A SIMPLIFIED CALCULATION FOR
THE FUNDAMENTAL SOLUTION TO THE HEAT EQUATION
ON THE HEISENBERG GROUP

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ABSTRACT. Let $\mathcal{L}_\gamma = -\frac{1}{4}\left(\sum_{j=1}^n (X_j^2 + Y_j^2) + i\gamma T\right)$ where $\gamma \in \mathbb{C}$, and X_j, Y_j and T are the left-invariant vector fields of the Heisenberg group structure for $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. We explicitly compute the Fourier transform (in the spatial variables) of the fundamental solution of the heat equation $\partial_s \rho = -\mathcal{L}_\gamma \rho$. As a consequence, we have a simplified computation of the Fourier transform of the fundamental solution of the $\Box_b$-heat equation on the Heisenberg group and an explicit kernel of the heat equation associated to the weighted $\overline{\partial}$-operator in $\mathbb{C}^n$ with weight $\exp(-\tau P(z_1, \ldots, z_n))$, where $P(z_1, \ldots, z_n) = \frac{1}{2}(|\operatorname{Im} z_1|^2 + \cdots + |\operatorname{Im} z_n|^2)$ and $\tau \in \mathbb{R}$.

0. INTRODUCTION

The purpose of this note is to present a simplified calculation of the Fourier transform of the fundmental solution of the $\Box_b$-heat equation on the Heisenberg group. The Fourier transform of the fundamental solution has been computed by a number of authors [Gav77, Hul76, CT00, Tie06]. We use the approach of [CT00, Tie06] and compute the heat kernel using Hermite functions but differ from the earlier approaches by working on a different, though biholomorphically equivalent, version of the Heisenberg group. The simplification in the computation occurs because the differential operators on this equivalent Heisenberg group take on a simpler form. Moreover, in the proof of Theorem 1.2, we reduce the n-dimensional heat equation to a 1-dimensional heat equation, and this technique would also be useful when analyzing the heat equation on the nonisotropic Heisenberg group (e.g., see [CT00]). We actually use the same version of the Heisenberg group as Hulanicki [Hul76], but he computes the fundamental solution of the heat equation associated to the sub-Laplacian and not the Kohn Laplacian acting on $(0, q)$-forms.

A consequence of our fundamental solution computation is that we can explicitly compute the heat kernel associated to the weighted $\overline{\partial}$-problem in $\mathbb{C}^n$ when the weight is given by $\exp(-\tau P(z_1, \ldots, z_n))$, where $\tau \in \mathbb{R}$ and $P(z_1, \ldots, z_n) = \frac{1}{2}(|\operatorname{Im} z_1|^2 + \cdots + |\operatorname{Im} z_n|^2)$. When $n = 1$ and $p(z_1)$ is a subharmonic, nonharmonic

polynomial, the weighted $\overline{\partial}$-problem (with weight $\exp(-p(z_1))$) and the explicit construction of Bergman and Szegö kernels have been studied by a number of authors in different contexts (for example, see [Chr91, Has94, Has95, Has98, FS91, Ber92]). In addition, Raich has estimated the heat kernel and its derivatives [Rai06b, Rai06a, Rai07, Rai].

1. The Heisenberg group and the $\square_b$-heat equation

Definition 1.1. The Heisenberg group is the set $\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the following group structure:

$$g * g' = (x, y, t) * (x', y', t') = (x + x', y + y', t + t' + x \cdot y'),$$

where $(x, y, t), (x', y', t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $\cdot$ denotes the standard dot product in $\mathbb{R}^n$.

The left-invariant vector fields for this group structure are:

$$X_j^g = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t} \quad \text{and} \quad Y_j^g = \frac{\partial}{\partial y_j}, \quad 1 \leq j \leq n, \quad \text{and} \quad T^g = \frac{\partial}{\partial t}.$$

