INTEGERS REPRESENTED AS THE SUM OF ONE PRIME,
TWO SQUARES OF PRIMES AND POWERS OF 2

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Abstract. In this short paper we prove that every sufficiently large odd integer can be written as a sum of one prime, two squares of primes and 83 powers of 2.

1. Introduction and main results

It was shown by Linnik [9], [10] that each large even integer \( N \) is a sum of two primes and a bounded number of powers of 2,
\[
N = p_1 + p_2 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k},
\]
where \( p \) and \( v \), with or without subscripts, denote a prime number and a positive integer respectively. Later Gallagher [1] established a stronger result by a different method. An explicit value for the number \( k \) of powers of 2 was first established by Liu, Liu and Wang [11], who found that \( k = 54000 \) is acceptable. The original value for the number \( k \) was subsequently improved by Li [6], Wang [20] and Li [7]. In 2002, Heath-Brown and Puchta [3] applied a rather different approach to this problem and showed that \( k = 13 \) is acceptable. In 2003, Pintz and Ruzsa [16] announced that \( k = 8 \) is acceptable.

There are other similar problems. In 1938, Hua [4] proved that almost all \( n \) satisfying a certain necessary condition are representable as sums of a prime and two squares of primes,
\[
n = p_1^2 + p_2^2 + p_3,
\]
where the necessary condition is that
\[
n \in A = \{ n : n \in \mathbb{N}, n \not\equiv 0 \pmod{2}, n \not\equiv 2 \pmod{3} \}.
\]
Motivated by Hua’s result and the works of Linnik and Gallagher, Liu, Liu and Zhan [12], among other important results, proved that every large odd integer \( N \) can be written as a sum of one prime, two squares of primes and \( k \) powers of 2, namely
\[
N = p_1^2 + p_2^2 + p_3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}.
\]

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In 2004, Liu [14] proved that \( k = 22000 \) is acceptable in (1.2). In 2007, Li [8] further showed that \( k = 106 \) is acceptable in (1.2). However when we compare these results with the former result of Heath-Brown and Puchta [3] (or Pintz and Ruzsa [16]), it is a pity that a value for the number \( k \) with two digits cannot be obtained.

In this short paper we shall show that the current techniques are able to obtain such a result.

**Theorem 1.1.** Every sufficiently large odd integer can be written as a sum of one prime, two squares of primes and \( 83 \) powers of \( 2 \).

Unlike the previous works, we use a different idea to treat the second integral in (3.1). This results in the improvement.

2. **Preliminaries**

In order to prove Theorem 1.1 it suffices to estimate the number of solutions of the equation

\[
N = p_1^2 + p_2^2 + p_3 + 2^{v_1} + 2^{v_2} + \cdots + 2^{v_k}.
\]

Suppose \( N \) is sufficiently large. We write

\[
P = N^{\frac{1}{6}} - \varepsilon, \quad Q = NP^{-1}L^{-10}, \quad M = NL^{-9}, \quad L = \log_2 N.
\]

We use \( c \) and \( \varepsilon \) to denote an absolute constant and a sufficiently small positive number respectively, not necessarily the same at each occurrence.

By Dirichlet’s lemma on rational approximation, each \( \alpha \in \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \) can be written as

\[
\alpha = \frac{a}{q} + \beta, \quad |\beta| \leq \frac{1}{qQ},
\]

for some integers \( a, q \) with \( 1 \leq a \leq q \leq Q, (a, q) = 1 \). We define the major arcs \( \mathcal{M} \) and minor arcs \( C(\mathcal{M}) \) as usual, namely

\[
\mathcal{M} = \bigcup_{q \leq P} \bigcup_{1 \leq a \leq q \atop (a, q) = 1} \left[ \frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ} \right], \quad C(\mathcal{M}) = \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus \mathcal{M}.
\]
On the minor arcs, we need estimates for the measure of the set
\[(2.9) \quad \mathcal{E}_\lambda := \{\alpha \in [0, 1] : |h(\alpha)| \geq \lambda L\}.\]
The following lemma is due to Heath-Brown and Puchta [3].

**Lemma 2.1.** We have
\[
\text{meas}(\mathcal{E}_\lambda) \ll N^{-E(\lambda)} \quad \text{with} \quad E(0.887167) > \frac{3}{4} + 10^{-10}.
\]

**Proof.** Let
\[
T_h(\alpha) = \sum_{0 \leq n \leq h-1} e(\alpha 2^n),
\]
\[
F(\xi, h) = \frac{1}{2^h} \sum_{r=0}^{2^h-1} \exp\{\xi \text{Re}(T_h(r/2^h))\},
\]
and
\[
E(\lambda) = \frac{\xi \lambda}{\log 2} - \frac{\log F(\xi, h)}{h \log 2} - \frac{\varepsilon}{\log 2}.
\]
Then for any \(\xi, \varepsilon > 0\), and any \(h \in \mathbb{N}\), we have
\[
\text{meas}(\mathcal{E}_\lambda) \ll N^{-E(\lambda)}.
\]
This was proved in Section 7 of Heath-Brown and Puchta [3]. Taking \(\xi = 1.21\), \(h = 22\), we get on a PC that
\[
E(0.887167) > \frac{3}{4} + 10^{-10}.
\]
This completes the proof of the lemma. \(\square\)

To control the minor arcs we also need three other lemmas.

