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# AN ALGEBRAIC INDEPENDENCE RESULT FOR EULER PRODUCTS OF FINITE DEGREE

ALEXANDRU ZAHARESCU AND MOHAMMAD ZAKI

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ABSTRACT. We investigate the algebraic independence of some derivatives of certain multiplicative arithmetical functions over the field $\mathbb{C}$ of complex numbers.

## 1. INTRODUCTION

In this paper we consider arithmetical functions defined over the field of complex numbers, and their associated Dirichlet series. Let $r \geq 1$ be an integer and write $A_r = A_r(\mathbb{C}) = \{f : \mathbb{N}^r \to \mathbb{C}\}$. Given $f, g \in A_r$, define the convolution $f * g$ of $f$ and $g$ by

$$(f * g)(n_1, ..., n_r) = \sum_{d_1|n_1} ... \sum_{d_r|n_r} f(d_1, ..., d_r) g(\frac{n_1}{d_1}, ..., \frac{n_r}{d_r}). \tag{1.1}$$

Then $\mathbb{C}$ has a natural embedding in the ring $A_r$, and $A_r$ with addition and convolution defined as above becomes a $\mathbb{C}$-algebra. The ring $A_1$ has been studied from various points of view by a number of authors. We mention in this connection the work of Cashwell and Everett [4], who proved that $(A_1, +, .)$ is a unique factorization domain. Schwab and Silberberg [12] constructed an extension of $(A_1, +, .)$ which is a discrete valuation ring. Alkan and the authors [1] generalized this construction and provided a family of extensions of $A_r$ which are discrete valuation rings. For other work on rings of arithmetical functions the reader is referred to [5], [6], [9], [12], [13], [10], [11], [2]. In [1], it was shown that for any completely additive arithmetical function $\psi \in A_r$, the map $D_\psi : A_r \to A_r$ defined by $D_\psi(f)(n_1, \ldots, n_r) = f(n_1, \ldots, n_r)\psi(n_1, \ldots, n_r)$, for all $n_1, \ldots, n_r \in \mathbb{N}$, is a derivation on $A_r$. It was also proved in [1] that for any multiplicative function $f \in A_r$, any completely additive function $\psi \in A_r$, and any $n_1, \ldots, n_r \in \mathbb{N}$ not all prime powers, $\frac{D_\psi(f)}{f}(n_1, \ldots, n_r) = 0$, where $\frac{D_\psi(f)}{f}$ is viewed as $D_\psi(f) * f^{-1}$. In this connection, a natural line of investigation would be to study the action of $D_\psi$ on the subring $\mathbb{C}[f]$ of $A_r$ generated over $\mathbb{C}$ by a given multiplicative function $f \in A_r$, for any $\psi$ as above. From this point of view, the first issue that arises is to consider the image of $\mathbb{C}[f]$ through $D_\psi$, and identify the intersection of $D_\psi(\mathbb{C}[f])$ and $\mathbb{C}[f]$. We will do this for a special class of multiplicative functions $f$ which are of particular interest, namely, those which have Euler factors of finite degree.

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Fix $\psi \in A_r$. Assume that $\psi$ is completely additive and satisfies

$$|\psi(n_1, \ldots, n_r)| \to \infty,$$

as $n_1 + \cdots + n_r \to \infty$. For any $g \in A_r$, any prime number $p$, and any integer $k \in \{1, \ldots, r\}$, let $g_{p,k,r} \in A_1$ be the function defined as follows. Let $m \in \mathbb{N}$. If $m$ is not a power of the prime $p$, then $g_{p,k,r}(m) = 0$. If $m = p^n$ for some nonnegative integer $n$, let

