$ with Lie algebras $\mathfrak{g}_0$ and $\mathfrak{p}$. Namely, we let P consist of all elements whose adjoint actions preserve $\mathfrak{p}$ and $\mathfrak{g}_1$, while the adjoint actions of elements of G_0 preserve any of the components $\mathfrak{g}_i$, $i = 1, 0, 1$. It turns out that the exponential mapping restricts to a diffeomorphism from $\mathfrak{g}_1$ onto a closed normal subgroup $P_+ \subset P$ and that P is the semidirect product of G_0 and P_+.

TABLE 1. Real and complex $|1|$–graded simple Lie algebras

$\mathfrak{g}$	$\mathfrak{g}_0$	$\mathfrak{g}_{-1}$
$\mathfrak{sl}(n+1,\mathbb{K})$, $n \geq 2$	$\mathfrak{gl}(n,\mathbb{K})$	$\mathbb{K}^n$
$\mathfrak{sl}(p+q,\mathbb{K})$, $p,q \geq 2$	$\mathfrak{s}(\mathfrak{gl}(p,\mathbb{K}) \times \mathfrak{gl}(q,\mathbb{K}))$	$\mathbb{K}^{p*} \otimes \mathbb{K}^q$
$\mathfrak{sl}(p+q,\mathbb{H})$, $p,q \geq 1$	$\mathfrak{s}_{\mathbb{R}}(\mathfrak{gl}(p,\mathbb{H}) \times \mathfrak{gl}(q,\mathbb{H}))$	$L_{\mathbb{H}}(\mathbb{H}^p,\mathbb{H}^q)$
$\mathfrak{su}(p,p)$, $p \geq 2$	$\mathfrak{s}_{\mathbb{R}}(\mathfrak{gl}(p,\mathbb{C}))$	$\mathfrak{u}(p)$
$\mathfrak{sp}(2n,\mathbb{K})$, $n \geq 3$	$\mathfrak{gl}(n,\mathbb{K})$	$S^2\mathbb{K}^n$
$\mathfrak{so}(p+1,q+1)$, $p+q \geq 3$	$\mathfrak{cso}(p,q)$	$\mathbb{R}^{p+q}$
$\mathfrak{so}(n+2,\mathbb{C})$, $n \geq 3$	$\mathfrak{cso}(n,\mathbb{C})$	$\mathbb{C}^n$
$\mathfrak{so}(n,n)$, $n \geq 4$	$\mathfrak{gl}(n,\mathbb{R})$	$\Lambda^2\mathbb{R}^n$
$\mathfrak{so}(2n,\mathbb{C})$, $n \geq 4$	$\mathfrak{gl}(n,\mathbb{C})$	$\Lambda^2\mathbb{C}^n$
$\mathfrak{so}^*(4n)$, $n \geq 2$	$\mathfrak{gl}(n,\mathbb{H})$	$\Lambda^2\mathbb{H}^n$
EI (split E_6)	$\mathfrak{cspin}(5,5)$	$\mathbb{R}^{16}$
EIV (non-split E_6)	$\mathfrak{cspin}(9,1)$	$\mathbb{R}^{16}$
E_6 (complex)	$\mathfrak{cspin}(10,\mathbb{C})$	$\mathbb{C}^{16}$
EV (split E_7)	$EI \oplus \mathbb{R}$	$\mathbb{R}^{27}$
$EVII$ (non-split E_7)	$EIV \oplus \mathbb{R}$	$\mathbb{R}^{27}$
E_7 (complex)	$E_6 \oplus \mathbb{C}$	$\mathbb{C}^{27}$

The representation $G_0 \to GL(\mathfrak{g}_{-1})$ obtained from the adjoint representation is infinitesimally injective, so the notion of a first order structure with structure group G_0 makes sense on manifolds of dimension $\dim(\mathfrak{g}_{-1})$. The structures obtained in that way are called *AHS–structures*, *generalized conformal structures*, *irreducible parabolic geometries*, or *abelian parabolic geometries* in the literature. It turns out that they are equivalent to normal Cartan geometries of type (G,P); see e.g. [5].