The Heisenberg group can also be identified with the following hypersurface in $\mathbb{C}^{n+1}$: $H^n = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \operatorname{Im} z_{n+1} = (1/2) \sum_{j=1}^n (\operatorname{Im} z_j)^2\}$, where we identify $(z_1, \ldots, z_n, t + i(1/2) \sum_{j=1}^n (\operatorname{Im} z_j)^2) \in H^n$ with $(z_1, \ldots, z_n, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$ where $z_j = x_j + iy_j \in \mathbb{C}$. With this identification, the left-invariant vector fields of types $(0, 1)$ and $(1, 0)$, respectively, are:

$$\overline{Z}_j^g = \frac{1}{2}(X_j + iY_j) = \frac{\partial}{\partial \overline{z}_j} + \frac{y_j}{2} \frac{\partial}{\partial t}, \quad Z_j^g = \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j} + \frac{y_j}{2} \frac{\partial}{\partial t}$$

for $g = (x, y, t) \in \mathbb{H}^n$ and $1 \leq j \leq n$.

The heat equation. The Kohn Laplacian $\square_b$ acting on $(0, q)$-forms on $H^n \approx \mathbb{H}^n$ can be easily described in terms of these left-invariant vector fields. Suppose $f = \sum_{J \in \mathcal{I}_q} f_J d\overline{z}_J$ is a $(0, q)$-form where $\mathcal{I}_q$ is the set of all increasing q-tuples $J = (j_1, \ldots, j_q)$, $1 \leq j_k \leq n$. Then

$$\square_b f = \sum_{J \in \mathcal{I}_q} \mathcal{L}_{n-2q} f_J \, d\overline{z}_J,$$

where

$$(1) \qquad \mathcal{L}_\gamma = -\frac{1}{4} \left(\sum_{j=1}^n (X_j^2 + Y_j^2) + i\gamma T \right).$$

See Stein ([Ste93], XIII §2) for details on computing $\square_b$. For comparison, the box operator (or Laplacian) in Hulanicki ([Hul76]) is $-\frac{1}{2} \sum_{j=1}^n (X_j^2 + Y_j^2)$.

The heat equation is defined on $(0, q)$-forms ρ on $\mathbb{H}^n$ with coefficient functions that depend on $s \in (0, \infty)$ and $(x, y, t) \in \mathbb{H}^n$. It is

$$\frac{\partial \rho}{\partial s} = -\square_b \rho$$

(note that here s is the "time" variable and t is a spatial variable). Since $\square_b$ acts diagonally, we can restrict ourselves to a fixed component and look for a fundamental

solution ρ that satisfies

(2)
$$\begin{cases} \dfrac{\partial \rho}{\partial s} = -\mathcal{L}_\gamma \rho \quad \text{for} \quad s > 0, \ (x, y, t) \in \mathbb{H}^n, \\ \\ \rho(s = 0, x, y, t) = \delta_0(x, y, t) \end{cases}$$

(i.e., the delta function at the origin in the spatial variables).

Fourier transformed variables. We will use a Fourier transform in the spatial (x, y, t) variables (i.e., *not* the s-variable): let (α, β, τ) be the transform variables corresponding to (x, y, t), and define:

$$\widehat{f}(\alpha, \beta, \tau) = \int_{\mathbb{H}^n} f(x, y, t) \, e^{-i(\alpha \cdot x + \beta \cdot y + \tau t)} \, dx \, dy \, dt.$$

Our main result is the following:

Theorem 1.2. *For any $\gamma \in \mathbb{C}$, the spatial Fourier transform of the fundamental solution to the heat equation* (2) *is given by*

(3)
$$\hat{\rho}^\gamma(s, \alpha, \beta, \tau) = \frac{e^{-\gamma s \tau / 4}}{(\cosh(s\tau/2))^{n/2}} e^{-A(|\alpha|^2 + |\beta|^2)/2 + iB\alpha \cdot \beta},$$

where

$$A = \frac{\sinh(s\tau/2)}{\tau \cosh(s\tau/2)}, \qquad B = \frac{2\sinh^2(s\tau/4)}{\tau \cosh(s\tau/2)}.$$

Note that γ may be any complex number, but $\gamma = n - 2q$ is the value where $\mathcal{L}_\gamma$ corresponds to $\Box_b$ on $(0, q)$-forms.

