AN ALGEBRAIC INDEPENDENCE RESULT FOR EULER PRODUCTS OF FINITE DEGREE

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Abstract. We investigate the algebraic independence of some derivatives of certain multiplicative arithmetical functions over the field \( \mathbb{C} \) of complex numbers.

1. Introduction

In this paper we consider arithmetical functions defined over the field of complex numbers, and their associated Dirichlet series. Let \( r \geq 1 \) be an integer and write

\[
A_r(\mathbb{C}) = \{ f : \mathbb{N}^r \to \mathbb{C} \}.
\]

Given \( f, g \in A_r \), define the convolution \( f * g \) of \( f \) and \( g \) by

\[
(f * g)(n_1, \ldots, n_r) = \sum_{d_1|n_1} \cdots \sum_{d_r|n_r} f(d_1, \ldots, d_r) g\left(\frac{n_1}{d_1}, \ldots, \frac{n_r}{d_r}\right).
\]

Then \( \mathbb{C} \) has a natural embedding in the ring \( A_r \), and \( A_r \) with addition and convolution defined as above becomes a \( \mathbb{C} \)-algebra. The ring \( A_1 \) has been studied from various points of view by a number of authors. We mention in this connection the work of Cashwell and Everett [4], who proved that \((A_1, +, \cdot)\) is a unique factorization domain. Schwab and Silberberg [12] constructed an extension of \((A_1, +, \cdot)\) which is a discrete valuation ring. Alkan and the authors [1] generalized this construction and provided a family of extensions of \( A_r \) which are discrete valuation rings. For other work on rings of arithmetical functions the reader is referred to [5], [6], [9], [12], [13], [10], [11], [2]. In [1], it was shown that for any completely additive arithmetical function \( \psi \in A_r \), the map \( D_\psi : A_r \to A_r \) defined by \( D_\psi(f)(n_1, \ldots, n_r) = f(n_1, \ldots, n_r) \psi(n_1, \ldots, n_r) \), for all \( n_1, \ldots, n_r \in \mathbb{N} \), is a derivation on \( A_r \). It was also proved in [1] that for any multiplicative function \( f \in A_r \), any completely additive function \( \psi \in A_r \), and any \( n_1, \ldots, n_r \in \mathbb{N} \) not all prime powers, \( D_\psi(f)(n_1, \ldots, n_r) = 0 \), where \( D_\psi(f) \) is viewed as \( D_\psi(f) * f^{-1} \). In this connection, a natural line of investigation would be to study the action of \( D_\psi \) on the subring \( \mathbb{C}[f] \) of \( A_r \) generated over \( \mathbb{C} \) by a given multiplicative function \( f \in A_r \), for any \( \psi \) as above. From this point of view, the first issue that arises is to consider the image of \( \mathbb{C}[f] \) through \( D_\psi \), and identify the intersection of \( D_\psi(\mathbb{C}[f]) \) and \( \mathbb{C}[f] \). We will do this for a special class of multiplicative functions \( f \) which are of particular interest, namely, those which have Euler factors of finite degree.
Fix $\psi \in A_r$. Assume that $\psi$ is completely additive and satisfies

$$|\psi(n_1, \ldots, n_r)| \to \infty,$$

as $n_1 + \cdots + n_r \to \infty$. For any $g \in A_r$, any prime number $p$, and any integer $k \in \{1, \ldots, r\}$, let $g_{p,k,r} \in A_1$ be the function defined as follows. Let $m \in \mathbb{N}$. If $m$ is not a power of the prime $p$, then $g_{p,k,r}(m) = 0$. If $m = p^n$ for some nonnegative integer $n$, let

$$g_{p,k,r}(p^n) = g(1, \ldots, 1, p^n, 1, \ldots, 1),$$

where $p^n$ occurs at the $k$-th component of the tuple $(1, \ldots, 1, p^n, 1, \ldots, 1)$ on the right side of (1.2). Given a multiplicative function $f \in A_r$, we say that $f$ has an Euler factor of finite degree at a prime number $p$ provided there exists $k \in \{1, \ldots, r\}$ and $m \in \mathbb{N}$ and nonzero complex numbers $a_1, \ldots, a_m$ such that the Dirichlet series associated to the arithmetical function $f_{p,k,r}$ is given by

$$\sum_{n=1}^{\infty} \frac{f_{p,k,r}(n)}{n^s} = \frac{1}{(1 - a_1 \frac{1}{p}) \cdots (1 - a_m \frac{1}{p^m})}.$$

As a matter of terminology, we will call the above Euler factor trivial if $m = 0$ and respectively nontrivial if $m \geq 1$. We will prove the following result.

