PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 4, April 2009, Pages 1185–1191
S 0002-9939(08)09603-2
Article electronically published on September 26, 2008

INTEGERS REPRESENTED AS THE SUM OF ONE PRIME, TWO SQUARES OF PRIMES AND POWERS OF 2

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(Communicated by Ken Ono)

ABSTRACT. In this short paper we prove that every sufficiently large odd integer can be written as a sum of one prime, two squares of primes and 83 powers of 2.

1. INTRODUCTION AND MAIN RESULTS

It was shown by Linnik [9], [10] that each large even integer N is a sum of two primes and a bounded number of powers of 2,

$$N = p_1 + p_2 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}, \tag{1.1}$$

where p and v, with or without subscripts, denote a prime number and a positive integer respectively. Later Gallagher [1] established a stronger result by a different method. An explicit value for the number k of powers of 2 was first established by Liu, Liu and Wang [11], who found that $k = 54000$ is acceptable. The original value for the number k was subsequently improved by Li [6], Wang [20] and Li [7]. In 2002, Heath-Brown and Puchta [3] applied a rather different approach to this problem and showed that $k = 13$ is acceptable. In 2003, Pintz and Ruzsa [16] announced that $k = 8$ is acceptable.

There are other similar problems. In 1938, Hua [4] proved that almost all n satisfying a certain necessary condition are representable as sums of a prime and two squares of primes,

$$n = p_1^2 + p_2^2 + p_3,$$

where the necessary condition is that

$$n \in \mathcal{A} = \{n : n \in \mathbb{N}, n \not\equiv 0(\mathrm{mod} 2), n \not\equiv 2(\mathrm{mod} 3)\}.$$

Motivated by Hua's result and the works of Linnik and Gallagher, Liu, Liu and Zhan [12], among other important results, proved that every large odd integer N can be written as a sum of one prime, two squares of primes and k powers of 2, namely

$$N = p_1^2 + p_2^2 + p_3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}. \tag{1.2}$$

Received by the editors January 30, 2008, and, in revised form, April 4, 2008.
2000 *Mathematics Subject Classification.* Primary 11P32, 11P05, 11N36, 11P55.
Key words and phrases. Squares of primes, powers of 2, circle method.
This work is supported by the National Natural Science Foundation of China (Grant No. 10701048).

In 2004, Liu [14] proved that $k = 22000$ is acceptable in (1.2). In 2007, Li [8] further showed that $k = 106$ is acceptable in (1.2). However when we compare these results with the former result of Heath-Brown and Puchta [3] (or Pintz and Ruzsa [16]), it is a pity that a value for the number k with two digits cannot be obtained.

In this short paper we shall show that the current techniques are able to obtain such a result.

Theorem 1.1. *Every sufficiently large odd integer can be written as a sum of one prime, two squares of primes and* 83 *powers of* 2.

Unlike the previous works, we use a different idea to treat the second integral in (3.1). This results in the improvement.

2. Preliminaries

In order to prove Theorem 1.1, it suffices to estimate the number of solutions of the equation

$$N = p_1^2 + p_2^2 + p_3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}. \tag{2.1}$$

Suppose N is sufficiently large. We write

$$P = N^{\frac{1}{6}-\varepsilon}, \quad Q = NP^{-1}L^{-10}, \quad M = NL^{-9}, \quad L = \log_2 N. \tag{2.2}$$

We use c and ε to denote an absolute constant and a sufficiently small positive number respectively, not necessarily the same at each occurrence.

