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PIECEWISE CONTRACTIONS ARE ASYMPTOTICALLY PERIODIC

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ABSTRACT. We show that, given a finite partition of the plane $\mathbb{C}$ such that the map G acts as a linear contraction on each part, for almost every choice of parameters every orbit of G is (asymptotically) periodic.

1. INTRODUCTION

Piecewise isometries are a class of dynamical systems which exhibit complicated behaviour without being chaotic in the classical sense; they have zero Lyapunov exponents, their topological entropy is zero [5], and they usually have islands of quasi-periodic motion. However, they also tend to have 'exceptional sets' on which the dynamical behaviour is of a fascinating complexity. Two well-known examples of piecewise isometries are (i) the class of piecewise affine maps of the torus, studied for example in [1, 2, 6] and (ii) the 'Goetz map' [10]. The latter consists of piecewise rotations of the positive and negative half planes around different centres of rotation. For angles satisfying specific number-theoretical properties, both 'toral maps' and Goetz maps can be understood in terms of substitution shifts [1, 4, 12], but otherwise the dynamics remain mostly not understood.

Piecewise isometries, including those mentioned above, appear in many applications, for instance, as descriptions of at least three electronic circuits [2, 6, 8], and also in relation to impact oscillators, as first return maps of polygonal billiards and in queueing theory [13]. In two of the three electronic circuits alluded to, the (not realistic) assumption of zero dissipation has been made; allowing nonzero dissipation forces one to consider piecewise contractions instead of piecewise isometries, and it is this that motivates the present work.

Whereas in piecewise isometries, the discontinuities are responsible for complicated behaviour, we show in this paper that for typical piecewise contractions, the contracting behaviour dominates the complexity introduced by discontinuities, so that we only see (asymptotic) periodic motion. This is illustrated by a 'contracting Goetz map' in Figure 1. When the Goetz map has a contraction factor $\lambda < 1$, we are left with only finitely many preimages of pieces of the discontinuity line in a finite area. In fact, for sufficiently small λ, it has been shown in [9] that every point

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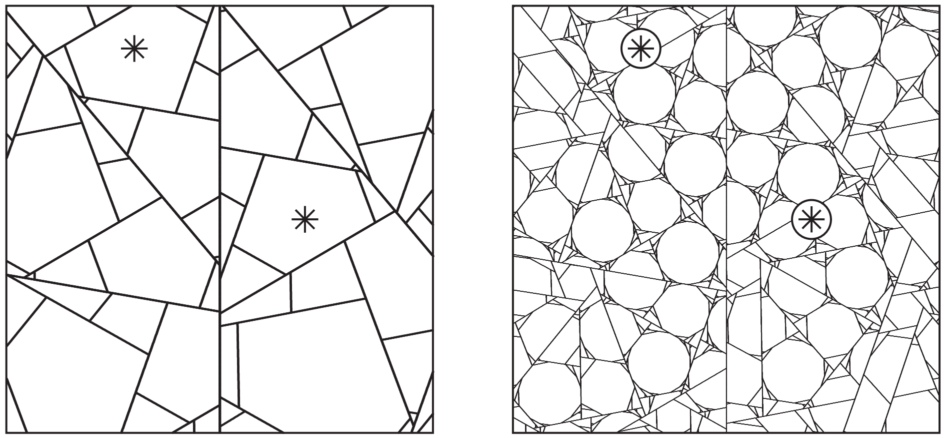

FIGURE 1. Approximations to the exceptional set for the Goetz map with rotation angle 0.7 radians and $\lambda = 0.9$ (left) and $\lambda = 1$ (right). The rotation angle and contraction factor apply to both partitions. The centres of rotation, shown as stars, are $w_1 = -1 + 2\mathrm{i}$, $w_2 = 1$ and the left/right-hand half planes are rotated about w_2, w_1 respectively.

is attracted to a single periodic orbit. The case of $\lambda < 1$ is to be contrasted with the same mapping with $\lambda = 1$, also shown in Figure 1.