**Theorem 1.** Let $\psi \in A_r$ be completely additive and satisfy

$$|\psi(n_1, \ldots, n_r)| \to \infty,$$

as $n_1 + \cdots + n_r \to \infty$. Let $f \in A_r$ be multiplicative and such that for infinitely many prime numbers $p$, $f$ has an Euler product of finite degree at $p$ as defined above. Then for any distinct nonnegative integers $i$, and $j$, the derivations $D_{\psi}^i(f)$ and $D_{\psi}^j(f)$ of $f$ of orders $i$ and $j$ respectively are algebraically independent over $\mathbb{C}$.

As a consequence of this result, for $\psi$ and $f$ as above, the arithmetical function which is constant and equal to zero is the only common element of $D_{\psi}(\mathbb{C}[f])$ and $\mathbb{C}[f]$.

**Corollary 1.** Let $\psi$ and $f$ be elements of $A_r$ satisfying the assumptions in Theorem 1. Let $\mathbb{C}[f]$ be the subring of $A_r$ generated over $\mathbb{C}$ by $f$. Then,

$$D_{\psi}(\mathbb{C}[f]) \cap \mathbb{C}[f] = 0.$$

We end this section with some examples. Let $r = 1$, and let $\psi_0 \in A_1$ be the completely additive function given by $\psi_0(n) = -\log n$ for all $n \in \mathbb{N}$. Then condition (1.3) is satisfied. Next, let $f = \chi$ be a Dirichlet character. So $f$ satisfies the condition in Theorem 1 with $m = 1$, for all but finitely many primes (where the corresponding Euler factor is trivial). Then Theorem 1 applies, and it shows that the derivations $D_{\psi_0}^i(\chi)$ and $D_{\psi_0}^j(\chi)$ of $\chi$ of orders $i$ and $j$ are algebraically independent for any nonnegative distinct integers $i$ and $j$. Moreover, by the standard isomorphism which sends any arithmetical function $h \in A_1(\mathbb{C})$ to its associated Dirichlet series $H(s) = L(s, h) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$, and also sends $D_{\psi_0}(h)$ to $\frac{d}{ds}(H(s))$, we see that for any nonnegative distinct integers $i$ and $j$, the functions $L^{(i)}(s, \chi)$ and $L^{(j)}(s, \chi)$ are algebraically independent over $\mathbb{C}$.
Thus Corollary 1, and therefore also Theorem 1, fails in this case. But the Fourier coefficients 
\[ \alpha \]
where the product is taken over all primes, satisfy the hypothesis of Theorem 1 either.

Let 
\[ t \]
be a newform (or normalized Hecke eigenform) of weight \( k \) in \( S_k(\Gamma_1(N), \chi) \) which has Fourier expansion

\[ f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}, \quad \text{Im} \, z > 0. \]

The Fourier coefficients \( a_f(n) \) form a multiplicative arithmetical function. The associated \( L \)-function is given by

\[ L(s, f) = \sum_{n=1}^{\infty} a_f(n) n^{-s}, \]

where \( s \in \mathbb{C} \) is a complex variable. Here \( L(s, f) \) has an Euler product expansion

\[ L(s, f) = \prod_p (1 - a_f(p) p^{-s} + \chi(p) p^{k-1-2s})^{-1} = \prod_p \frac{1}{1 - \frac{\alpha p^{k-2} + \beta}{p^s}} \frac{1}{1 - \frac{\beta p^{k-2}}{p^s}}, \]

where the product is taken over all primes, \( \alpha_p + \beta_p = a_f(p) p^{1-2s} \), and \( \alpha_p \beta_p = \chi(p) \).

