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POWERS OF COXETER ELEMENTS IN INFINITE GROUPS ARE REDUCED

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ABSTRACT. Let W be an infinite irreducible Coxeter group with $(s_1, \ldots, s_n)$ the simple generators. We give a short proof that the word $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots s_1 s_2 \cdots s_n$ is reduced for any number of repetitions of $s_1 s_2 \cdots s_n$. This result was proved for simply laced, crystallographic groups by Kleiner and Pelley using methods from the theory of quiver representations. Our proof uses only basic facts about Coxeter groups and the geometry of root systems. We also prove that, in finite Coxeter groups, there is a reduced word for w_0 which is obtained from the semi-infinite word $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots$ by interchanging commuting elements and taking a prefix.

Let W be a Coxeter group with S the generating set of simple reflections. An element $c \in W$ of the form $s_1 \cdots s_n$, with $s_1, \ldots, s_n$ some ordering of the elements of S, is called a Coxeter element. It is a result of Howlett [4] that, if W is infinite, then any Coxeter element has infinite order. Our main result is a dramatic strengthening of this result:

Theorem 1. *Let W be an infinite, irreducible Coxeter group and let $(s_1, \ldots, s_n)$ be any ordering of the simple generators. Then the word $s_1 \cdots s_n s_1 \cdots s_n \cdots s_1 \cdots s_n$ is reduced for any number of repetitions of $s_1 \cdots s_n$.*

There are several earlier results which are special cases of Theorem 1. In [9], it is shown that, in each classical affine group, there is an ordering $(s_1, \ldots, s_n)$ of the simple generators such that the word $s_1 \cdots s_n s_1 \cdots s_n \cdots s_1 \cdots s_n$ is reduced for any number of repetitions of $s_1 \cdots s_n$.[1] Fomin and Zelevinsky [2, Corollary 9.6] proved a version of Theorem 1 for Coxeter groups with bipartite diagrams; they show that, if $S = I \sqcup J$ is a partition of S into two sets so that all the elements in each set commute, and if W is irreducible and infinite, then the word $\prod_{i \in I} s_i \prod_{j \in J} s_j \prod_{i \in I} s_i \prod_{j \in J} s_j \cdots \prod_{i \in I} s_i \prod_{j \in J} s_j$ is reduced for any number of repetitions of $\prod_{i \in I} s_i \prod_{j \in J} s_j$. Krammer [8] defines an element w of W to be ***straight*** if $\ell(w^m) = |m| \ell(w)$ for all integers m. In his terminology, w is straight if and only if the identity element is contained in the axis of w. This gives a finite procedure to check whether any particular w is straight, but it is not clear how

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[1]More specifically, the authors of [9] define four properties of a sequence $r_1, r_2, \ldots$ of simple reflections; property (IV) is that the word $r_1 r_2 \cdots r_N$ is reduced for any N. For each classical affine type, they exhibit a sequence of reflections which satisfies their properties and the ordering is periodic in each case.

Krammer's methods could be used to show that infinitely many elements, in different Coxeter groups, are straight. It is trivial to extend any of these results to the case where $W \cong W_1 \times W_2 \times \cdots \times W_r$, with each W_i a Coxeter group meeting the above conditions.

Most relevant to us is recent work of Kleiner and Pelley [5], who show that if W is a simply laced, crystallographic Coxeter group which is irreducible and infinite, then the word $s_1 \cdots s_n s_1 \cdots s_n \cdots s_1 \cdots s_n$ is reduced for any number of repetitions of $s_1 \cdots s_n$. Their argument relies heavily on the theory of quiver representations, and on the work of Kleiner and Tyler [6]. Our proof is inspired by that of Kleiner and Pelley, but we strip out the quiver theory and simplify several arguments. We also eliminate the need for the assumptions that W is crystallographic and simply laced.