We also seek the fundamental solution to the heat equation associated to the weighted $\overline{\partial}$ operator in (s, x, y)-space. Given a function f on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, let

$$\tilde{f}_\tau(x, y) = \int_{\mathbb{R}} e^{-i\tau t} f(x, y, t) \, dt$$

be the partial Fourier transform in t. Define

$$\overline{L}_j = \frac{\partial}{\partial \overline{z}_j} + \frac{i}{2} y_j \tau = \frac{1}{2} \Big(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} + iy\tau \Big), \quad L_j = \frac{\partial}{\partial z_j} + \frac{i}{2} y_j \tau = \frac{1}{2} \Big(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} + iy\tau \Big).$$

Note that these operators are just the Fourier transform of $\overline{Z}_j$ and Z_j in the t-direction. If $\triangle_{x,y}$ is the Laplacian in both the x and y variables, the partial t-Fourier transform of $\mathcal{L}_\gamma$ is

$$\tilde{\mathcal{L}}_\gamma = -\frac{1}{4} \Big(\triangle_{x,y} + 2i\tau \, y \cdot \nabla_x - (\tau^2 y \cdot y + \gamma\tau) \Big).$$

The operator $\tilde{\mathcal{L}}_\gamma$ acts on functions, but it can be extended to $(0, q)$-forms by acting on each component function of the form. If $\gamma = n - 2q$, then $\tilde{\mathcal{L}}_\gamma$ is the higher dimensional analog of the $\Box_{\tau p}$-operator from [Rai06a, Rai07, Rai] associated to the weighted $\overline{\partial}$ operator in $\mathbb{C}^n$ with weight $\exp(-\tau P(z_1, \ldots, z_n))$, where $P(z_1, \ldots, z_n) = \frac{1}{2}(|\operatorname{Im} z_1|^2 + \cdots + |\operatorname{Im} z_n|^2)$ and $\tau \in \mathbb{R}$. As a corollary to our main theorem, we compute the fundamental solution to the heat operator associated to this weighted $\overline{\partial}$.

Corollary 1.3. *For any $\gamma \in \mathbb{C}$, $\tau \in \mathbb{R}$, the function*

$$\tilde{\rho}_\tau^\gamma(s, x, y) = \frac{e^{-\gamma s\tau/4}}{(2\pi)^n (\cosh(s\tau/2))^{n/2}(A^2 + B^2)^{n/2}} e^{-\frac{A}{2(A^2+B^2)}(|x|^2 + |y|^2) - i\frac{B}{A^2+B^2}x\cdot y}$$

is the fundamental solution to the weighted $\overline{\partial}$ heat equation: $(\frac{\partial}{\partial s} + \tilde{\mathcal{L}}_\gamma)\tilde{\rho}_\tau^\gamma(s, x, y) = 0$ with $\tilde{\rho}_\tau^\gamma(s = 0, x, y) = \delta_{(0,0)}(x, y)$.

Finally, we use $\tilde{\rho}_\tau^\gamma$ to derive the heat kernel, as studied in [Rai06a, Rai07, Rai, NS01].

Corollary 1.4. *For any $\gamma \in \mathbb{C}$, $\tau \in \mathbb{R}$, let*

$$H_\tau^\gamma(s, x, y, x', y') = \frac{\tau^n e^{-\gamma s\tau/4}}{(4\pi)^n \sinh^n(s\tau/4)} e^{-\frac{\tau}{4}\coth(s\tau/4)\left(|x-x'|^2 + |y-y'|^2\right) - i\frac{\tau}{2}(x-x')\cdot(y+y')}.$$

Then H_τ^γ is the heat kernel which satisfies the following property: if $f \in L^2(\mathbb{C})$, then

$$H_\tau^\gamma[f](s, x, y) = \int_{\mathbb{R}^n \times \mathbb{R}^n} H_\tau^\gamma(s, x, y, x', y') f(x', y') \, dx' dy'$$

is a solution to the following initial value problem for the heat equation:

(4)
$$\begin{cases} \left(\dfrac{\partial}{\partial s} + \tilde{\mathcal{L}}_\gamma\right) H_\tau^\gamma[f] = 0 \\ H_\tau^\gamma[f](s = 0, x, y) = f(x, y). \end{cases}$$

Note that H_τ^γ is conjugate symmetric in $z = x + iy$ and $z' = x' + iy'$ (i.e., switching z with z' results in a conjugate).