$$g_{p,k,r}(p^n) = g(1, \ldots, 1, p^n, 1, \ldots, 1), \tag{1.2}$$

where $p^n$ occurs at the $k$-th component of the tuple $(1, \ldots, 1, p^n, 1, \ldots, 1)$ on the rightside of (1.2). Given a multiplicative function $f \in A_r$, we say that $f$ has an Euler factor of finite degree at a prime number $p$ provided there exists $k \in \{1, \ldots, r\}$ and $m \in \mathbb{N}$ and nonzero complex numbers $a_1, \ldots, a_m$ such that the Dirichlet series associated to the arithmetical function $f_{p,k,r}$ is given by

$$\sum_{n=1}^{\infty} \frac{f_{p,k,r}(n)}{n^s} = \frac{1}{\left(1 - \frac{a_1}{p^s}\right) \cdots \left(1 - \frac{a_m}{p^s}\right)}.$$

As a matter of terminology, we will call the above Euler factor trivial if $m = 0$ and respectively nontrivial if $m \geq 1$. We will prove the following result.

**Theorem 1.** *Let $\psi \in A_r$ be completely additive and satisfy*

$$|\psi(n_1, \ldots, n_r)| \to \infty, \tag{1.3}$$

*as $n_1 + \cdots + n_r \to \infty$. Let $f \in A_r$ be multiplicative and such that for infinitely many prime numbers $p$, $f$ has an Euler product of finite degree at $p$ as defined above. Then for any distinct nonnegative integers $i$, and $j$, the derivations $D_\psi^i(f)$ and $D_\psi^j(f)$ of $f$ of orders $i$ and $j$ respectively are algebraically independent over $\mathbb{C}$.*

As a consequence of this result, for $\psi$ and $f$ as above, the arithmetical function which is constant and equal to zero is the only common element of $D_\psi(\mathbb{C}[f])$ and $\mathbb{C}[f]$.

**Corollary 1.** *Let $\psi$ and $f$ be elements of $A_r$ satisfying the assumptions in Theorem* 1. *Let $\mathbb{C}[f]$ be the subring of $A_r$ generated over $\mathbb{C}$ by $f$. Then,*

$$D_\psi(\mathbb{C}[f]) \cap \mathbb{C}[f] = 0.$$

We end this section with some examples. Let $r = 1$, and let $\psi_0 \in A_1$ be the completely additive function given by $\psi_0(n) = -\log n$ for all $n \in \mathbb{N}$. Then condition (1.3) is satisfied. Next, let $f = \chi$ be a Dirichlet character. So $f$ satisfies the condition in Theorem 1 with $m = 1$, for all but finitely many primes (where the corresponding Euler factor is trivial). Then Theorem 1 applies, and it shows that the derivations $D_{\psi_0}^{(i)}(\chi)$ and $D_{\psi_0}^{(j)}(\chi)$ of $\chi$ of orders $i$ and $j$ are algebraically independent for any nonnegative distinct integers $i$ and $j$. Moreover, by the standard isomorphism which sends any arithmetical function $h \in A_1(\mathbb{C})$ to its associated Dirichlet series $H(s) = L(s, h) = \sum_1^\infty \frac{h(n)}{n^s}$, and also sends $D_{\psi_0}(h)$ to $\frac{d}{ds}(H(s))$, we see that for any nonnegative distinct integers $i$ and $j$, the functions $L^{(i)}(s, \chi)$ and $L^{(j)}(s, \chi)$ are algebraically independent over $\mathbb{C}$.

For another example, let us again take $r = 1$, $f = \chi$, and $\psi_0$ as above. Also fix a prime $p$ such that $\chi(p) \neq 0$. Next, let $\chi_p \in A_1(\mathbb{C})$ be defined by

$$\chi_p(n) = \begin{cases} \chi(n), & \text{if } n = p^m,\ m \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that

$$\begin{aligned} D_{\psi_0}(\chi_p)(p^m) &= (-m\log p)(\chi_p(p))^m \\ &= (-\log p)(\chi_p(p))^m((\sum_{d|p^m} 1) - 1) \\ &= (\log p)(\chi_p^2(p^m) - \chi_p(p^m)). \end{aligned}$$

One finds that

$$D_{\psi_0}(\mathbb{C}[\chi_p]) = (\chi_p^2 - \chi_p)\mathbb{C}[\chi_p].$$

Thus Corollary 1, and therefore also Theorem 1, fails in this case. But $\chi_p$ does not satisfy the hypothesis of Theorem 1 either.