In the present paper, we assume that $\{X_k\}_{k=1}^K$ is a finite partition of $\mathbb{C}$ such that for each k, $G|_{X_k}$ is an affine map contracting distances. By this we mean that $G|_{X_k}$ extends to an affine contraction on $\mathbb{C}$ with a fixed point $w_k \in \mathbb{C}$. (Note that w_k need not belong to X_k.) Let $w = \{w_1, \dots, w_K\} \in \mathbb{C}^K$ and $\lambda = \{\lambda_1, \dots, \lambda_K\} \in \mathbb{D}^K$, where $\lambda_k \in \mathbb{D}$ are contraction factors and $\mathbb{D}$ is the open unit disc in $\mathbb{C}$.

Thus we arrive at a piecewise continuous map $G : \mathbb{C} \to \mathbb{C}$ defined as

$$G(z) := G_k(z) = \lambda_k z + (1 - \lambda_k) w_k \quad \text{if} \quad z \in X_k. \tag{1}$$

Lemma 1. *There exists an R such that the disc $B_R = \{|z| \leq R\}$ is forward invariant, and for every z there is an n such that $G^n(z) \in B_R$.*

Proof. Let $\lambda_{\max} = \max_k |\lambda_k|$ and $w_{\max} = \max_k |w_k|$. Then it is straightforward to show that taking $R = 2w_{\max}/(1 - \lambda_{\max})$ satisfies the lemma. □

Let $S = \bigcup_k \partial X_k \cap B_R$; by definition of the partition, S consists of finitely many curves, which we assume to be *rectifiable*; i.e., they have finite length (finite one-dimensional Hausdorff measure). In what follows, it is unimportant how G is defined on S.

The *exceptional set* $\mathcal{E}$ is defined as

$$\mathcal{E}_n = \bigcup_{0 \leq i \leq n} G^{-i}(S) \cap B_R \quad \text{and} \quad \mathcal{E} = \overline{\bigcup_{n \geq 0} \mathcal{E}_n}.$$

Each $\mathcal{E}_n$ consists of a finite set of rectifiable arcs.

Theorem 2. *For all $\lambda \in \mathbb{D}^K$ and Lebesgue a.e. $w \in \mathbb{C}^K$, there exists a finite N such that $\mathcal{E} = \mathcal{E}_N$.*

Corollary 3. *For all $\lambda \in \mathbb{D}^K$ and Lebesgue a.e. $w \in \mathbb{C}^K$, G has a finite number of attracting periodic orbits, and every point is attracted to one of them.*

Remark. Not all piecewise contractions have asymptotic periodic behaviour. Simple examples occur in the family $f_a : [0,1) \to [0,1)$, $f(x) = \lambda x + a \pmod 1$, for a fixed $\lambda \in (0,1)$ and parameter $a \in [0,1)$. The rotation number $\rho(f_a)$ depends continuously on a, and is not constant. In fact, the point 0 has period one for $a = 0$ and two for $a = 1/(1+\lambda)$, so $\rho(f_0) = 0 < \frac{1}{2} = \rho(f_{1/(1+\lambda)})$. By continuity, if we vary a, we will obtain irrational rotation numbers, and in such a case, no periodic orbits exist, and the asymptotic dynamics is an irrational rotation on a Cantor set.

Most applications of piecewise contractions that we are familiar with are in the plane. Theorem 2 has higher-dimensional generalisations, where (1) is replaced by e.g.

$$G_k(x) = \Lambda_k x + (I - \Lambda_k) w_k \quad \text{if} \quad x \in X_k$$

for regions $X_k \subset \mathbb{R}^d$ and linear contractions $\Lambda_k : \mathbb{R}^d \to \mathbb{R}^d$ translated over $w_k \in \mathbb{R}^d$. However, since the geometry of the boundaries ∂X_k and the possible eccentricities of Λ_k create technicalities that only obscure the main idea, we prefer to deal only with the planar case in this paper.