For example, one can take the Ramanujan tau function \( \tau(n) \), defined in terms of the Delta function

\[ \Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z}, \]

which is the unique normalized cusp form of weight 12 on \( SL_2(\mathbb{Z}) \). The Euler product expansion of the \( L \)-series associated to \( \Delta(z) \) is given by

\[ L(s, \Delta) = \prod_p (1 - \tau(p) p^{-s} + p^{11-2s})^{-1} = \prod_p \frac{1}{1 - \frac{\alpha_p + \beta_p}{p^s}} \frac{1}{1 - \frac{\beta p^{11}}{p^s}}, \]

where the product is taken over all primes, \( \alpha_p + \beta_p = \tau(p) p^{-11+2s} \), and \( \alpha_p \beta_p = 1 \).

The conditions in Theorem 1 are satisfied in this case, and therefore any two derivatives of \( L(s, f) \) are algebraically independent over \( \mathbb{C} \).
Theorem 1 applies, more generally, to the case when $f$ is an automorphic cusp form on $GL_m/\mathbb{Q}$, $m \geq 1$. Its $L$-function $L(s, f)$ has an Euler product of degree $m$:

$$L(s, f) = \prod_{p} L(s, f_p),$$

where

$$L(s, f_p) = \frac{1}{\prod_{\ell=1}^{m} \left( 1 - \frac{\alpha_{p, f}(\ell)}{p^s} \right)}.$$

By Theorem 1 any two derivatives of $L(s, f)$ are algebraically independent over $\mathbb{C}$.

2. Preliminaries

Let $r$ be a positive integer and denote as above $A_r = \{ f : N^r \rightarrow \mathbb{C} \}$. We say that an arithmetical function $f \in A_r$ is multiplicative provided one has

$$f(n_1 m_1, \ldots, n_r m_r) = f(n_1, \ldots, n_r) f(m_1, \ldots, m_r),$$

for any $n_1, \ldots, n_r, m_1, \ldots, m_r \in \mathbb{N}$ satisfying $(n_1, m_1) = \cdots = (n_r, m_r) = 1$. We say that $f \in A_r$ is completely multiplicative provided

$$f(n_1 m_1, \ldots, n_r m_r) = f(n_1, \ldots, n_r) f(m_1, \ldots, m_r),$$

for any $n_1, \ldots, n_r, m_1, \ldots, m_r \in \mathbb{N}$. Similarly we say that a function $f \in A_r(R)$ is additive provided

$$f(n_1 m_1, \ldots, n_r m_r) = f(n_1, \ldots, n_r) + f(m_1, \ldots, m_r),$$

for any $n_1, \ldots, n_r, m_1, \ldots, m_r \in \mathbb{N}$ satisfying $(n_1, m_1) = \cdots = (n_r, m_r) = 1$. We call a function $f \in A_r$ completely additive provided

$$f(n_1 m_1, \ldots, n_r m_r) = f(n_1, \ldots, n_r) + f(m_1, \ldots, m_r),$$

for any $n_1, \ldots, n_r, m_1, \ldots, m_r \in \mathbb{N}$. For any completely additive function $\psi \in A_r$, the map $D_\psi : A_r \rightarrow A_r$ defined by

$$D_\psi(f)(n_1, \ldots, n_r) = f(n_1, \ldots, n_r) \psi(n_1, \ldots, n_r),$$

for all $n_1, \ldots, n_r \in \mathbb{N}$, satisfies the following properties (see [1]). For all $f, g \in A_r$ and $c \in \mathbb{C}$,

(a) $D_\psi(f + g) = D_\psi(f) + D_\psi(g),$

(b) $D_\psi(cf) = c D_\psi(f),$

(c) $D_\psi(gf) = D_\psi(g) + D_\psi(f).$

Consequently, $D_\psi$ is a derivation on $A_r$ over $\mathbb{C}$.

Every $f \in A_r$ has an associated formal Dirichlet series

$$\tilde{f}(s_1, \ldots, s_r) = \sum_{n_1, \ldots, n_r \in \mathbb{N}} \frac{f(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}}.$$

Let $\mathcal{A}_r$ be the ring of all such series with the usual addition and multiplication of series. The map $f \rightarrow \tilde{f}$ is a ring isomorphism.

