A SHORT PROOF OF PITT’S COMPACTNESS THEOREM

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Abstract. We give a short proof of Pitt’s theorem that every bounded linear operator from $\ell_p$ or $c_0$ into $\ell_q$ is compact whenever $1 \le q < p < \infty$.

A bounded linear operator between two Banach spaces $X$ and $Y$ is said to be compact if it maps the closed unit ball of $X$ into a relatively compact subset of $Y$.

Theorem (Pitt; see for example [1], p. 175). Let $1 \le q < p < +\infty$, and put $X_p = \ell_p$ if $p < +\infty$ and $X_\infty = c_0$. Then every bounded linear operator from $X_p$ into $\ell_q$ is compact.

Proof. Let $T : X_p \to \ell_q$ be a norm-one operator. As $1 < p$, the dual of $X_p$ is separable. Hence every bounded sequence in $X_p$ has a weakly Cauchy subsequence. Thus, for proving the compactness of $T$, it is enough to show that $T$ is weak-to-norm continuous. So, let us consider a weakly null sequence $(h_n)$ in $X_p$. We have to show that $\lim_{n \to \infty} \|T(h_n)\| = 0$. We claim that

1. for every $x \in c_0$ and for every weakly null sequence $(w_n)$ in $c_0$,
   $$\lim_{n \to \infty} \|x + w_n\| = \max(\|x\|, \limsup_{n \to \infty} \|w_n\|),$$
2. for every $x \in \ell_r$, $1 \le r < \infty$, and for every weakly null sequence $(w_n)$ in $\ell_r$,
   $$\lim_{n \to \infty} \|x + w_n\|^r = \|x\|^r + \limsup_{n \to \infty} \|w_n\|^r.$$

Indeed this is obvious when $x$ is finitely supported, because the coordinates of $(w_n)$ along the support of $x$ tend to 0 in norm. The general case is true by the density of finitely supported elements in $X_p$ and since the norm is a Lipschitzian function.

Fix $0 < \varepsilon < 1$. By definition of the norm of $T$, there exists $x_\varepsilon \in X_p$ such that $\|x_\varepsilon\| = 1$ and $1 - \varepsilon \le \|T(x_\varepsilon)\| \le 1$. Moreover, for all $n \in \mathbb{N}$ and for all $t > 0$,

$$\|T(x_\varepsilon) + T(th_n)\| \le \|x_\varepsilon + th_n\|. \quad (0)$$

In the left-hand side of $(0)$, we apply claim (2) in $\ell_q$, with $x = T(x_\varepsilon)$ and the weakly null sequence $(T(th_n))$.

First, assume $p < +\infty$. We apply claim (2) to the right-hand side of $(0)$ with $r = p$, $x = x_\varepsilon$ and the weakly null sequence $(th_n)$ to obtain

$$\left[\|T(x_\varepsilon)\|^q + t^q \limsup_{n \to \infty} \|T(h_n)\|^q\right]^{\frac{1}{q}} \le \left[\|x_\varepsilon\|^p + t^p \limsup_{n \to \infty} \|h_n\|^p\right]^{\frac{1}{p}}.$$
Recall that $\| x_\varepsilon \| = 1$, $1 - \varepsilon \leq \| T(x_\varepsilon) \| \leq 1$ and that $(h_n)$ is weakly convergent, thus bounded by some $M > 0$. This gives

$$
\limsup_{n \to \infty} \| T(h_n) \|^q \leq \frac{1}{t^q} \left[ 1 + \frac{1}{p} MP \varepsilon - (1 - q\varepsilon) \right].
$$

Taking $t = \varepsilon^{\frac{1}{p}}$ here, we get

$$
\limsup_{n \to \infty} \| T(h_n) \|^q \leq \frac{1}{\varepsilon^{q/p}} \left[ 1 + \frac{1}{p} M \varepsilon - (1 - q\varepsilon) + o(\varepsilon) \right].
$$

Now, letting $\varepsilon \to 0$ here, we get that $\limsup_{n \to \infty} \| T(h_n) \|^q \leq 0$, and therefore the sequence $(T(h_n))$ norm-converges to 0.

Second, assume $p = +\infty$. We apply claim (1) to the right-hand side of (0) to obtain

$$
\limsup_{n \to \infty} \| T(h_n) \|^q \leq \frac{1}{t^q} \left[ \max (1, t^q M^q) - (1 - q\varepsilon) \right].
$$

Considering here any $0 < \varepsilon < M^{-2q}$ and then taking $t = \varepsilon^{\frac{1}{q}}$, we get that

$$
\limsup_{n \to \infty} \| T(h_n) \|^q \leq \frac{1}{\varepsilon^{1/2}} \left[ 1 - (1 - \varepsilon)^q \right].
$$

Now, letting $\varepsilon \to 0$ here, we get as before that the sequence $(T(h_n))$ norm-converges to 0.

The framework of this paper was inspired by [2]. The proof given in [2], devoted to the case $p < +\infty$, uses Stegall’s variational principle.

**References**