2. Proof of Theorem 2

Define the *itinerary* of z as a sequence $e(z) = e_0 e_1 \cdots \in \{1, \ldots, K\}^{\mathbb{N}_0}$ ($\mathbb{N}_0 = \mathbb{N} \cup \{0\}$), where $e_n = k$ if $G^n(z) \in X_k$. Let I_n be a collection of strings in $\{1, \ldots, K\}^n$ to be specified later, but satisfying the properties:

- $\sigma(I_n) \subset I_{n-1}$, where σ denotes the left shift,
- $\{e_0(z) \ldots e_{n-1}(z) : z \in B_R\} \subset I_n$.

Let $I = \bigcup_n I_n$. Define a multivalued image of z by

$$\widetilde{G^n}(z) := \{G_{e_{n-1}} \circ \cdots \circ G_{e_0}(z) : e_0 \ldots e_{n-1} \in I_n\}.$$

The omega-limit set is the set of accumulation points of an orbit, i.e., $\omega(z) = \bigcap_m \overline{\bigcup_{n \geq m} G^n(z)}$. Let us define the multivalued omega-limit set analogously:

$$\widetilde{\omega}(z) = \bigcap_m \overline{\bigcup_{n \geq m} \widetilde{G^n}(z)}.$$

Lemma 4. *For every $z \in B_R$, $\omega(z) \subset \widetilde{\omega}(0)$.*

Proof. If $y \in \omega(z)$, then there is a sequence $(n_i)_{i \in \mathbb{N}}$ such that $G^{n_i}(z) \to y$. Take $x_i \in I_{n_i}$ obtained as $G_{e_{n_i - 1}} \circ G_{e_{n_i - 2}} \circ \cdots \circ G_{e_0}(0)$, where $e = e(z)$ is the itinerary of z. Since $|G^{n_i}(z) - x_i| \leq 2R\lambda_{\max}^{n_i}$, we have $x_{n_i} \to y$ and the lemma follows. $\square$

Let S_ε be an ε-neighbourhood of S.

Lemma 5. *For all $\lambda \in \mathbb{D}^K$ and Lebesgue a.e. $w \in \mathbb{C}^K$, the following holds: for every $L \in \mathbb{N}$, there exists $\varepsilon > 0$ and a neighbourhood $U \ni w$ such that for every $x \in S$ and $w' \in U$, there is at most one integer $r_1 \leq L$ such that $S_\varepsilon \cap G^{r_1}(B_\varepsilon(x)) \neq \varnothing$.*

Proof. Suppose first that the conclusion fails for w. Then for every $m \in \mathbb{N}$, there is $x_m \in S$ and $r_1 < r_2 \leq L$ such that for $\varepsilon = 1/m$, $S_\varepsilon \cap G^{r_i}(B_\varepsilon(x)) \neq \varnothing$, $i \in \{1, 2\}$. Since S is compact, and by passing to a subsequence if necessary, we can say that $x_m \to x \in S$ and there is a pair $r_1 < r_2 \leq L$ such that $G^{r_i}(x) \in S$ for $i \in \{1, 2\}$.

This is a condition that happens with positive co-dimension, so for Lebesgue a.e. w, it will not occur. Finally, because G depends continuously on w, and by decreasing ε if necessary, there is a neighbourhood U of w on which the conclusion remains true on U. $\square$

Lemma 6. *If $\widetilde{\omega}(0) \cap S = \varnothing$, then there exists $N \in \mathbb{N}$ such that $\mathcal{E}_N = \mathcal{E}$.*

Proof. Since $\widetilde{\omega}(0) \cap S = \varnothing$, there is $N \in \mathbb{N}$ and $\varepsilon > 0$ such that $\bigcup_{n \geq N} \widetilde{G^n}(0) \cap S_\varepsilon = \varnothing$. Additionally, assume that $\lambda_{\max}^N R < \varepsilon$.