For any $g \in A_r$, a prime number $p$, and an integer $k \in \{1, \ldots, r\}$, let us denote by $\phi_{p,k}$ the map from $A_r$ into $A_1$ which sends $g$ to $g_{p,k} = g_{p,k} \in A_1$, where $g_{p,k}$ is defined as in Section 1. The mapping $\phi_{p,k}$ is a homomorphism of $\mathbb{C}$-algebras: for any $c \in \mathbb{C}$ and $g, h \in A_r$, $(cg)_{p,k} = cg_{p,k}$, $(g + h)_{p,k} = g_{p,k} + h_{p,k}$, and $(g * h)_{p,k} = g_{p,k} * h_{p,k}$. To see this, let $n \in \mathbb{N}$ and consider the $r$-tuple
(1, \ldots, 1, p^n, 1, \ldots, 1) \in \mathbb{N}^r$, where $p^n$ occurs at the $k$-th component of the tuple. Then,
\[
(g \ast h)_{p,k,r}(p^n) = (g \ast h)(1, \ldots, 1, p^n, 1, \ldots, 1).
\]
\[
= \sum_{d|p^n} g(1, \ldots, d, 1, \ldots, 1)h(1, \ldots, \frac{p^n}{d}, 1, \ldots, 1)
\]
\[
= \sum_{d|p^n} g_{p,k,r}(d)h_{p,k,r}(\frac{p^n}{d})
\]
\[
= g_{p,k,r} \ast h_{p,k,r}(p^n).
\]

On the other hand, if $n$ is not a power of the prime $p$, then we have that
\[
(g \ast h)_{p,k,r}(n) = 0 = g_{p,k,r} \ast h_{p,k,r}(n).
\]

Therefore, $(g \ast h)_{p,k,r} = g_{p,k,r} \ast h_{p,k,r}$. Similarly, one sees that $(cg)_{p,k,r} = cg_{p,k,r}$ and $(g + h)_{p,k,r} = g_{p,k,r} + h_{p,k,r}$.

Note that the homomorphism sending any $g \in A_r$ to $g_{p,k,r} \in A_1$ induces a homomorphism of $\mathcal{A}_r$ onto $\mathcal{A}_1$ which sends $f(s_1, \ldots, s_r)$ to $f_{p,k,r}(s)$. As an example, for $r = 1$, this map sends the Riemann zeta function
\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_q \left(1 - \frac{1}{q^s}\right)
\]
to the function $\zeta_p(s) = \sum_{k=0}^{\infty} \frac{1}{p^k} = \frac{1}{1 - \frac{1}{p^s}}$. Also, $-\zeta'(s) \zeta_p(s)$ is sent to $-\frac{\zeta'(s)}{\zeta_p(s)} = \frac{\log p}{p^s - 1}$.

3. THE CASE OF THE RIEMANN ZETA FUNCTION

In order to present the main idea behind the proof of Theorem 1 in terms as simple as possible, in this section we show that the Riemann zeta function $\zeta(s)$ and its derivative $\zeta'(s)$ are algebraically independent over $\mathbb{C}$. In doing this, we will avoid the use of any analytic properties of the Riemann zeta function, so that we later have a chance of generalizing this reasoning in the context of Theorem 1 where one does not have any assumptions on the convergence of the Dirichlet series associated to $f$, or its Euler product. Returning to the Riemann zeta function, let us assume that $\zeta(s)$ and $\zeta'(s)$ are algebraically dependent, and let $Q(x, y)$ be a nonzero polynomial in two variables $x$ and $y$ with coefficients in $\mathbb{C}$ such that $Q(\zeta(s), \zeta'(s)) = 0$. Let $P(x, y) = Q(x, xy)$. Then $P(x, y)$ is a nonzero polynomial and $P \left( \zeta(s), \frac{\zeta'(s)}{\zeta(s)} \right) = 0$. Next, this gives us an equality in $A_1$, namely
\[
P(I, -D_{\psi_0(I)} \ast I^{-1}) = 0,
\]
where $I \in A_1$ denotes the arithmetical function given by $I(n) = 1$, and $\psi_0$ is the completely additive function given by $\psi_0(n) = \log(n)$ for all $n \in \mathbb{N}$. Now for any prime $p$, we apply the homomorphism $\phi_{p,1,1}$ to the equality (3.1) and find that $P(I_p, -D_{\psi_0(I_p)} \ast I_p^{-1}) = 0$. This in turn gives us an equality between the corresponding Dirichlet series, namely
\[
P \left( \zeta_p(s), \frac{-\zeta'_p(s)}{\zeta_p(s)} \right) = 0.
\]
This is a nontrivial relation which needs to be satisfied by each Euler factor $\zeta_p(s)$ of $\zeta(s)$ with the same polynomial $P$. On the other hand, one checks by a direct computation that

