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A SHORT PROOF OF PITT'S COMPACTNESS THEOREM

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ABSTRACT. We give a short proof of Pitt's theorem that every bounded linear operator from ℓ_p or c_0 into ℓ_q is compact whenever $1 \leq q < p < \infty$.

A bounded linear operator between two Banach spaces X and Y is said to be *compact* if it maps the closed unit ball of X into a relatively compact subset of Y.

Theorem (Pitt; see for example [1], p. 175)**.** *Let $1 \leq q < p \leq +\infty$, and put $X_p = \ell_p$ if $p < +\infty$ and $X_\infty = c_0$. Then every bounded linear operator from X_p into ℓ_q is compact.*

Proof. Let $T : X_p \to \ell_q$ be a norm-one operator. As $1 < p$, the dual of X_p is separable. Hence every bounded sequence in X_p has a weakly Cauchy subsequence. Thus, for proving the compactness of T, it is enough to show that T is weak-to-norm continuous. So, let us consider a weakly null sequence (h_n) in X_p. We have to show that $\lim_{n\to\infty} \|T(h_n)\| = 0$. We claim that

(1) for every $x \in c_0$ and for every weakly null sequence (w_n) in c_0,
$$\limsup_{n\to\infty} \|x + w_n\| = \max(\|x\|, \limsup_{n\to\infty} \|w_n\|),$$
(2) for every $x \in \ell_r$, $1 \leq r < \infty$, and for every weakly null sequence (w_n) in ℓ_r,
$$\limsup_{n\to\infty} \|x + w_n\|^r = \|x\|^r + \limsup_{n\to\infty} \|w_n\|^r.$$

Indeed this is obvious when x is finitely supported, because the coordinates of (w_n) along the support of x tend to 0 in norm. The general case is true by the density of finitely supported elements in X_p and since the norm is a Lipschitzian function.

Fix $0 < \varepsilon < 1$. By definition of the norm of T, there exists $x_\varepsilon \in X_p$ such that $\|x_\varepsilon\| = 1$ and $1 - \varepsilon \leq \|T(x_\varepsilon)\| \leq 1$. Moreover, for all $n \in \mathbb{N}$ and for all $t > 0$
$$\|T(x_\varepsilon) + T(th_n)\| \leq \|x_\varepsilon + th_n\|. \tag{0}$$
In the left-hand side of (0), we apply claim (2) in ℓ_q, with $x = T(x_\varepsilon)$ and the weakly null sequence $(T(th_n))$.

First, assume $p < +\infty$. We apply claim (2) to the right-hand side of (0) with $r = p$, $x = x_\varepsilon$ and the weakly null sequence (th_n) to obtain
$$\left[\|T(x_\varepsilon)\|^q + t^q \limsup_{n\to\infty} \|T(h_n)\|^q\right]^{\frac{1}{q}} \leq \left[\|x_\varepsilon\|^p + t^p \limsup_{n\to\infty} \|h_n\|^p\right]^{\frac{1}{p}}.$$

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Recall that $\|x_\varepsilon\| = 1$, $1-\varepsilon \le \|T(x_\varepsilon)\| \le 1$ and that (h_n) is weakly convergent, thus bounded by some $M > 0$. This gives

$$\limsup_{n\to\infty} \|T(h_n)\|^q \le \frac{1}{t^q}\left[(1+t^pM^p)^{q/p} - (1-\varepsilon)^q\right].$$

Taking $t = \varepsilon^{\frac{1}{p}}$ here, we get

$$\limsup_{n\to\infty} \|T(h_n)\|^q \le \frac{1}{\varepsilon^{q/p}}\left[1 + \frac{q}{p}M^p\varepsilon - (1-q\varepsilon) + o(\varepsilon)\right].$$

Now, letting $\varepsilon \to 0$ here, we get that $\limsup_{n\to\infty} \|T(h_n)\|^q \le 0$, and therefore the sequence $\big(T(h_n)\big)$ norm-converges to 0.

Second, assume $p = +\infty$. We apply claim (1) to the right-hand side of (0) to obtain

$$\limsup_{n\to\infty} \|T(h_n)\|^q \le \frac{1}{t^q}\left[\max\left(1, t^qM^q\right) - (1-\varepsilon)^q\right].$$

Considering here any $0 < \varepsilon < M^{-2q}$ and then taking $t = \varepsilon^{\frac{1}{2q}}$, we get that

$$\limsup_{n\to\infty} \|T(h_n)\|^q \le \frac{1}{\varepsilon^{1/2}}\left[1 - (1-\varepsilon)^q\right].$$

Now, letting $\varepsilon \to 0$ here, we get as before that the sequence $\big(T(h_n)\big)$ norm-converges to 0. □

The framework of this paper was inspired by [2]. The proof given in [2], devoted to the case $p < +\infty$, uses Stegall's variational principle.

References

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