Let A be any arc in $\mathcal{E}_n \setminus \mathcal{E}_{n-1}$, so $G^n(A) \subset S$. Moreover, there is $x \in \widetilde{G^n}(0)$ such that $d(x, G^n(A)) < \lambda_{\max}^n R$. Yet, if $n \geq N$, then $\lambda_{\max}^n R < \varepsilon$, and no such arc A can exist. This proves the lemma. $\square$

Lemma 7. *Suppose that $I = \bigcup_n I_n$ has the following property: there is an N such that for $n \geq N$ and every $e \in I_n$, there are at most L_0 strings in I_{n+L} that coincide with e on the first n coordinates. Then the Hausdorff dimension $\dim_H(\widetilde{\omega}(0)) \leq \frac{\log L_0}{-L \log \lambda_{\max}}$.*

Proof. Let $a_n = \#\widetilde{G^n}(0)$. By the condition in the lemma, $a_{n+L} \leq L_0 a_n$ for every $n \geq N$, so $a_{N+iL} \leq K^N L_0^i$.

Take $\delta > \frac{\log L_0}{-L \log \lambda_{\max}}$, so $\lambda_{\max}^{L\delta} L_0 < 1$. Let $\varepsilon > 0$ be arbitrary, and i so large that $2R\lambda_{\max}^m < \varepsilon$, where $m = N + iL$. We will argue that $\widetilde{\omega}(0)$ is contained in the union of closed discs D_x of radius $2R\lambda_{\max}^m$ centred at the points $x \in \widetilde{G^m}(0)$.

Indeed, let $y \in \widetilde{\omega}(0)$, and let $y_k \in \widetilde{G^{n_k}}(0)$ be such that $y_k \to y$. By passing to a subsequence, we may assume that

$$e_{n_k - m}(y_k) \dots e_{n_k - 1}(y_k) = d_0 \dots d_{m-1};$$

i.e., the itinerary of y_k ends in the same m coordinates for all sufficiently large m. Since $\sigma(I_n) \subset I_{n-1}$ for all n, it follows that $d_0 \dots d_{m-1} \subset I_m$, and there exists $x = G_{d_{m-1}} \circ \dots \circ G_{d_0}(0) \in \widetilde{G^m}(0)$. Therefore $|x - y_k| \leq \lambda_{\max}^m R$ for all k, and hence $y \in D_x$.

Now sum over all such discs to get

$$\sum_{x \in \widetilde{G^m}(0)} \operatorname{diam}(D_x)^\delta = \sum_{x \in \widetilde{G^m}(0)} (2R)^\delta \lambda_{\max}^{m\delta} \leq K^N \lambda_{\max}^{N\delta} L_0^i (2R)^\delta \lambda_{\max}^{L\delta i} \leq K^N \lambda_{\max}^{N\delta} (2R)^\delta$$

independently of m, where the last inequality follows from the choice of δ above. Hence we have found a cover of $\widetilde{\omega}(0)$ with discs D_x of diameter $< \varepsilon$ and with $\sum_x \operatorname{diam}(D_x)^\delta < \infty$. Since this holds for any ε and $\delta > \frac{\log L_0}{-L \log \lambda_{\max}}$ is arbitrary, the Hausdorff dimension $\dim_H(\widetilde{\omega}(0)) \leq \frac{\log L_0}{-L \log \lambda_{\max}}$ as required. $\square$

Remark. The idea of the proof of Theorem 2 is that since $\dim_H(\widetilde{\omega}(0)) < 1$, $\widetilde{\omega}(0)$ should be disjoint from S for each λ and Lebesgue a.e. $w \in \mathbb{C}^K$ and 'generically parametrised' families of piecewise contractions. In the proof below, we use linearity in w to show that for a fixed $\lambda \in \mathbb{D}^K$, the family $\{G_w\}_{w \in \mathbb{C}^K}$ is indeed 'generically parametrised'; however, the result should hold for piecewise contractions that are nonlinear in w as well.