Other interesting examples arise from the theory of modular forms. For a nice treatment of this subject the reader is referred to the recent monograpgh of Ono [7]. Let $f(z)$ be a newform (or normalized Hecke eigenform) of weight $k$ in $S_k(\Gamma_1(N), \chi)$ which has Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z},\ \operatorname{Im} z > 0.$$

The Fourier coefficients $a_f(n)$ form a multiplicative arithmetical function. The associated $L$-function is given by

$$L(s, f) = \sum_{n=1}^{\infty} a_f(n) n^{-s},$$

where $s \in \mathbb{C}$ is a complex variable. Here $L(s, f)$ has an Euler product expansion

$$L(s,f) = \prod_p (1 - a_f(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1} = \prod_p \frac{1}{\left(1 - \frac{\alpha_p p^{\frac{k-1}{2}}}{p^s}\right)\left(1 - \frac{\beta_p p^{\frac{k-1}{2}}}{p^s}\right)},$$

where the product is taken over all primes, $\alpha_p + \beta_p = a_f(p)p^{\frac{1-k}{2}}$, and $\alpha_p\beta_p = \chi(p)$.

For example, one can take the Ramanujan tau function $\tau(n)$, defined in terms of the Delta function

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z}, \tag{1.4}$$

which is the unique normalized cusp form of weight 12 on $SL_2(\mathbb{Z})$. The Euler product expansion of the $L$-series associated to $\Delta(z)$ is given by

$$L(s,\Delta) = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1} = \prod_p \frac{1}{\left(1 - \frac{\alpha_p p^{\frac{11}{2}}}{p^s}\right)\left(1 - \frac{\beta_p p^{\frac{11}{2}}}{p^s}\right)},$$

where the product is taken over all primes, $\alpha_p + \beta_p = \tau(p)p^{-\frac{11}{2}}$, and $\alpha_p\beta_p = 1$.

The conditions in Theorem 1 are satisfied in this case, and therefore any two derivatives of $L(s, f)$ are algebraically independent over $\mathbb{C}$.

Theorem 1 applies, more generally, to the case when $f$ is an automorphic cusp form on $GL_m/\mathbb{Q}$, $m \geq 1$. Its $L$-function $L(s,f)$ has an Euler product of degree $m$: $L(s,f) = \prod_p L(s,f_p)$, where

$$L(s,f_p) = \frac{1}{\prod_{j=1}^{m}\left(1 - \frac{\alpha_{j,f}(p)}{p^s}\right)}.$$

By Theorem 1, any two derivatives of $L(s,f)$ are algebraically independent over $\mathbb{C}$.

## 2. Preliminaries

Let $r$ be a positive integer and denote as above $A_r = \{f : \mathbb{N}^r \to \mathbb{C}\}$. We say that an arithmetical function $f \in A_r$ is multiplicative provided one has

$$f(n_1m_1, ..., n_rm_r) = f(n_1, ..., n_r)f(m_1, ..., m_r),$$

for any $n_1, ..., n_r, m_1, ..., m_r \in \mathbb{N}$ satisfying $(n_1, m_1) = \cdots = (n_r, m_r) = 1$. We say that $f \in A_r$ is completely multiplicative provided

$$f(n_1m_1, ..., n_rm_r) = f(n_1, ..., n_r)f(m_1, ..., m_r),$$

for any $n_1, ..., n_r, m_1, ..., m_r \in \mathbb{N}$. Similarly we say that a function $f \in A_r(R)$ is additive provided