$$
-\frac{\zeta_p'(s)}{\zeta_p(s)} = (1 + \zeta_p(s)) \log p.
$$

Using equation (3.3) in [3.2], we derive that $\zeta_p(s)$ is a zero of the polynomial $U_p(t)$ which is given by $U_p(t) = P(t, (t + 1) \log p)$. Since $\zeta_p(s)$ is transcendental over $\mathbb{C}$, $U_p(t)$ has to be identically zero. But, since $P(x, y)$ is a nonzero polynomial, $P(t, (t + 1) \log p)$ can be identically zero only for finitely many values of $p$, and this completes the proof that $\zeta(s)$ and $\zeta'(s)$ are algebraically independent over $\mathbb{C}$.

4. Proof of Theorem 1

Let $\psi$ and $f$ be as in the statement of Theorem 1. By our assumptions, we know that there is an infinite set $\mathcal{P}$ of prime numbers with the following property. For each prime $p \in \mathcal{P}$, there exists a component $k_p \in \{1, \ldots, r\}$ such that the Dirichlet series associated to the arithmetical function $f_{p, k, r}$ is given by

$$
F_{p, k, r}(s) = \frac{1}{\prod_{n=1}^{\infty} \left(1 - \frac{a_n}{p^n}\right)},
$$

for some $m \in \mathbb{N}$ and nonzero complex numbers $a_1, \ldots, a_m$. Therefore, there exists a component $k \in \{1, \ldots, r\}$ and an infinite subset $\mathcal{P}_k \subseteq \mathcal{P}$ of prime numbers $p$ such that the corresponding values $k_p$ are the same and equal $k$.

Fix such an integer $k$ and a prime number $p$ in the subset $\mathcal{P}_k$. Let $F(t)$ be defined by $F(t) = F_{p, k, r}(t) = \left(\frac{1}{1 - t}\right) \prod_{n=1}^{\infty} \left(1 - \frac{a_n}{p^n}\right)$. Then, we see that $F_{p, k, r}(s) = F(p^{-s})$. Let $\psi_k \in A_1$ be the function defined by $\psi_k(n) = \psi(1, \ldots, 1, n, 1, \ldots, 1)$ for all $n \geq 1$, where $n$ occurs at the $k$-th component of the tuple $(1, \ldots, 1, n, 1, \ldots, 1)$ on the right side.

Let $\mathbb{C}(t)$ denote, as usual, the field of rational functions in $t$ over $\mathbb{C}$, and $R(t)$ the derivative of $R(t) \in \mathbb{C}(t)$ as a rational function. Define $\Gamma : \mathbb{C}(t) \rightarrow \mathbb{C}(t)$ by $\Gamma(R(t)) = \psi_k(p)tR'(t)$, for $R(t) \in \mathbb{C}(t)$.

Also define

$$
F_{p, k, r}(s) = \prod_{n=1}^{\infty} \left(1 - \frac{a_n}{p^n}\right),
$$

and inductively $F_{p, k, r}(s) = \Gamma(F_{p, k, r}(t))((p^{-s})^l)$ for any positive integer $l$, where $\Gamma(l)$ denotes the composition of $\Gamma$ with itself $l$ times.

Now let $G(t) = G_{p, k, r}(t) = \frac{1}{F_{p, k, r}(t)}$. Then, we find that $G_{p, k, r}(t)$ is a polynomial $G_{p, k, r}(t) = \alpha_p t^m + \cdots$ with leading coefficient $\alpha_p = (-1)^m a_1 \cdots a_m$, and its derivative is given by $G'_{p, k, r}(t) = m \alpha_p t^{m-1} + \cdots$.