The *homogeneous model* of the AHS–structure of type (G,P) is the homogeneous space G/P. Here the first order G_0–structure is given by the canonical projection $G/P_+ \to G/P$, which is a principal bundle with structure group $P/P_+ = G_0$, and the soldering form induced by the Maurer–Cartan form on G. The corresponding Cartan geometry is the natural principal P–bundle $G \to G/P$ with the Maurer–Cartan form as the Cartan connection. This Cartan connection is flat by the Maurer–Cartan equation.

On any manifold endowed with an AHS–structure, there is a family of preferred principal connections on the principal bundle defining the G_0–structure. The theory of these connections is developed in [4]. Let us briefly describe the case of conformal structures, which motivated the whole theory.

Let M be a smooth manifold of dimension ≥ 3 endowed with a conformal class $[\gamma]$ of Riemannian metrics. A *Weyl connection* on TM is a torsion free linear connection ∇ such that $\nabla_\xi \gamma = f\gamma$ for one (or equivalently any) metric γ in the conformal class and any vector field $\xi \in \mathfrak{X}(M)$. Here f is some smooth function (depending on ξ). The Levi–Civita connection of any of the metrics in the conformal class is an example of a Weyl connection, but these do not exhaust all Weyl connections. The Weyl connections can equivalently be considered as principal connections on the conformal frame bundle.

One can consider the conformal class of metrics as a ray subbundle in S^2T^*M, which can be viewed as a principal bundle with structure group $\mathbb{R}_+$. This is called

the bundle of scales. Any Weyl connection induces a principal connection on the bundle of scales, and it turns out that this induces a bijection between the set of Weyl connections and the set of all principal connections on the bundle of scales. The Levi–Civita connections of metrics in the conformal class exactly correspond to the flat connections induced by global sections of the bundle of scales. The space of principal connections on the bundle of scales is an affine space modelled on the vector space $\Omega^1(M)$ of one-forms on M, so one can carry over the the affine structure to the space of all Weyl connections (obtaining a non-trivial action of one-forms on Weyl connections). The subspace of Levi–Civita connections thereby becomes an affine space modelled on the space of exact one-forms. The totally trace-free part of the curvature is the same for all Weyl connections. This is the Weyl curvature of the conformal structure. The trace part of the curvature of a Weyl connection is best described by the *Rho tensor* (a trace adjusted version of the Ricci curvature).

Now all this generalizes to all AHS–structures (and further to parabolic geometries). One always has a distinguished family of principal connections on the bundle defining the G_0–structure, which is in bijective correspondence with the space of all connections on a principal $\mathbb{R}_+$–bundle, and hence forms an affine space modelled on one-forms. These are called *Weyl structures* or *Weyl connections*. The Weyl structures coming from global sections of the $\mathbb{R}_+$–bundle are called *exact*. It will be important in the sequel that (as for conformal structures) exact Weyl connections actually are induced from a further reduction of the principal bundle defining the G_0–structure. The structure group of this reduced bundle has Lie algebra $\mathfrak{g}_0^{ss}$, the semisimple part of $\mathfrak{g}_0$. The exact Weyl structure forms an affine space modelled on the space of exact one-forms.

In the case of AHS–structures, the set of all Weyl connections is easy to describe. There is a basic invariant of such a geometry called the harmonic torsion, and the Weyl connections are exactly those connections on the principal bundle defining the G_0–structure which have that torsion. (For some geometries, like conformal structures, this torsion always has to vanish.) Finally, any Weyl structure comes with a Rho tensor, a T^*M–valued one-form which describes a part of the curvature of the Weyl connection. For our purposes, the following special case of these facts will be sufficient.