$$f(n_1m_1, ..., n_rm_r) = f(n_1, ..., n_r) + f(m_1, ..., m_r),$$

for any $n_1, ..., n_r, m_1, ..., m_r \in \mathbb{N}$ satisfying $(n_1, m_1) = \cdots = (n_r, m_r) = 1$. We call a function $f \in A_r$ completely additive provided

$$f(n_1m_1, ..., n_rm_r) = f(n_1, ..., n_r) + f(m_1, ..., m_r),$$

for any $n_1, ..., n_r, m_1, ..., m_r \in \mathbb{N}$. For any completely additive function $\psi \in A_r$, the map $D_\psi : A_r \to A_r$ defined by

$$D_\psi(f)(n_1, \ldots, n_r) = f(n_1, \ldots, n_r)\psi(n_1, \ldots, n_r),$$

for all $n_1, \ldots, n_r \in \mathbb{N}$, satisfies the following properties (see [1]). For all $f, g \in A_r$ and $c \in \mathbb{C}$,

(a) $D_\psi(f+g) = D_\psi(f) + D_\psi(g)$,
(b) $D_\psi(fg) = fD_\psi(g) + gD_\psi(f)$,
(c) $D_\psi(cf) = cD_\psi(f)$.

Consequently, $D_\psi$ is a derivation on $A_r$ over $\mathbb{C}$.

Every $f \in A_r$ has an associated formal Dirichlet series

$$\overline{f}(s_1, \ldots, s_r) = \sum_{n_1, \ldots, n_r \in \mathbb{N}} \frac{f(n_1, \ldots, n_r)}{n_1^{s_1} \cdots n_r^{s_r}}.$$

Let $\overline{A}_r$ be the ring of all such series with the usual addition and multiplication of series. The map $f \to \overline{f}$ is a ring isomorphism.

For any $g \in A_r$, a prime number $p$, and an integer $k \in \{1, \ldots, r\}$, let us denote by $\phi_{p,k,r}$ the map from $A_r$ into $A_1$ which sends $g$ to $g_{p,k} = g_{p,k,r} \in A_1$, where $g_{p,k,r}$ is defined as in Section 1. The mapping $\phi_{p,k,r}$ is a homomorphism of $\mathbb{C}$-algebras: for any $c \in \mathbb{C}$ and $g, h \in A_r$, $(cg)_{p,k,r} = cg_{p,k,r}$, $(g+h)_{p,k,r} = g_{p,k,r} + h_{p,k,r}$, and $(g * h)_{p,k,r} = g_{p,k,r} * h_{p,k,r}$. To see this, let $n \in \mathbb{N}$ and consider the $r$-tuple

$(1, \ldots, 1, p^n, 1, \ldots, 1) \in \mathbb{N}^r$, where $p^n$ occurs at the $k$-th component of the tuple. Then,

$$\begin{aligned}(g * h)_{p,k,r}(p^n) &= (g * h)(1, \ldots, 1, p^n, 1, \ldots, 1) \\ &= \sum_{d|p^n} g(1, \ldots, d, 1, \ldots, 1)h(1, \ldots, \frac{p^n}{d}, 1, \ldots, 1) \\ &= \sum_{d|p^n} g_{p,k,r}(d)h_{p,k,r}(\frac{p^n}{d}) \\ &= g_{p,k,r} * h_{p,k,r}(p^n).\end{aligned}$$

On the other hand, if $n$ is not a power of the prime $p$, then we have that

$$(g * h)_{p,k,r}(n) = 0 = g_{p,k,r} * h_{p,k,r}(n).$$

Therefore, $(g * h)_{p,k,r} = g_{p,k,r} * h_{p,k,r}$. Similarly, one sees that $(cg)_{p,k,r} = cg_{p,k,r}$ and $(g + h)_{p,k,r} = g_{p,k,r} + h_{p,k,r}$.