To apply the circle method, we begin with the observation

$$\begin{aligned} R(N) : \ &= \sum_{\substack{N=p_1^2+p_2^2+p_3+2^{v_1}+2^{v_2}+\cdots+2^{v_k} \\ M<p_1^2,p_2^2,p_3\le N}} (\log p_1)(\log p_2)(\log p_3) \\ &= \int_0^1 f^2(\alpha)g(\alpha)h^k(\alpha)e(-\alpha N)d\alpha, \end{aligned} \tag{2.3}$$

where

$$f(\alpha) = \sum_{M<p^2\le N} (\log p)e(\alpha p^2), \tag{2.4}$$

$$g(\alpha) = \sum_{M<p\le N} (\log p)e(\alpha p), \tag{2.5}$$

and

$$h(\alpha) = \sum_{2^v\le N} e(\alpha 2^v) = \sum_{v\le L} e(\alpha 2^v). \tag{2.6}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1+1/Q]$ can be written as

$$\alpha = \frac{a}{q} + \beta, \qquad |\beta| \le \frac{1}{qQ}, \tag{2.7}$$

for some integers a, q with $1 \le a \le q \le Q$, $(a, q) = 1$. We define the major arcs $\mathcal{M}$ and minor arcs $C(\mathcal{M})$ as usual, namely

$$\mathcal{M} = \bigcup_{q\le P} \bigcup_{\substack{1\le a\le q \\ (a,q)=1}} \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right], \qquad C(\mathcal{M}) = \left[\frac{1}{Q}, 1+\frac{1}{Q}\right] \setminus \mathcal{M}. \tag{2.8}$$

On the minor arcs, we need estimates for the measure of the set

$$\mathcal{E}_\lambda := \{\alpha \in [0,1] : |h(\alpha)| \geq \lambda L\}. \tag{2.9}$$

The following lemma is due to Heath-Brown and Puchta [3].

Lemma 2.1. *We have*

$$\mathrm{meas}(\mathcal{E}_\lambda) \ll N^{-E(\lambda)} \qquad \text{with} \qquad E(0.887167) > \frac{3}{4} + 10^{-10}.$$

Proof. Let

$$T_h(\alpha) = \sum_{0 \leq n \leq h-1} e(\alpha 2^n),$$

$$F(\xi, h) = \frac{1}{2^h} \sum_{r=0}^{2^h-1} \exp\{\xi \mathrm{Re}(T_h(r/2^h))\},$$

and

$$E(\lambda) = \frac{\xi\lambda}{\log 2} - \frac{\log F(\xi,h)}{h \log 2} - \frac{\varepsilon}{\log 2}.$$

Then for any ξ, $\varepsilon > 0$, and any $h \in \mathbb{N}$, we have

$$\mathrm{meas}(\mathcal{E}_\lambda) \ll N^{-E(\lambda)}.$$

This was proved in Section 7 of Heath-Brown and Puchta [3]. Taking $\xi = 1.21$, $h = 22$, we get on a PC that

$$E(0.887167) > \frac{3}{4} + 10^{-10}.$$

This completes the proof of the lemma. □

To control the minor arcs we also need three other lemmas.

Lemma 2.2. *Suppose that α is a real number and that there exist integers a and q satisfying*

$$1 \leq q \leq Y, \quad (a,q) = 1, \quad |q\alpha - a| < Y^{-1},$$

with $Y = X^{\frac{3}{2}}$. Then for any fixed $\varepsilon > 0$ one has

$$\sum_{X < p \leq 2X} (\log p) e(\alpha p^2) \ll X^{\frac{7}{8}+\varepsilon} + \frac{q^\varepsilon X (\log X)^c}{(q + X^2|q\alpha - a|)^{\frac{1}{2}}}.$$

Proof. This is Theorem 3 for the case $k = 2$ in Kumchev [5], which is a powerful tool to control the contribution from the minor arcs when one applies the circle method to the Waring-Goldbach problems. □

Lemma 2.3. *Let $f(\alpha)$ and $h(\alpha)$ be as in* (2.4) *and* (2.6)*. Then*

$$\int_0^1 |f(\alpha)h(\alpha)|^4 d\alpha \leq c_1 \frac{\pi^2}{16} N L^4,$$

where

$$c_1 \leq \left(\frac{32^4 \cdot 101 \cdot 1.620767}{3} + \frac{8 \cdot \log^2 2}{\pi^2}\right)(1+\varepsilon)^9.$$

Proof. The first version of this lemma was established in Liu and Liu [13]. Then the constant was subsequently refined in [15] and [8]. □