Proof of Theorem 2. We can assume without loss of generality that $0 \notin S$, so $\eta := \inf\{|s| : s \in S\} > 0$. Fix $\lambda \in \mathbb{D}^K$, take $w \in \mathbb{C}^K$ arbitrary so that Lemma 5 holds, and let $\varepsilon > 0$ and neighbourhood U be taken from that lemma; U can be arbitrarily small. Recall that $w_{\max} = \max_k\{|w_k|\}$ and $\lambda_{\max} = \max_k\{|\lambda_k|\}$. Let

$$w_* := 1 + \sup\{|w'_{\max}| : w' \in U\}.$$

Then for $R := 2w_*/(1-\lambda_{\max})$, the disc B_R satisfies Lemma 1 for every $(w', \lambda') \in U$.

Let $L_0 := K^2$ and take L so large that $\log L_0/(-L\log\lambda_{\max}) < 1$. Now take $N \in \mathbb{N}$ such that $2R\lambda_{\max}^N < \varepsilon$. If $n \geq N$ and $Y \subset B_R$ is a neighbourhood on which G^n is continuous, then $\operatorname{diam}(G^n(Y)) < \varepsilon$, so by Lemma 5, each $x \in G^n(Y)$ can visit S_ε at most twice in the next L iterates. On such a visit, say the ith, $G^{n+i}(Y)$ can intersect all K regions X_k, but as this happens at most twice, there are at most K^2 subregions of Y on which G^{n+L} is continuous. This is true for all $w' \in U$.

It follows that at most $b_n := K^N L_0^{(n-N)/L}$ discs of radius $\varepsilon_n := 2R\lambda_{\max}^n$ are sufficient to cover $\omega(0)$, uniformly over $(w', \lambda) \in U$. Let I_n be the collection of all possible itineraries of points $x \in B_R$ and $w' \in U$. For each $e \in I_n$, let

$$\begin{array}{rcl} H_{e,n}(w') & := & G_{e_{n-1}} \circ \cdots \circ G_{e_0}(0) \\ & = & \lambda_{e_{n-1}}\lambda_{e_{n-2}} \cdots \lambda_{e_1}(1-\lambda_{e_0})w'_{e_0} + \cdots \\ & & \cdots + \lambda_{e_{n-1}}(1-\lambda_{e_{n-2}})w'_{e_{n-2}} + (1-\lambda_{e_{n-1}})w'_{e_{n-1}}. \end{array}$$

Since this expression is linear in w, the partial derivative $\frac{\partial H_{e,n}}{\partial w_k}$ is the sum of all coefficients of terms which contain w_k. Thus

$$D_{e,n} := \max_{k\in\{1,\ldots,K\}} \left|\frac{\partial H_{e,n}(w')}{\partial w_k}\right|$$

is independent of w'. Take

$$\begin{array}{rcl} A & := & \{w' \in \mathbb{C}^K \ : \ \widetilde{\omega}(0) \cap S \neq \varnothing\} \\ & \subset & \{w' \in \mathbb{C}^K \ : \ \bigcup_{e\in I_n} B_{\varepsilon_n}(H_{e,n}(w')) \cap S \neq \varnothing\}. \end{array}$$

We show that A has no Lebesgue density points, whence $\operatorname{Leb}(A) = 0$.