Proposition 3. *For the homogeneous model G/P of an AHS–structure, the Weyl connections are exactly the torsion-free linear connections on $T(G/P)$ which are induced from the principal G_0–bundle $G/P_+ \to G/P$. Any Weyl connection has holonomy Lie algebra contained in $\mathfrak{g}_0$, and for exact Weyl connections the holonomy Lie algebra is even contained in $\mathfrak{g}_0^{ss}$.*

An important feature of the theory of Weyl structures is that there is an explicit description of the behavior of all relevant quantities under a change of Weyl structure, which is valid for all the geometries in question. This needs some tensorial maps coming from the AHS–structure.

For a G_0–structure $E \to M$ one by definition has $TM \cong E \times_{G_0} \mathfrak{g}_{-1}$. We have already noted that $\mathfrak{g}_1$ is dual to $\mathfrak{g}_{-1}$ as a G_0–representation, so $T^*M \cong E \times_{G_0} \mathfrak{g}_1$. Finally, $\mathfrak{g}_0$ can be viewed as a Lie subalgebra of $L(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$, so $E \times_{G_0} \mathfrak{g}_0$ can be naturally viewed as a subbundle $\mathrm{End}_0(TM)$ of $L(TM, TM)$. Now the Lie bracket on $\mathfrak{g}$ is a G_0–equivariant map, so passing to associated bundles, the components of this

bracket induce tensorial maps $TM \times T^*M \to \mathrm{End}_0(TM)$, $\mathrm{End}_0(TM) \times TM \to TM$ and $\mathrm{End}_0(TM) \times T^*M \to T^*M$, all of which we denote by $\{\ ,\ \}$.

In terms of these brackets, one can now easily write a formula for the affine structure on Weyl connections which is uniform for all the AHS–structures as well as a formula for the change of the Rho tensor. The change of Weyl connections is most conveniently expressed in terms of linear connections on associated vector bundles. Given a representation V of G_0 and the corresponding vector bundle $F := E \times_{G_0} V \to M$, we find that the infinitesimal representation of $\mathfrak{g}_0$ on V induces a bilinear bundle map $\bullet : \mathrm{End}_0(TM) \times F \to F$. The following formulae are taken from [4, 3.6].

Proposition 4. *Let $E \to M$ be a first order G_0–structure, fix a Weyl structure and denote the induced connections on all associated bundles by ∇. Let $\Upsilon \in \Omega^1(M)$ be a one-form and let us indicate by hats the quantities associated to the Weyl structure obtained by modifying the initial structure by Υ. Then for any vector field $\xi \in \mathfrak{X}(M)$ we have:*

(1) *The modified Weyl connection on an associated vector bundle $F = E \times_{G_0} V$ is given by*

$$\hat{\nabla}_\xi s = \nabla_\xi s - \{\Upsilon, \xi\} \bullet s,$$

for all $s \in \Gamma(F)$.

(2) *The Rho tensor of the modified Weyl connection is given by*

$$\hat{\mathsf{P}}(\xi) = \mathsf{P}(\xi) + \nabla_\xi \Upsilon + \tfrac{1}{2}\{\Upsilon, \{\Upsilon, \xi\}\}.$$

3. Realizing affine holonomies

As we have seen in Proposition 3, for any Weyl connection (respectively exact Weyl connection) on the homogeneous model G/P of an AHS–structure, the holonomy Lie algebra is contained in $\mathfrak{g}_0$ (respectively $\mathfrak{g}_0^{ss}$). Our aim is to show that there are Weyl connections for which the holonomy Lie algebras equal these two subalgebras. Our strategy will be to first construct an exact Weyl connection on G/P which is flat on an open neighborhood of $o = eP \in G/P$. Then we exploit the affine structure on the spaces of Weyl connections, respectively exact Weyl connections. We construct a one-form, respectively an exact one-form, such that modifying by these one-forms one obtains connections with the full holonomy Lie algebra. In view of the explicit formulae for the changes of the data associated to a Weyl structure under a change of Weyl structures, this can be deduced from rather simple algebraic facts.