Note that the homomorphism sending any $g \in A_r$ to $g_{p,k,r} \in A_1$ induces a homomorphism of $\overline{A}_r$ onto $\overline{A}_1$ which sends $\overline{g}(s_1, \ldots, s_r)$ to $\overline{g}_{p,k,r}(s)$. As an example, for $r = 1$, this map sends the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_q \frac{1}{1 - \frac{1}{q^s}}$$

to the function $\zeta_p(s) = \sum_{k=0}^{\infty} \frac{1}{p^{ks}} = \frac{1}{1-\frac{1}{p^s}}$. Also, $\frac{-\zeta'(s)}{\zeta(s)}$ is sent to $\frac{-\zeta_p'(s)}{\zeta_p(s)} = \frac{\log p}{p^s-1}$.

## 3. The case of the Riemann zeta function

In order to present the main idea behind the proof of Theorem 1 in terms as simple as possible, in this section we show that the Riemann zeta function $\zeta(s)$ and its derivative $\zeta'(s)$ are algebraically independent over $\mathbb{C}$. In doing this, we will avoid the use of any analytic properties of the Riemann zeta function, so that we later have a chance of generalizing this reasoning in the context of Theorem 1, where one does not have any assumptions on the convergence of the Dirichlet series associated to $f$, or its Euler product. Returning to the Riemann zeta function, let us assume that $\zeta(s)$ and $\zeta'(s)$ are algebraically dependent, and let $Q(x, y)$ be a nonzero polynomial in two variables $x$ and $y$ with coefficients in $\mathbb{C}$ such that $Q(\zeta(s), \zeta'(s)) = 0$. Let $P(x, y) = Q(x, xy)$. Then $P(x, y)$ is a nonzero polynomial and $P\left(\zeta(s), \frac{-\zeta'(s)}{\zeta(s)}\right) = 0$. Next, this gives us an equality in $A_1$, namely

$$P(I, -D_{\psi_{lg(I)}} * I^{-1}) = 0, \tag{3.1}$$

where $I \in A_1$ denotes the arithmetical function given by $I(n) = 1$, and $\psi_0$ is the completely additive function given by $\psi_0(n) = \log(n)$ for all $n \in \mathbb{N}$. Now for any prime $p$, we apply the homomorphism $\phi_{p,1,1}$ to the equality (3.1) and find that $P(I_p, -D_{\psi_0(I_p)} * I_p^{-1}) = 0$. This in turn gives us an equality between the corresponding Dirichlet series, namely

$$P\left(\zeta_p(s), \frac{-\zeta_p'(s)}{\zeta_p(s)}\right) = 0. \tag{3.2}$$

This is a nontrivial relation which needs to be satisfied by each Euler factor $\zeta_p(s)$ of $\zeta(s)$ with the same polynomial $P$. On the other hand, one checks by a direct computation that

$$\frac{-\zeta_p'(s)}{\zeta_p(s)} = (1+\zeta_p(s))\log p. \tag{3.3}$$

Using equation (3.3) in (3.2), we derive that $\zeta_p(s)$ is a zero of the polynomial $U_p(t)$ which is given by $U_p(t) = P(t,(t+1)\log p)$. Since $\zeta_p(s)$ is transcendental over $\mathbb{C}$, $U_p(t)$ has to be identically zero. But, since $P(x,y)$ is a nonzero polynomial, $P(t,(t+1)\log p)$ can be identically zero only for finitely many values of $p$, and this completes the proof that $\zeta(s)$ and $\zeta'(s)$ are algebraically independent over $\mathbb{C}$.

## 4. Proof of Theorem 1

Let $\psi$ and $f$ be as in the statement of Theorem 1. By our assumptions, we know that there is an infinite set $\mathcal{P}$ of prime numbers with the following property. For each prime $p \in \mathcal{P}$, there exists a component $k_p \in \{1,\dots,r\}$ such that the Dirichlet series associated to the arithmetical function $f_{p,k,r}$ is given by

$$\overline{f}_{p,k,r}(s) = \overline{f}_{p,k}(s) = \sum_{n=1}^{\infty} \frac{f_{p,k,r}(n)}{n^s} = \frac{1}{\left(1-\frac{a_1}{p^s}\right)\cdots\left(1-\frac{a_m}{p^s}\right)},$$

for some $m \in \mathbb{N}$ and nonzero complex numbers $a_1,\dots,a_m$. Therefore, there exists a component $k \in \{1,\dots,r\}$ and an infinite subset $\mathcal{P}_k \subseteq \mathcal{P}$ of prime numbers $p$ such that the corresponding values $k_p$ are the same and equal $k$.