Next, define inductively $B_0 = B_{p, k, r, 0} = 1$ and

$$
B_{n+1}(t) = B_{p, k, r, n+1}(t) = t \left( G_{p, k, r}(t) B_n(t) - (n + 1) G'_{p, k, r}(t) B_n(t) \right).
$$

We claim that

$$
\Gamma^n(F_{p, k, r}(t)) = \frac{B_n(t)}{(G(t))^{n+1}} = \frac{B_n(t)}{(1 - a_1 t)^{n+1} \cdots (1 - a_m t)^{n+1}}.
$$
To prove this claim, first notice that $\Gamma^0(F_{p,k,r}(t)) = F_{p,k,r} = \frac{B_0(t)}{G(t)}$ since $B_0 = 1$. Next, assume that $n \geq 1$ and $\Gamma^n(F_{p,k,r}(t)) = \frac{B_n(t)}{G(t)^{n+1}}$. Then,

$$
\Gamma^{n+1}(F_{p,k,r}(t)) = \Gamma(\Gamma^n(F_{p,k,r}(t)))
$$

$$
= \Gamma \left( \frac{B_n(t)}{(G(t))^{n+1}} \right)
$$

$$
= t' \left( \frac{B_n(t)}{(G(t))^{n+1}} \right)
$$

$$
= \frac{b'_n(t)G(t)^{n+1} - (n+1)B_n(t)G(t)^nu'(t)}{G(t)^{2n+2}}
$$

$$
= \frac{b'_n(t)G(t) - (n+1)B_n(t)u'(t)}{G(t)^{n+2}}.
$$

This completes the proof of the claim.

Observe that $\deg(G_{p,k,r}(t)) = m$. Now we show inductively that $B_n(t)$ is a polynomial of degree $\deg(B_n(t)) = mn$ with leading coefficient $(-1)^n\alpha_p^nm^n\psi_k(p)$ for all $n \geq 1$. Clearly, $B_0(t)$ satisfies this claim. Assume that $n \geq 1$, and $B_n(t)$ satisfies the claim. We would like to prove that $B_{n+1}(t)$ satisfies the claim as well; i.e., $B_{n+1}(t)$ is a polynomial of degree $\deg(B_{n+1}(t)) = (n+1)m$ with leading coefficient $(-1)^{n+1}\alpha_p^{n+1}m^{n+1}\psi_k(p)$. Since

$$
B_{n+1}(t) = B_{(p,k),n+1}(t) = t \left( G_{p,k,r}(t)B'_n(t) - (n+1)G'_{p,k,r}(t)B_n(t) \right),
$$

its leading term can be written in the form

$$
t(\alpha_p t^n)nm(-1)^n\alpha_p^nm^n\psi_k(p)t^{nm-1} - t(\alpha_p(n+1)mt^{m-1})(-1)^n\alpha_p^nm^n\psi_k(p)t^{nm}
$$

$$
= \alpha_p nm(-1)^n\alpha_p^nm^n\psi_k(p)t^{(n+1)m} - \alpha_p(n+1)m(-1)^n\alpha_p^nm^n\psi_k(p)t^{(n+1)m}
$$

$$
= (-1)^n\alpha_p^{n+1}m^{n+1}\psi_k(p)(nm - m(n+1))t^{(n+1)m}.
$$

Hence the desired claim holds.

Now let $i,j$ be nonnegative integers such that $i \neq j$. We have that

$$
\overline{\mathcal{F}}^{(i)}_{p,k}(s) = (\Gamma^{(i)}(F(t)))(p^{-s})
$$

and

$$
\overline{\mathcal{F}}^{(j)}_{p,k}(s) = (\Gamma^{(j)}(F(t)))(p^{-s}).
$$

Let $S$ denote a finite set of pairs $(u,v)$ of positive integers. Let $P(X,Y) \in \mathbb{C}[X,Y]$ and $P(X,Y) = \sum_{(u,v) \in S} C_{uv}X^uY^v$, where $C_{uv}$ is a nonzero complex number for every $(u,v) \in S$.