2. Proof of Theorem 1.2

It is easy to verify the following calculations. Recall that $\widehat{}$ refers to the spatial Fourier transform

$$\begin{aligned} \widehat{X_j^2 f}(\alpha, \beta, \tau) &= (-\alpha_j^2 - 2i\alpha_j\tau\frac{\partial}{\partial\beta_j} + \tau^2\frac{\partial^2}{\partial\beta_j^2})\widehat{f} \\ \widehat{Y_j^2 f}(\alpha, \beta, \tau) &= -\beta_j^2\widehat{f} \\ \widehat{Tf}(\alpha, \beta, \tau) &= i\tau\widehat{f}. \end{aligned}$$

We first reduce the problem down to dimension one. Define $\hat{\rho}^{\gamma,1}$ by the same formula as given in (3), but for dimension one (i.e., $n = 1$ and α, $\beta \in \mathbb{R}$). From (3), note that

(5)
$$\hat{\rho}^\gamma(s, \alpha, \beta, \tau) = \prod_{j=1}^n \hat{\rho}^{\gamma/n,1}(s, \alpha_j, \beta_j, \tau), \quad \alpha = (\alpha_1, \ldots, \alpha_n), \ \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$$

(note the γ on the left and the γ/n on the right). Once we show that $\rho^{\gamma,1}$ satisfies the transformed heat equation in dimension one, i.e.,

(6)
$$\left(\frac{\partial}{\partial s} - (1/4)(\widehat{X^2} + \widehat{Y^2} + i\gamma\widehat{T})\right)\{\widehat{\rho}^{\gamma,1}(s, \cdot, \cdot)\} = 0$$

with initial condition $\hat{\rho}^{\gamma,1}(s = 0, \cdot, \cdot, \cdot) = 1$ (the Fourier transform of the delta function), then by using (5), it is an easy exercise to show that $\widehat{\rho}^\gamma$ in dimension n satisfies Theorem 1.2.

From now on, we assume the dimension n is one and so x, y, α and β are all real variables. Also, γ will be suppressed as a superscript. Define

$$(7) \qquad u(s, \alpha, \beta, \tau) = \widehat{\rho}(s, \alpha, \beta, \tau) e^{-i\frac{\alpha\beta}{\tau}}.$$

Then, the following equations are easily verified:

$$(8) \qquad u(s = 0, \alpha, \beta, \tau) \;=\; e^{-i\frac{\alpha\beta}{\tau}}$$

$$(9) \qquad \frac{\partial u}{\partial s} \;=\; \frac{1}{4}\left(\tau^2 \frac{\partial^2}{\partial \beta^2} - \beta^2 - \gamma\tau\right) u.$$

The first equation follows from the fact that the Fourier transform of the delta function is the constant one. The second equation follows from the heat equation for $\widehat{\rho}$ (from (6)) and the above formulas for the transformed differential operators $\widehat{X}, \widehat{Y}$ and $\widehat{T}$. We will refer to the above differential equation as the *transformed heat equation*.

Solution of heat equation using Hermite special functions. For $m = 0, 1, 2,$ $\ldots$ and $x \in \mathbb{R}$, let

$$\psi_m(x) \;=\; \frac{(-1)^m}{\sqrt{2^m m! \sqrt{\pi}}} e^{x^2/2} \frac{d^m}{dx^m}\{e^{-x^2}\}.$$

For $\tau \in \mathbb{R}$, let

$$\Psi_m^\tau(x) = |\tau|^{-1/4} \psi_m(x/\sqrt{|\tau|}).$$

It is a fact that ψ_m and hence Ψ_m^τ form an orthonormal system for $L^2(\mathbb{R})$ (see [Tha93], pp. 1-7). It is also a fact (again see [Tha93], (1.1.28)) that

$$\psi_m''(x) = x^2 \psi_m(x) - (2m+1)\psi_m(x).$$

We first assume that $\tau > 0$ and later indicate the minor changes needed in the case that $\tau \leq 0$. Replacing x by $\beta/\sqrt{\tau}$ in the previous equation yields:

$$(10) \qquad \left(\tau^2 \frac{\partial^2}{\partial \beta^2} - \beta^2 - \gamma\tau\right)\{\Psi_m^\tau\}(\beta) = -(2m+1+\gamma)\tau \Psi_m^\tau(\beta).$$

In other words, Ψ_m^τ is an eigenfunction of the differential operator on the right side of (9) with eigenvalue $-\frac{1}{4}(2m+1+\gamma)\tau$.