Fix such an integer $k$ and a prime number $p$ in the subset $\mathcal{P}_k$. Let $F(t)$ be defined by $F(t) = F_{p,k,r}(t) = \frac{1}{(1-a_1t)\cdots(1-a_mt)}$. Then, we see that $\overline{f}_{p,k,r}(s) = F(p^{-s})$. Let $\psi_k \in A_1$ be the function defined by $\psi_k(n) = \psi(1,\dots,1,n,1,\dots,1)$ for all $n \geq 1$, where $n$ occurs at the $k$-th component of the tuple $(1,\dots,1,n,1,\dots,1)$ on the right side.

Let $\mathbb{C}(t)$ denote, as usual, the field of rational functions in $t$ over $\mathbb{C}$, and $R'(t)$ the derivative of $R(t) \in \mathbb{C}(t)$ as a rational function. Define $\Gamma : \mathbb{C}(t) \to \mathbb{C}(t)$ by $\Gamma(R(t)) = \psi_k(p)tR'(t)$, for $R(t) \in \mathbb{C}(t)$.

Also define

$$\overline{f}'_{p,k,r}(s) = \overline{f}'_{p,k}(s) = (\Gamma(F_{p,k,r}(t)))(p^{-s}),$$

and inductively $\overline{f}^{(l)}_{p,k,r}(s) = \overline{f}^{(l)}_{p,k}(s) = (\Gamma^{(l)}(F_{p,k,r}(t)))(p^{-s})$ for any positive integer $l$, where $\Gamma^{(l)}$ denotes the composition of $\Gamma$ with itself $l$ times.

Now let $G(t) = G_{p,k,r}(t) = \frac{1}{F_{p,k,r}(t)}$. Then, we find that $G_{p,k,r}(t)$ is a polynomial $G_{p,k,r}(t) = \alpha_p t^m + \cdots$ with leading coefficient $\alpha_p = (-1)^m a_1 \cdots a_m$, and its derivative is given by $G'_{p,k,r}(t) = m\alpha_p t^{m-1} + \cdots$.

Next, define inductively $B_0 = B_{(p,k,r),0} = 1$ and

$$B_{n+1}(t) = B_{(p,k,r),n+1}(t) = t\left(G_{p,k,r}(t)B'_n(t) - (n+1)G'_{p,k,r}(t)B_n(t)\right).$$

We claim that

$$\begin{aligned}\Gamma^n(F_{p,k,r}(t)) &= \frac{B_n(t)}{(G(t))^{n+1}}\\ &= \frac{B_n(t)}{(1-a_1t)^{n+1}\cdots(1-a_mt)^{n+1}}.\end{aligned}$$

To prove this claim, first notice that $\Gamma^0(F_{p,k,r}(t)) = F_{p,k,r} = \frac{B_0(t)}{G(t)}$ since $B_0 = 1$. Next, assume that $n \geq 1$ and $\Gamma^n(F_{p,k,r}(t)) = \frac{B_n(t)}{(G(t))^{n+1}}$. Then,

$$\begin{aligned}\Gamma^{n+1}(F_{p,k,r}(t)) &= \Gamma(\Gamma^n(F_{p,k,r}(t)))\\ &= \Gamma\left(\frac{B_n(t)}{(G(t))^{n+1}}\right)\\ &= t\frac{b_n'(t)G(t)^{n+1} - (n+1)B_n(t)G(t)^n u'(t)}{G(t)^{2n+2}}\\ &= t\frac{b_n'(t)G(t) - (n+1)B_n(t)u'(t)}{G(t)^{n+2}}.\end{aligned}$$

This completes the proof of the claim.