Suppose that

$$
P(D_{\psi}^i(f), D_{\psi}^j(f)) = 0.
$$

By applying the homomorphism $\phi_{p,k,r}$ to both sides of equality (4.1), we find that $P(D_{\psi}^i(f)_{p,k,r}, D_{\psi}^j(f)_{p,k,r}) = 0$. This in turn gives us an equality between the corresponding Dirichlet series, namely,

$$
P(\overline{\mathcal{F}}^i_{p,k}(s), \overline{\mathcal{F}}^j_{p,k}(s)) = 0.
$$

Thus,

$$
\sum_{(u,v) \in S} C_{uv} \left( \frac{B_i(t)}{(G(t))^{i+1}} \right)^u \left( \frac{B_j(t)}{(G(t))^{j+1}} \right)^v = 0.
$$

Let $N = \max_{(u,v) \in S} \{(i+1)u + (j+1)v\}$. We have that

$$
\sum_{(u,v) \in S} C_{uv}B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v} = 0.
$$
Note that 
\[ \text{deg}(B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v}) = ium + vjm + (N-(i+1)u-(j+1)v)m = m(N-u-v). \]

Let \( L = \min_{(u,v) \in S} \{ u + v \} \). Then equality (4.3) can be written as
\[
\sum_{(u,v) \in S, u+v=L} C_{uv} B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v} + \sum_{(u,v) \in S, u+v>L} C_{uv} B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v} = 0.
\]

For \( f \in A_1 \), consider the support of \( f \) given by \( \text{supp}(f) = \{ n \in \mathbb{N} | f(n) \neq 0 \} \). By abuse of notation, let us denote by \( B_i, B_j, \) and \( G \) the arithmetical functions whose Dirichlet series are given respectively by \( B_i(p^{-s}), B_j(p^{-s}), \) and \( G(p^{-s}) \). Note that the support of the arithmetical function \( (B_i^a B_j^b G^{N-(i+1)u-(j+1)v}) \) is a subset of \( \{1, p, p^2, \ldots, p^m(N-L)\} \). So the arithmetical function corresponding to the second sum in the above equation, that is, the function given by the sum
\[
\sum_{(u,v) \in S, u+v>L} C_{uv} B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v},
\]
vanishes at \( p^m(N-L) \). Since this must hold for infinitely many primes, we conclude that the second sum in the equation above vanishes, and thus
\[
\sum_{(u,v) \in S, u+v=L} C_{uv} B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v} = 0.
\]

In this equation, the coefficient of \( t^{m(N-L)} \) is
\[
\sum_{(u,v) \in S, u+v=L} C_{uv}((-\psi_k(p))^j \alpha_p^m)^u ((-\psi_k(p))^j \alpha_p^m)^v (\alpha_p^{N-(i+1)u-(j+1)v}),
\]
which equals
\[
\sum_{(u,v) \in S, u+v=L} C_{uv}(-\psi_k(p))^i u^j v^m \alpha_p^{N-u-v} m^{i+u+jv}.
\]

We rewrite this sum as
\[
\sum_{(u,v) \in S, u+v=L} C_{uv}(-m\psi_k(p))^{i+u+jv} \alpha_p^{N-L}.
\]

Since the coefficient of \( t^{m(N-L)} \) must equal zero, we have that
\[
\sum_{(u,v) \in S, u+v=L} C_{uv}(-m\psi_k(p))^{i+u+jv} \alpha_p^{N-L} = 0.
\]
But, \( \alpha_p \neq 0 \), and so we must have
\[
(4.4) \quad \sum_{(u,v) \in S, u+v=L} C_{uv}(-m\psi_k(p))^{i+u+jv} = 0.
\]
By our assumption on $\psi$ and our observation on the set $\mathcal{P}_k$ of prime numbers at the beginning of this section, it follows that there exist infinitely many distinct values $\psi_k(p)$ for primes in $\mathcal{P}_k$. Each of these values $\psi_k(p)$ must satisfy (4.4), which is not possible. This completes the proof of Theorem 1.

Proof of Corollary 1. Let $f$ and $\psi$ be as in the statement of Corollary 1. Let $Q \in \mathbb{C}[f]$ be such that $D_{\psi}(Q) \in \mathbb{C}[f]$. Since for any $c \in \mathbb{C}$ and $n \in \mathbb{N}$, $D_{\psi}(cf^n) = cnf^{n-1}D_{\psi}(f)$, $D_{\psi}(Q)$ equals $D_{\psi}(f)$ times a polynomial in $f$. But, $f$ and $D_{\psi}(f)$ being algebraically independent, the only multiple of $D_{\psi}(f)$ inside $\mathbb{C}[f, D_{\psi}(f)]$ which belongs to $\mathbb{C}[f]$ is zero, and this proves the corollary. □

References


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