Lemma 2.4. *Let* $g(\alpha)$ *and* $h(\alpha)$ *be as in* (2.5) *and* (2.6)*. Then*

$$\int_0^1 |g(\alpha)h(\alpha)|^2 d\alpha \le 12.3238 c_0 N L^2,$$

where

$$c_0 = \prod_{p\ge 3}\left(1 - \frac{1}{(p-1)^2}\right) = 0.6601....$$

Proof. This lemma is actually Lemma 10 in [3]. By Lemma 2 of [17], we can replace (41) of [3] by $C_2 \le 1.93657$, and by the result of Wu [21] we can replace (32) of [3] by 7.8209. Then by the proof of Lemma 9 of [3] this lemma follows. □

To treat the major arcs, we need the following three lemmas.

Lemma 2.5. *For all integers* $n \in \mathcal{A}$*, we have*

$$\int_{\mathcal{M}} f^2(\alpha)g(\alpha)e(-\alpha n)d\alpha = (\pi/4 + o(1))\mathfrak{S}(n,P)n + O(N/\log N). \tag{2.10}$$

Proof. This lemma is Lemma 4 in [8] or Theorem 2 in [19]. These results are based on the new approach to treat the enlarged major arcs in the circle method, which was developed by Liu, Liu and Zhan [12]. □

Lemma 2.6. *For all integers* $n \in \mathcal{A}$*, we have*

$$\mathfrak{S}(n,P) \ge 2.27473966. \tag{2.11}$$

Proof. This lemma is Lemma 5 in Li [8]. □

Lemma 2.7. *Let* $\mathcal{A}(N,k) = \{n \ge 2 : n = N - 2^{v_1} - \cdots - 2^{v_k}\}$ *with* $k \ge 80$*. Then for odd* N*, we have*

$$\sum_{\substack{n\in\mathcal{A}(N,k)\\ n\not\equiv 2 (\mathrm{mod}\, 3)}} n \ge (\frac{2}{3} - 2^{-70})NL^k.$$

Proof. This lemma is actually Lemma 6 in Li [8]. We make the corresponding change according to the range of k. □

3. Proof of Theorem 1.1

Let $\mathcal{E}_\lambda$ be as defined in (2.9), and $\mathcal{M}$ and $C(\mathcal{M})$ be as in (2.8), with P, Q determined in (2.2). Then (2.3) becomes

$$R(N) = \int_0^1 f^2(\alpha)g(\alpha)h^k(\alpha)e(-\alpha N)d\alpha = \int_{\mathcal{M}} + \int_{C(\mathcal{M})\cap\mathcal{E}_\lambda} + \int_{C(\mathcal{M})\setminus\mathcal{E}_\lambda}. \tag{3.1}$$

For the major arcs, by Lemma 2.5 we have

(3.2)
$$\begin{aligned}\int_{\mathcal{M}} f^2(\alpha)g(\alpha)h^k(\alpha)e(-\alpha N)d\alpha &= \sum_{n\in\mathcal{A}(N,k)}\int_{\mathcal{M}} f^2(\alpha)g(\alpha)e(-\alpha n)d\alpha\\ &= \left(\frac{\pi}{4}+o(1)\right)\sum_{n\in\mathcal{A}(N,k)}\mathfrak{S}(n,P)n+O(NL^{k-1})\\ &\ge 2.27473966\left(\frac{\pi}{4}+o(1)\right)\sum_{n\in\mathcal{A}(N,k)} n+O(NL^{k-1})\\ &\ge 1.516492\frac{\pi}{4}NL^k,\end{aligned}$$

where we have used Lemmas 2.6 and 2.7.