Observe that $deg(G_{p,k,r}(t)) = m$. Now we show inductively that $B_n(t)$ is a polynomial of degree $deg(B_n(t)) = nm$ with leading coefficient $(-1)^n\alpha_p^n m^n \psi_k(p)$ for all $n \geq 1$. Clearly, $B_0(t)$ satisfies this claim. Assume that $n \geq 1$, and $B_n(t)$ satisfies the claim. We would like to prove that $B_{n+1}(t)$ satisfies the claim as well; i.e., $B_{n+1}(t)$ is a polynomial of degree $deg(B_{n+1}(t)) = (n+1)m$ with leading coefficient $(-1)^{n+1}\alpha_p^{n+1}m^{n+1}\psi_k(p)$. Since

$$B_{n+1}(t) = B_{(p,k,r),n+1}(t) = t\left(G_{p,k,r}(t)B_n'(t) - (n+1)G_{p,k,r}'(t)B_n(t)\right),$$

its leading term can be written in the form

$$\begin{aligned}&t(\alpha_p t^m)nm(-1)^n\alpha_p^n m^n\psi_k(p)t^{nm-1} - t(\alpha_p(n+1)mt^{m-1})(-1)^n\alpha_p^n m^n\psi_k(p)t^{nm}\\ &= \alpha_p nm(-1)^n\alpha_p^n m^n\psi_k(p)t^{(n+1)m} - \alpha_p(n+1)m(-1)^n\alpha_p^n m^n\psi_k(p)t^{(n+1)m}\\ &= (-1)^n\alpha_p^{n+1}m^n\psi_k(p)(nm - m(n+1))t^{(n+1)m}\\ &= (-1)^{n+1}\alpha_p^{n+1}m^{n+1}\psi_k(p)t^{(n+1)m}.\end{aligned}$$

Hence the desired claim holds.

Now let $i$, $j$ be nonnegative integers such that $i \neq j$. We have that $\overline{f}_{p,k}^{(i)}(s) = (\Gamma^{(i)}(F(t)))(p^{-s})$ and $\overline{f}_{p,k}^{(j)}(s) = (\Gamma^{(j)}(F(t)))(p^{-s})$. Let $S$ denote a finite set of pairs $(u,v)$ of positive integers. Let $P(X,Y) \in \mathbb{C}[X,Y]$ and $P(X,Y) = \sum_{(u,v)\in S} C_{uv}X^uY^v$, where $C_{uv}$ is a nonzero complex number for every $(u,v) \in S$.

Suppose that

$$P(D_\psi^i(f), D_\psi^j(f)) = 0. \tag{4.1}$$

By applying the homomorphism $\phi_{p,k,r}$ to both sides of equality (4.1), we find that $P(D_\psi^i(f)_{p,k,r}, D_\psi^j(f)_{p,k,r}) = 0$. This in turn gives us an equality between the corresponding Dirichlet series, namely,

$$P(\overline{f}_{p,k}^i(s), \overline{f}_{p,k}^j(s)) = 0. \tag{4.2}$$

Thus,

$$\sum_{(u,v)\in S} C_{uv}\left(\frac{B_i(t)}{(G(t))^{i+1}}\right)^u\left(\frac{B_j(t)}{(G(t))^{j+1}}\right)^v = 0.$$

Let $N = \max_{(u,v)\in S}\{(i+1)u + (j+1)v\}$. We have that

$$\sum_{(u,v)\in S} C_{uv}B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v} = 0. \tag{4.3}$$

Note that

$$\begin{aligned} deg(B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v}) &= ium + vjm + (N-(i+1)u-(j+1)v)m \\ &= m(N-u-v). \end{aligned}$$