Proposition 5. *Let G be a Lie group with $|1|$–graded simple Lie algebra $\mathfrak{g}$ and let $G_0 \subset P \subset G$ be the subgroups determined by the grading. Then there is a globally defined Weyl connection ∇ for G/P which is flat with identically vanishing Rho tensor locally around $o = eP \in G/P$.*

Proof. By Proposition 3.2 of [4] there always exist global exact Weyl connections. On the other hand, since $\mathfrak{g}_{-1}$ is transverse to $\mathfrak{p}$ in $\mathfrak{g}$, there is an open neighborhood U of 0 in $\mathfrak{g}_{-1}$ such that $X \mapsto \exp(X)g$ defines diffeomorphisms from $U \times P$ to an open neighborhood of P in G as well as from $U \times G_0$ to an open neighborhood of $G_0 = P/P_+$ in G/P_+. Via these diffeomorphisms, the inclusion $G_0 \hookrightarrow P$ defines a G_0–equivariant smooth section of $G \to G/P_+$, and hence a local Weyl structure as defined in [4], over $V := \exp(U)P \subset G/P$. (In fact, one may take $U = \mathfrak{g}_{-1}$, but

this is not necessary for our purposes.) The local trivialization of G/P_+ over V gives rise to a local trivialization of any bundle of scales, and by construction the corresponding section of that bundle is parallel for the induced Weyl connection, so we have constructed an *exact* local Weyl structure.

Applying Proposition 2.10 of [3] to the opposite grading, we see that $\exp(\mathfrak{g}_{-1})G_0$ is a closed subgroup of G (the parabolic subgroup opposite to P). Our section thus is simply the restriction of the inclusion of this subgroup to some open subset, so the pull-back of the Maurer–Cartan form of G along this inclusion is the Maurer–Cartan form of the subgroup. Hence it has values in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, which implies that the Rho tensor vanishes identically, and the Maurer–Cartan equation shows that the resulting connection is flat.

In view of the affine structure on exact Weyl connections, there is an exact one-form df on V, which describes the change from the restriction of the initial global Weyl connection to our local flat Weyl connection. Now multiply f by a smooth function with support in V which is identically one locally around o. Then the result can be extended by zero to a globally defined smooth function on G/P, and the corresponding exact one-form transforms the initial global Weyl connection to an exact Weyl connection with the required properties. □

The classical example for this construction is again provided by conformal structures. Here G/P is the sphere S^n, and one may use the round metric as the initial exact Weyl structure. The local flat Weyl structure is obtained by pulling back the flat metric on $\mathbb{R}^n$ via stereographic projection, which is well known to be conformal. Then one glues the two metrics using a partition of unity.

Now let ∇ be a Weyl connection as described in Proposition 5; i.e. the curvature R and the Rho tensor P of ∇ both vanish identically on an open neighborhood U of $o = eP \in G/P$. Now let us change the Weyl connection by a one-form $\Upsilon \in \Omega^1(G/P)$ and let us indicate the resulting quantities by hats. Using that the geometry on G/P is flat, Proposition 4.3 of [4] shows that on U we have

$$\hat{R}(\xi,\eta) = (\partial\hat{\mathsf{P}})(\xi,\eta) = \{\hat{\mathsf{P}}(\xi),\eta\} - \{\hat{\mathsf{P}}(\eta),\xi\}.$$

On the other hand, $\hat{\mathsf{P}}$ is determined by part (2) of Proposition 4. Now the bundle map

$$\partial : T^*(G/P) \otimes T^*(G/P) \to \Lambda^2 T^*(G/P) \otimes \mathrm{End}_0(T(G/P))$$

is induced by a G_0–equivariant map on the representation spaces inducing the bundles (which we will denote by the same letter), so it is parallel for any Weyl connection. This implies that on U we get $\hat{\nabla}^k\hat{R} = (\mathrm{id}\otimes\partial)(\hat{\nabla}^k\hat{\mathsf{P}})$. Now we are ready to state the main algebraic input.

Lemma 6. *The map* $(\mathsf{P},X,Y) \mapsto \partial\mathsf{P}(X,Y)$ *induces surjections* $S^2\mathfrak{g}_1\otimes\mathfrak{g}_{-1}\otimes\mathfrak{g}_{-1} \to \mathfrak{g}_0^{ss}$ *and* $\mathfrak{g}_1\otimes\mathfrak{g}_1\otimes\mathfrak{g}_{-1}\otimes\mathfrak{g}_{-1} \to \mathfrak{g}_0$.