Let l_S be the length of S, i.e., the sum of the lengths of all rectifiable curves that comprise $S \cap B_R$. Next take $n \geq N$ such that

$$b_n w_* \varepsilon_n l_S < \frac{\eta}{8}\operatorname{Leb}(U).$$

Each of the at most b_n discs $B_{\varepsilon_n}(H_{e,n})$ needed to cover $\widetilde{\omega}(0)$ moves slightly as w' moves in U. For each such disc, i.e., for each $e \in I_n$, there are two cases.
(i) If $D_{e,n} < \eta/2w_*$, then

$$\sup\{|H_{e,n}(w')| : w' \in U\} \leq w_* D_{e,n} < \eta/2,$$

so $B_{\varepsilon_n}(H_{e,n}) \cap S = \varnothing$ for each $w' \in U$. In this case, the disc $B_{\varepsilon_n}(H_{e,n})$ is 'harmless'; it doesn't contribute to the set A.
(ii) If $D_{e,n} \geq \eta/(2w_*)$, then we can take $k \in \{1,\ldots,K\}$ such that $\left|\frac{\partial H_{e,n}}{\partial w_k}\right| \geq \eta/(2w_*)$. The ε_n-neighbourhood S_{ε_n} has area $\leq 2\varepsilon_n l_S$, and the disc $B_{\varepsilon_n}(H_{e,n}(w'))$ intersects S only if its centre $H_{e,n}(w', \lambda')$ belongs to S_{ε_n}. Thus if we fix the other w_i and all λ_i, then

$$\operatorname{Leb}(\{w'_k \in \mathbb{C} \ : \ w' \in U, B_{\varepsilon_n}(H_{e,n}(w')) \cap S \neq \varnothing\}) \leq \frac{2l_S\varepsilon_n}{D_n} \leq \frac{4w_* l_S \varepsilon_n}{\eta}.$$

Integrating over the remaining w'_i (using Fubini's theorem) gives

$$\mathrm{Leb}(\{w' \in U \ : \ B_{\varepsilon_n}(H_{e,n}(w')) \cap S \neq \varnothing\}) \leq \frac{4w_* l_S \varepsilon_n}{\eta}.$$

Summing over all the b_n discs, we obtain

$$\mathrm{Leb}(\{w' \in U \ : \ \bigcup_{e \in I_n} B_{\varepsilon_n}(H_{e,n}(w')) \cap S \neq \varnothing\}) \leq \frac{4w_* l_S b_n \varepsilon_n}{\eta} < \frac{1}{2}\mathrm{Leb}(U).$$

Since this holds for all sufficiently small neighbourhoods U of w (adjusting n if necessary), it follows that w cannot be a Lebesgue density point of A. Since w was arbitrary in a set of full measure, $\mathrm{Leb}(A) = 0$, as required.

Next we claim that if $\omega(z) \cap S = \varnothing$, then z is asymptotically periodic. To prove this, let $y \in \omega(z)$, and let the sequence $(n_k)_{k \in \mathbb{N}}$ be such that $G^{n_k}(z) =: z_k \to y$. We have $\omega(y) \cap S = \varnothing$, so there is $\delta > 0$ such that $G^n(B_\delta(y)) \cap S = \varnothing$ for all $n \geq 0$. Take $k < k'$ such that $z_k, z_{k'} \in B_{\delta/2}(y)$ and $\lambda_{\max}^{n_{k'}-n_k} < \frac{1}{4}$. Then, since

$$z_{k'} \in G^{n_{k'}-n_k}(B_\delta(x)) \subset B_{2\delta\lambda_{\max}^{n_{k'}-n_k}}(y_{k'}) \subset B_\delta(x),$$

the disc $B_\delta(y)$ is mapped continuously into itself under $G^{n_{k'}-n_k}$. So it contains a single attracting periodic point attracting the orbit of z.

Finally, apply Lemma 6 to complete the proof. □

Proof of Corollary 3. On each component Y of $B_R \setminus \mathcal{E}_N$, G^n is continuous and contracting for all $n \geq 0$, and therefore Y contains at most one periodic point p_Y. If it does contain such a point, then every point in Y is asymptotic to $\mathrm{orb}(p_Y)$. Since $\mathcal{E}_N$ consists of a finite number of arcs, there are finitely many periodic orbits, and every point in B_R, and hence every point in $\mathbb{C}$, is asymptotic to one of them. □

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