Let $L = \min_{(u,v)\in S}\{u+v\}$. Then equality (4.3) can be written as

$$\sum_{\substack{(u,v)\in S \\ u+v=L}} C_{uv} B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v} + \sum_{\substack{(u,v)\in S \\ u+v>L}} C_{uv} B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v} = 0.$$

For $f \in A_1$, consider the support of $f$ given by $supp(f) = \{n \in \mathbb{N} | f(n) \neq 0\}$. By abuse of notation, let us denote by $B_i$, $B_j$, and $G$ the arithmetical functions whose Dirichlet series are given respectively by $B_i(p^{-s})$, $B_j(p^{-s})$, and $G(p^{-s})$. Note that the support of the arithmetical function $(B_i^u B_j^v G^{N-(i+1)u-(j+1)v})$ is a subset of $\{1, p, p^2, \ldots, p^{m(N-L)}\}$. So the arithmetical function corresponding to the second sum in the above equation, that is, the function given by the sum

$$\sum_{\substack{(u,v)\in S \\ u+v>L}} C_{uv} B_i^u B_j^v G^{N-(i+1)u-(j+1)v},$$

vanishes at $p^{m(N-L)}$. Since this must hold for infinitely many primes, we conclude that the second sum in the equation above vanishes, and thus

$$\sum_{\substack{(u,v)\in S \\ u+v=L}} C_{uv} B_i(t)^u B_j(t)^v (G(t))^{N-(i+1)u-(j+1)v} = 0.$$

In this equation, the coefficient of $t^{m(N-L)}$ is

$$\sum_{\substack{(u,v)\in S \\ u+v=L}} C_{uv} ((-\psi_k(p))^i \alpha_p^i m^i)^u ((-\psi_k(p))^j \alpha_p^j m^j)^v (\alpha_p)^{N-(i+1)u-(j+1)v},$$

which equals

$$\sum_{\substack{(u,v)\in S \\ u+v=L}} C_{uv} (-\psi_k(p))^{iu+jv} \alpha_p^{N-u-v} m^{iu+jv}.$$

We rewrite this sum as

$$\sum_{\substack{(u,v)\in S \\ u+v=L}} C_{uv} (-m\psi_k(p))^{iu+jv} \alpha_p^{N-L}.$$

Since the coefficient of $t^{m(N-L)}$ must equal zero, we have that

$$\sum_{\substack{(u,v)\in S \\ u+v=L}} C_{uv} (-m\psi_k(p))^{iu+jv} \alpha_p^{N-L} = 0.$$

But, $\alpha_p \neq 0$, and so we must have

$$\sum_{\substack{(u,v)\in S \\ u+v=L}} C_{uv} (-m\psi_k(p))^{iu+jv} = 0. \tag{4.4}$$

By our assumption on $\psi$ and our observation on the set $\mathcal{P}_k$ of prime numbers at the beginning of this section, it follows that there exist infinitely many distinct values $\psi_k(p)$ for primes in $\mathcal{P}_k$. Each of these values $\psi_k(p)$ must satisfy (4.4), which is not possible. This completes the proof of Theorem 1.

*Proof of Corollary* 1. Let $f$ and $\psi$ be as in the statement of Corollary 1. Let $Q \in \mathbb{C}[f]$ be such that $D_\psi(Q) \in \mathbb{C}[f]$. Since for any $c \in \mathbb{C}$ and $n \in \mathbb{N}$, $D_\psi(cf^n) = cnf^{n-1}D_\psi(f)$, $D_\psi(Q)$ equals $D_\psi(f)$ times a polynomial in $f$. But, $f$ and $D_\psi(f)$ being algebraically independent, the only multiple of $D_\psi(f)$ inside $\mathbb{C}[f, D_\psi(f)]$ which belongs to $\mathbb{C}[f]$ is zero, and this proves the corollary. □

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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, Illinois 61801

*E-mail address*: zaharesc@math.uiuc.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, Illinois 61801

*E-mail address*: mzaki@math.uiuc.edu