**Lemma 2.2.** Suppose that \(\alpha\) is a real number and that there exist integers \(a\) and \(q\) satisfying
\[1 \leq q \leq Y, \quad (a, q) = 1, \quad |q\alpha - a| < Y^{-1},\]
with \(Y = X^{\frac{2}{7}}\). Then for any fixed \(\varepsilon > 0\) one has
\[
\sum_{X < p \leq 2X} \log p e(\alpha p^2) \ll X^{\frac{2}{7} + \varepsilon} + \frac{q^2 X (\log X)^\varepsilon}{(q + X^2 |q\alpha - a|)^{\frac{1}{7}}},
\]

**Proof.** This is Theorem 3 for the case \(k = 2\) in Kumchev [5], which is a powerful tool to control the contribution from the minor arcs when one applies the circle method to the Waring-Goldbach problems. \(\square\)

**Lemma 2.3.** Let \(f(\alpha)\) and \(h(\alpha)\) be as in (2.4) and (2.6). Then
\[
\int_0^1 |f(\alpha) h(\alpha)|^4 d\alpha \leq c_1 \frac{\pi^2}{16} NL^4,
\]
where
\[
c_1 \leq \left( \frac{32^4 \cdot 101 \cdot 1.620767}{3} + \frac{8 \cdot \log^2 2}{\pi^2} \right) (1 + \varepsilon)^9.
\]

**Proof.** The first version of this lemma was established in Liu and Liu [13]. Then the constant was subsequently refined in [15] and [8]. \(\square\)
Lemma 2.4. Let $g(\alpha)$ and $h(\alpha)$ be as in (2.5) and (2.6). Then
\[ \int_0^1 |g(\alpha)h(\alpha)|^2 d\alpha \leq 12.3238 c_0 NL^2, \]
where
\[ c_0 = \prod_{p \geq 3} \left( 1 - \frac{1}{(p-1)^2} \right) = 0.6601. \]

Proof. This lemma is actually Lemma 10 in [3]. By Lemma 2 of [17], we can replace (41) of [3] by $C_2 \leq 1.93657$, and by the result of Wu [21] we can replace (32) of [3] by 7.8209. Then by the proof of Lemma 9 of [3] this lemma follows. \hfill \Box

To treat the major arcs, we need the following three lemmas.

Lemma 2.5. For all integers $n \in \mathcal{A}$, we have
\[ \int_{\mathcal{A}} f^2(\alpha)g(\alpha)e(-\alpha n) d\alpha = (\pi/4 + o(1))\mathcal{G}(n, P)n + O(N/\log N). \]  

Proof. This lemma is Lemma 4 in [8] or Theorem 2 in [19]. These results are based on the new approach to treat the enlarged major arcs in the circle method, which was developed by Liu, Liu and Zhan [12]. \hfill \Box

Lemma 2.6. For all integers $n \in \mathcal{A}$, we have
\[ \mathcal{G}(n, P) \geq 2.27473966. \]

Proof. This lemma is Lemma 5 in Li [8]. \hfill \Box

Lemma 2.7. Let $\mathcal{A}(N, k) = \{ n \geq 2 : n = N - 2^{v_1} - \cdots - 2^{v_i} \}$ with $k \geq 80$. Then for odd $N$, we have
\[ \sum_{\substack{n \in \mathcal{A}(N, k) \\backslash n \equiv 2 (\text{mod 3})}} n \geq \left( \frac{2}{3} - 2^{-70} \right) NL^k. \]

Proof. This lemma is actually Lemma 6 in Li [8]. We make the corresponding change according to the range of $k$. \hfill \Box