Since $\{\Psi_m^\tau\}$ are an orthonormal basis for $L^2(\mathbb{R})$, u can be expressed as

$$u(s, \alpha, \beta, \tau) = \sum_{m=0}^{\infty} a_m(\alpha, \tau) e^{-\frac{1}{4}(2m+1+\gamma)s\tau} \Psi_m^\tau(\beta),$$

where $a_m(\alpha, \tau)$ will be determined later. Differentiating this with respect to s and using (10) gives

$$\frac{\partial}{\partial s} u(s, \alpha, \beta, \tau) = \sum_{m=0}^{\infty} a_m(\alpha, \tau) e^{-\frac{1}{4}(2m+1+\gamma)s\tau}\left(-\frac{1}{4}(2m+1+\gamma)\right)\tau \Psi_m^\tau(\beta)$$

$$= \frac{1}{4}\left(\tau^2 \frac{\partial^2}{\partial \beta^2} - \beta^2 - \gamma\tau\right)\{u(t, \alpha, \beta, \tau)\}.$$

So, u satisfies the transformed heat equation (9). To satisfy the initial condition (8), we must have

$$e^{-i\alpha\beta/\tau} = u(s = 0, \alpha, \beta, \tau) = \sum_{m=0}^{\infty} a_m(\alpha, \tau)\Psi_m^\tau(\beta).$$

Using the fact that the $\Psi_m^\tau(\beta)$ form an orthonormal system, we have

$$a_m(\alpha, \tau) = \int_{\mathbb{R}} e^{-i\alpha\beta/\tau}\Psi_m^\tau(\beta)\, d\beta = \tau^{1/4}\int_{\mathbb{R}} e^{-i\frac{\alpha}{\sqrt{\tau}}\beta}\psi_m(\beta)\, d\beta.$$

The integral on the right is just the Fourier transform of ψ_m at the point $\alpha/\sqrt{\tau}$. From Thangavelu ([Tha93], Lemma 1.1.3), the Fourier transform of ψ_m equals ψ_m up to a constant factor of $(-i)^m\sqrt{2\pi}$. Therefore,

$$a_m(\alpha, \tau) = (-i)^m(2\pi)^{\frac{1}{2}}\tau^{\frac{1}{4}}\psi_m(\alpha/\sqrt{\tau}).$$

Substituting this value of a_m into the expression for u and rearranging gives:

$$u(s, \alpha, \beta, \tau) = (2\pi)^{1/2}e^{-\frac{1}{4}(1+\gamma)s\tau}\sum_{m=0}^{\infty}(-i)^m\psi_m\left(\frac{\alpha}{\sqrt{\tau}}\right)\psi_m\left(\frac{\beta}{\sqrt{\tau}}\right)e^{-\frac{1}{2}ms\tau}.$$

Now solving for $\hat\rho$ (see equation (7)) yields

$$\hat\rho(s, \alpha, \beta, \tau) = e^{i\alpha\beta/\tau}u(s, \alpha, \beta, \tau)$$

$$= (2\pi)^{\frac{1}{2}}e^{-\frac{1}{4}(1+\gamma)s\tau}\sum_{m=0}^{\infty}(-i)^m\psi_m\left(\frac{\alpha}{\sqrt{\tau}}\right)\psi_m\left(\frac{\beta}{\sqrt{\tau}}\right)e^{-\frac{1}{2}ms\tau}e^{i\alpha\beta/\tau}.$$

Now let $S = e^{-s\tau/2}$, $x = \alpha/\sqrt{\tau}$, $y = \beta/\sqrt{\tau}$. Since $|iS| < 1$, we obtain (see [Tha93], (1.1.36))

$$\hat\rho(s, \alpha, \beta, \tau) = (2\pi)^{\frac{1}{2}}S^{\frac{1}{2}(1+\gamma)}\left(\sum_{m=0}^{\infty}(-iS)^m\psi_m(x)\psi_m(y)\right)e^{ixy}$$

$$= \frac{\sqrt{2}S^{\frac{1}{2}(1+\gamma)}}{(1+S^2)^{\frac{1}{2}}}e^{-\frac{1}{2}\frac{1-S^2}{1+S^2}(x^2+y^2)}e^{ixy(\frac{-2S}{1+S^2}+1)}.$$

Now substituting in for S, x and y, a short calculation finishes the proof for $\tau > 0$. Note that $\hat\rho(s = 0, \alpha, \beta, \tau) = 1$ (the Fourier transform of the delta function at the origin).