Proof. Both maps are evidently $\mathfrak{g}_0$–equivariant, so it suffices to show that their images meet each irreducible component of the target space. Since $\dim(\mathfrak{g}_{-1}) > 1$, we can choose linearly independent elements $X, Y \in \mathfrak{g}_{-1}$ and a linear map $\mathsf{P} : \mathfrak{g}_{-1} \to \mathfrak{g}_1$ such that $\mathsf{P}(Y) = 0$ and $B(\mathsf{P}(X),Y) \neq 0$, where B denotes the Killing form on $\mathfrak{g}$. Now it is well known that the center of $\mathfrak{g}_0$ is generated by the grading element E, i.e. the element whose adjoint action on each $\mathfrak{g}_j$ is multiplication by j, and that the decomposition $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathbb{K}E$ is orthogonal for B. Now, by construction,

$\partial\mathsf{P}(X,Y) = -[Y,\mathsf{P}(X)]$ and hence

$$B(\partial\mathsf{P}(X,Y),E) = B(\mathsf{P}(X),[Y,E]) = B(\mathsf{P}(X),Y) \neq 0.$$

Hence the image of the second map is not contained in $\mathfrak{g}_0^{ss}$, so it suffices to prove surjectivity of the first map.

Complexifying if necessary, we may assume that $\mathfrak{g}$ is a complex $|1|$–graded simple Lie algebra, and then we can use the root decomposition. (See [8] for the description of $|1|$–gradings in terms of roots.) There is a unique simple root α such that the root space $\mathfrak{g}_\alpha$ is contained in $\mathfrak{g}_1$. Now the Dynkin diagram of $\mathfrak{g}_0^{ss}$ is obtained by removing the node corresponding to α and all edges connecting to this node in the Dynkin diagram of $\mathfrak{g}$. Hence any simple ideal of $\mathfrak{g}_0^{ss}$ contains the root space $\mathfrak{g}_\beta$ for some simple root β such that the nodes corresponding to α and β in the Dynkin diagram of $\mathfrak{g}$ are connected. Equivalently, this means that β is not orthogonal to α and hence $\alpha+\beta$ is a root. By construction, the root space $\mathfrak{g}_{\alpha+\beta}$ is contained in $\mathfrak{g}_1$. Now choose a basis $\{X_\gamma\}$ of $\mathfrak{g}_{-1}$ such that each X_γ lies in the root space $\mathfrak{g}_{-\gamma}$ and let $\{Z_\gamma\}$ be the dual basis of $\mathfrak{g}_1$. Then for $\mathsf{P} = Z_\alpha^2 + Z_{\alpha+\beta}^2 \in S^2\mathfrak{g}_1$ we get

$$\partial\mathsf{P}(X_\alpha, X_{\alpha+\beta}) = [X_\alpha, Z_{\alpha+\beta}] - [X_{\alpha+\beta}, Z_\alpha].$$

This is the sum of a non-zero element of $\mathfrak{g}_\beta$ and a non-zero element of $\mathfrak{g}_{-\beta}$, so the image of our map meets the simple ideal containing $\mathfrak{g}_\beta$. □

Lemma 7. *There is an element of $S^6\mathfrak{g}_{-1}^*$ which, interpreted as a linear map $S^4\mathfrak{g}_{-1} \to S^2\mathfrak{g}_{-1}^*$, is surjective. Likewise, there is an element in $S^5\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^*$ which is surjective when viewed as a map $S^4\mathfrak{g}_{-1} \to \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^*$.*

Proof. Let $\{e_i\}$ be a basis of $\mathfrak{g}_{-1}$ with dual basis $\{\lambda_i\}$, and consider $\sum_{i<j} \lambda_i^4\lambda_j^2 \in S^6\mathfrak{g}_{-1}^*$. Then for $i < n = \dim(\mathfrak{g}_{-1})$, we obtain λ_i^2 as the image of $e_i^2 e_{i+1}^2$, while λ_n^2 is the image of e_{n-1}^4. For $i<j$ we get $\lambda_i\lambda_j$ as the image of $e_i^3 e_j$.