Now we consider the second integral in (3.1). By Dirichlet's lemma on rational approximation, any $\alpha\in C(\mathcal{M})$ can be written as

$$\alpha=\frac{a}{q}+\beta,\qquad |\beta|\le\frac{1}{qN^{\frac{3}{4}}},$$

for some integers a, q with $1\le a\le q\le N^{\frac{3}{4}}$, $(a,q)=1$. If $q\le P$, since $\alpha\in C(\mathcal{M})$, we have $PL^{10}<N|q\alpha-a|$; otherwise we have $q>P$. Hence we have that for $\alpha\in C(\mathcal{M})$,

$$q+N|q\alpha-a|>P.$$

Then by Lemma 2.2, we have

$$\max_{\alpha\in C(\mathcal{M})}|f(\alpha)|\ll N^{\frac{1}{2}-\frac{1}{16}+\varepsilon}. \tag{3.3}$$

It should be remarked that now (3.3) is a standard result, which has been used in [2], [18], [15] and [8], etc. For the second integral in (3.1), by Cauchy's inequality we have

$$\begin{aligned}\int_{C(\mathcal{M})\cap\mathcal{E}_\lambda} &\le \left(\int_{C(\mathcal{M})\cap\mathcal{E}_\lambda}|f^2(\alpha)g(\alpha)h^k(\alpha)|^2d\alpha\right)^{\frac{1}{2}}\left(\int_{C(\mathcal{M})\cap\mathcal{E}_\lambda}1d\alpha\right)^{\frac{1}{2}}\\ &\le \left(\int_{C(\mathcal{M})}|f^2(\alpha)g(\alpha)h^k(\alpha)|^2d\alpha\right)^{\frac{1}{2}}\left(\int_{\mathcal{E}_\lambda}1d\alpha\right)^{\frac{1}{2}}\\ &\le \left(L^{2k}\left(\max_{\alpha\in C(\mathcal{M})}|f(\alpha)|\right)^4\int_0^1|g(\alpha)|^2d\alpha\right)^{\frac{1}{2}}\left(\int_{\mathcal{E}_\lambda}1d\alpha\right)^{\frac{1}{2}}.\end{aligned}$$

Then by (3.3) and the well-known estimate

$$\int_0^1|g(\alpha)|^2d\alpha\ll NL,$$

we have

$$\begin{aligned}\int_{C(\mathcal{M})\cap\mathcal{E}_\lambda} &\ll (L^{2k}N^{\frac{7}{4}+\varepsilon}N)^{\frac{1}{2}}(\mathrm{meas}(\mathcal{E}_\lambda))^{\frac{1}{2}}\\ &\ll (L^{2k}N^{\frac{7}{4}+\varepsilon}N)^{\frac{1}{2}}N^{-\frac{E(\lambda)}{2}}\\ &\ll N^{\frac{11}{8}+\varepsilon}L^kN^{-\frac{E(\lambda)}{2}}\ll N^{1-\varepsilon},\end{aligned}\tag{3.4}$$

where we have used Lemma 2.1 with $\lambda = 0.887167$, namely

$$\operatorname{meas}(\mathcal{E}_{0.887167}) \ll N^{-E(0.887167)} < N^{-\frac{3}{4}-10^{-10}}.$$

For the last integral in (3.1) with the definition of $\mathcal{E}_\lambda$, and Lemmas 2.3 and 2.4, by Cauchy's inequality we have

$$\begin{aligned}\int_{C(\mathcal{M})\setminus\mathcal{E}_\lambda} &\leq (\lambda L)^{k-3}\left(\int_0^1 |f(\alpha)h(\alpha)|^4 d\alpha\right)^{\frac{1}{2}}\left(\int_0^1 |g(\alpha)h(\alpha)|^2 d\alpha\right)^{\frac{1}{2}} \\ &\leq 21576\lambda^{k-3}\frac{\pi}{4}NL^k.\end{aligned} \tag{3.5}$$

Combining this with (3.2) and (3.4), we get

$$R(N) \geq \frac{\pi}{4}NL^k(1.516492 - 21576\lambda^{k-3}). \tag{3.6}$$

When $k \geq 83$, for $\lambda = 0.887167$, by the above estimate we have

$$R(N) > 0.$$

This means that every large odd integer N can be written in the form of (1.2) for $k \geq 83$.

Acknowledgements

This work was completed when the first author visited Stanford University supported by CSC. The authors would like to thank Professor K. Soundararajan and Professor Jianya Liu for their encouragement.

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