When $\tau = 0$, the solution in (3) becomes $\widehat\rho(s, \alpha, \beta) = e^{-s(\alpha^2+\beta^2)/4}$, which is easily shown to satisfy (6).

If $\tau < 0$, then τ is replaced by $|\tau|$ on the right side of (10), which slightly changes the subsequent calculations. However the formula for the solution given Theorem 1.2 remains valid for $\tau < 0$.

3. PROOF OF THE COROLLARIES

Proof of Corollary 1.3. Again, we assume the dimension is $n = 1$. The fundamental solution to this heat operator must satisfy

$$\frac{\partial}{\partial s}\tilde\rho_\tau(s, x, y) + \tilde{\mathcal{L}}_\gamma\tilde\rho_\tau = 0$$

with the initial condition $\tilde\rho_\tau(s = 0, x, y) = \delta_0(x, y)$. Now since $\hat\rho$ is the Fourier transform of the fundamental solution to the original heat operator, clearly $\tilde\rho_\tau$ can

be obtained by taking the inverse Fourier transform of $\hat{\rho}$ in the α, β variables. This is a standard calculation involving Gaussian integrals and will be left to the reader. □

Proof of Corollary 1.4. If L_j and $\overline{L}_j$, $1 \le j \le n$, had constant coefficients, then the heat kernel would just be $\tilde{\rho}_\tau(s, x - x', y - y')$, an ordinary convolution. However, we must multiply by a "twist" factor $e^{-i\tau(x-x')\cdot y'}$ to account for the fact that L_j and $\overline{L}_j$ have variable coefficients. Let

$$(11) \qquad H_\tau(s, x, y, x', y', \tau) = \tilde{\rho}_\tau(s, x - x', y - y')e^{-i\tau(x-x')\cdot y'}.$$

Note that $H_\tau(f)$ satisfies the initial condition given in (4) in view of the initial condition satisfied by $\tilde{\rho}_\tau$ and noting that the twist term is 1 at $x' = x$. Showing that H_τ satisfies the heat equation in the s, x, y variables is a short calculation that uses the equation

$$\left(\frac{\partial}{\partial s} - \frac{1}{4}\Big(\triangle_{x,y} + 2i\tau(y - y')\cdot\nabla_x - (\tau^2(y - y')\cdot(y - y') + \gamma\tau)\Big)\right)$$
$$\times \{\tilde{\rho}_\tau(s, x - x', y - y')\} = 0,$$

which is just the equation $(\frac{\partial}{\partial s} + \tilde{\mathcal{L}}_\gamma)\tilde{\rho}_\tau = 0$ at the point $(s, x - x', y - y')$.

Simplification of the formula for H_τ. Note that the coefficient of the imaginary part of the exponent of $\tilde{\rho}_\tau$ is

$$\frac{-B}{A^2 + B^2} \quad \text{where} \quad A = \frac{\sinh(s\tau/2)}{\tau\cosh(s\tau/2)}, \quad B = \frac{2\sinh^2(s\tau/4)}{\tau\cosh(s\tau/2)}.$$

An easy calculation with cosh and sinh identities shows that

$$\frac{B}{A^2 + B^2} = \frac{\tau}{2} \quad \text{and} \quad \frac{A}{B} = \frac{\cosh(s\tau/4)}{\sinh(s\tau/4)}.$$

Consequently, the fundamental solution H_τ, from (11) and Corollary 1.3, can be rewritten

$$H_\tau(s, x, y, x', y') = \frac{\tau^n e^{-\gamma s\tau/4}}{(4\pi)^n\sinh^n(s\tau/4)}e^{-\frac{\tau}{4}\coth(s\tau/4)\left(|x-x'|^2+|y-y'|^2\right)-i\frac{\tau}{2}(x-x')\cdot(y+y')}.$$

□

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