On the other hand, consider $\sum_{i,j} \lambda_i^3\lambda_j^2 \otimes \lambda_i \in S^5\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^*$. The corresponding map sends e_i^4 to $\lambda_i \otimes \lambda_i$ and $e_i^3 e_j$ to $\lambda_j \otimes \lambda_i$. □

Theorem 8. *The space G/P admits global Weyl connections with holonomy Lie algebra $\mathfrak{g}_0$ as well as global exact Weyl connections with holonomy Lie algebra $\mathfrak{g}_0^{ss}$.*

Proof. Let ∇ be an exact Weyl connection on G/P as in Proposition 5, and change it by $\Upsilon \in \Omega^1(M)$. Then we have noted already that $\hat\nabla^k\hat R(o) = (\mathrm{id}\otimes\partial)(\hat\nabla^k\hat{\mathsf{P}}(o))$. Now suppose in addition that Υ has vanishing k–jet in o. Then part (2) of Proposition 4 shows that $j_o^{k-1}\hat{\mathsf{P}} = 0$ and that $\hat\nabla^k\hat{\mathsf{P}}(o) = \hat\nabla^k\nabla\Upsilon(o)$. Part (1) of Proposition 4 inductively shows that $\hat\nabla^k\nabla\Upsilon(o) = \nabla^{k+1}\Upsilon(o)$. Of course we can find a global smooth one-form Υ with $j_o^k\Upsilon = 0$ such that $\nabla^{k+1}\Upsilon(o)$ (which is automatically symmetric in the first $k+1$ entries by flatness of ∇) is any prescribed element of $S^{k+1}T_o(G/P)^* \otimes T_o(G/P)^*$. Likewise, this can be done for an exact one-form with any prescribed element of $S^{k+2}T_o(G/P)^*$.

Doing this for $k=4$ with elements which (via the isomorphism $T_o(G/P) \cong \mathfrak{g}_{-1}$) correspond to the ones described in Lemma 7, Lemma 6 shows that we get a Weyl connection for which the values of $\hat\nabla^4\hat R$ fill all of $\mathfrak{g}_0$ and an exact Weyl connection for which these values fill all of $\mathfrak{g}_0^{ss}$. Since these values are well known to lie in the holonomy Lie algebra, this completes the proof. □

This evidently implies Theorem 1 from the introduction.

Remark. Let M be a smooth manifold and let ∇ be a torsion-free linear connection on TM whose holonomy Lie algebra is contained in one of the Lie algebras $\mathfrak{g}_0$ from Table 1. If we further assume that there is a group G such that the holonomy group of ∇ is contained in the corresponding subgroup G_0 (which usually is no restriction), then ∇ is induced from a first order G_0–structure $E \to M$. It is a general fact that the harmonic torsion of such a structure which was mentioned in Section 2 can be obtain as a certain component of the torsion of any connection on the bundle. Hence we see that $E \to M$ has to have vanishing harmonic torsion, so ∇ is a Weyl connection for an AHS–structure.

For many choices of a $|1|$–graded Lie algebra $\mathfrak{g}$, vanishing of the harmonic torsion of an AHS–structure already implies local flatness, i.e. local isomorphism to the homogeneous model G/P. Indeed, the only instances in Table 1 for which there exist non-flat torsion-free geometries are the first and fifth lines, the second and fourth lines for $p = 2$ and the third line for $p = 1$. Hence in all other cases we are, at least up to local isomorphism, in the situation of a Weyl connection on G/P.

Acknowledgements

The idea to use Weyl connections for AHS–structures to realize special holonomies was originally brought up several years ago by J. Slovák and D. Alekseevsky. Conversations with S. Armstrong and J. Slovák have been very helpful.

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