3. Proof of Theorem 1.1

Let $\mathcal{E}_\lambda$ be as defined in (2.9), and $\mathcal{M}$ and $C(\mathcal{M})$ be as in (2.8), with $P$, $Q$ determined in (2.2). Then (2.3) becomes
\[ R(N) = \int_0^1 f^2(\alpha)g(\alpha)h^k(\alpha)e(-\alpha N) d\alpha = \int_{\mathcal{M}} + \int_{C(\mathcal{M}) \backslash \mathcal{E}_\lambda} + \int_{C(\mathcal{M}) \backslash \mathcal{E}_\lambda}. \]
For the major arcs, by Lemma 2.5 we have
\begin{equation}
\int_{\mathcal{M}} f^2(\alpha)g(\alpha)h^k(\alpha)e(-\alpha N)d\alpha = \sum_{n \in \mathbb{A}(N,k)} \int_{\mathcal{M}} f^2(\alpha)g(\alpha)e(-\alpha n)d\alpha
\end{equation}
\begin{align*}
&= \left( \frac{\pi}{4} + o(1) \right) \sum_{n \in \mathbb{A}(N,k)} \mathbb{S}(n, P)n + O(NL^{k-1}) \\
&\geq 2.2747396 \left( \frac{\pi}{4} + o(1) \right) \sum_{n \in \mathbb{A}(N,k)} n + O(NL^{k-1}) \\
&\geq 1.516492 \frac{\pi}{4} NL^k,
\end{align*}
where we have used Lemmas 2.6 and 2.7.

Now we consider the second integral in (3.1). By Dirichlet’s lemma on rational approximation, any $\alpha \in C(\mathcal{M})$ can be written as
$$\alpha = \frac{a}{q} + \beta, \quad |\beta| \leq \frac{1}{q N^\theta},$$
for some integers $a, q$ with $1 \leq a \leq q \leq N^{\theta}$. If $q \leq P$, since $\alpha \in C(\mathcal{M})$, we have $PL^0 < N|q\alpha - a|$; otherwise we have $q > P$. Hence we have that for $\alpha \in C(\mathcal{M})$,
$$q + N|q\alpha - a| > P.$$
Then by Lemma 2.2, we have
\begin{equation}
\max_{\alpha \in C(\mathcal{M})} |f(\alpha)| \ll N^{\frac{1}{2} - \frac{1}{4k} + \epsilon}.
\end{equation}
It should be remarked that now (3.3) is a standard result, which has been used in [2], [18], [15] and [8], etc. For the second integral in (3.1), by Cauchy’s inequality we have
\begin{align*}
\int_{C(\mathcal{M}) \cap \mathcal{E}_\lambda} &\leq \left( \int_{C(\mathcal{M}) \cap \mathcal{E}_\lambda} |f^2(\alpha)g(\alpha)h^k(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_{C(\mathcal{M}) \cap \mathcal{E}_\lambda} 1d\alpha \right)^{\frac{1}{2}} \\
&\leq \left( \int_{C(\mathcal{M})} |f^2(\alpha)g(\alpha)h^k(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_{\mathcal{E}_\lambda} 1d\alpha \right)^{\frac{1}{2}} \\
&\leq \left( L^{2k} \left( \max_{\alpha \in C(\mathcal{M})} |f(\alpha)| \right)^4 \int_0^1 |g(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \left( \int_{\mathcal{E}_\lambda} 1d\alpha \right)^{\frac{1}{2}}.
\end{align*}
Then by (3.3) and the well-known estimate
$$\int_0^1 |g(\alpha)|^2 d\alpha \ll NL,$$
we have
\begin{align*}
\int_{C(\mathcal{M}) \cap \mathcal{E}_\lambda} &\ll (L^{2k} N^{\frac{1}{2} + \epsilon} N)\left( \text{meas(}\mathcal{E}_\lambda) \right)^{\frac{1}{2}} \\
&\ll (L^{2k} N^{\frac{1}{2} + \epsilon} N)^{\frac{1}{2}} N^{-\frac{E(\lambda)}{2}} \\
&\ll N^{\frac{1}{2} + \epsilon} L^{k} N^{-\frac{E(\lambda)}{2}} \ll N^{1-\epsilon},
\end{align*}

\begin{equation}
\text{(3.4)}
\end{equation}
where we have used Lemma 2.1 with \( \lambda = 0.887167 \), namely
\[
\text{meas}(\mathcal{E}_{0.887167}) \ll N^{-\mathcal{E}(0.887167)} < N^{-\frac{4}{5} - 10^{-10}}.
\]
For the last integral in (3.1) with the definition of \( \mathcal{E}_\lambda \), and Lemmas 2.3 and 2.4, by Cauchy’s inequality we have
\[
\int_{C(M)\backslash \mathcal{E}_\lambda} \leq (\lambda L)^{k-3} \left( \int_0^1 |f(\alpha)h(\alpha)|^4 d\alpha \right)^{\frac{1}{8}} \left( \int_0^1 |g(\alpha)h(\alpha)|^2 d\alpha \right)^{\frac{1}{2}} \leq 21576\lambda^{k-3}\frac{\pi}{4}NL^k.
\]
Combining this with (3.2) and (3.4), we get
\[
R(N) \geq \frac{\pi}{4}NL^k(1.516492 - 21576\lambda^{k-3}).
\]
When \( k \geq 83 \), for \( \lambda = 0.887167 \), by the above estimate we have
\[
R(N) > 0.
\]
This means that every large odd integer \( N \) can be written in the form of (1.2) for \( k \geq 83 \).

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