The essential property of infinite Coxeter groups, in this proof, is that they have no maximal element. Therefore, it is reasonable to guess that our methods can be used to prove some property of the maximal element, w_0, in a finite Coxeter group. One guess might be that we could prove that there is some prefix of the semi-infinite word $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots$ which forms a reduced word for w_0, so that the partial products of this word climb all the way to the top of W before becoming nonreduced. This isn't quite true, but the following variant is: It is possible to interchange commuting elements of $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_n \cdots$ so that a reduced word for w_0 appears as a prefix. For example, let W be the symmetric group on four elements and let $s_1 = (12)$, $s_2 = (23)$ and $s_3 = (34)$. Then we can swap the second occurrence of s_3 with the third occurrence of s_1 in $s_1 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_3 \cdots$, and the reduced word $s_1 s_2 s_3 s_1 s_2 s_1$ for w_0 will appear as a prefix. We prove that this is always possible in Corollary 4.1.

Our primary technical tool is the introduction of a skew-symmetric form ω_c on the root space. In [13], Nathan Reading and the author use this form to generalize Reading's results on sortable elements to infinite Coxeter groups.

1. Conventions regarding Coxeter groups

Let W be a Coxeter group of rank n. That means that W is generated by $s_1, \ldots, s_n$, subject to the relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$ for $i \neq j$, where $2 \leq m_{ij} = m_{ji} \leq \infty$. Note that the m_{ij} may be any integers; we do not assume that W is crystallographic. The ***Coxeter diagram*** of W is the graph Γ whose vertices are labeled $1, \ldots, n$ and where there is an edge between i and j if $m_{ij} \neq 2$. The group W is called ***irreducible*** if Γ is connected. An element of W which is conjugate to one of the elements of S is called a ***reflection***; the elements of S are called ***simple reflections***. An element of the form $s_{x_1} \cdots s_{x_n}$ of W, for some permutation $x_1 \cdots x_n$ of $\{1, \ldots, n\}$, is called a ***Coxeter element***. Given such a permutation, direct Γ such that $x_j \to x_i$ if $i < j$. Two permutations yield the same Coxeter element if and only if they give rise to the same orientation of Γ, so in this way we may identify Coxeter elements and acyclic orientations of Γ.[2]

Let V be the n-dimensional real vector space with basis $\alpha_1, \ldots, \alpha_n$ and equip V with the symmetric bilinear form B such that $B(\alpha_i, \alpha_i) = 2$ and $B(\alpha_i, \alpha_j) = -2\cos(\pi/m_{ij})$ for $i \neq j$. Then W acts on V by $s_i : v \mapsto v - B(v, \alpha_i)\alpha_i$, and this action preserves the bilinear form B. The elements of V of the form $w\alpha_i$ are called

[2]This identification between Coxeter elements and acyclic orientations of Γ is the one used in [5]; it is the opposite of the one used in [10] and [11].

roots.[3] Every root is either in the positive real span of the α_i, in which case it is called a ***positive root***, or in the positive real span of the $-\alpha_i$, in which case it is called a ***negative root***. The positive roots are in bijection with the reflections, via $w\alpha_s \leftrightarrow wsw^{-1}$. We write α_t for the positive root associated to the reflection t. We have $w\alpha_t = \pm\alpha_{wtw^{-1}}$.

For any $w \in W$, the set of ***inversions*** of w is defined to be the set of reflections t such that $w^{-1}\alpha_t$ is a negative root. If we write w as $s_{x_1} \cdots s_{x_N}$, then the inversions of w are the reflections that occur an odd number of times in the sequence s_{x_1}, $s_{x_1}s_{x_2}s_{x_1}$, $s_{x_1}s_{x_2}s_{x_3}s_{x_2}s_{x_1}$, $\dots$, $s_{x_1}s_{x_2}\cdots s_{x_n}\cdots s_{x_2}s_{x_1}$. The ***length*** of w, written $\ell(w)$, is the length of the shortest expression for w as a product of the simple generators and a product which achieves this minimal length is called ***reduced***. If $s_{x_1}\cdots s_{x_N}$ is reduced, then $s_{x_1}\cdots s_{x_{i-1}}\alpha_{x_i} = \alpha_{s_{x_1}\cdots s_{x_{i-1}}s_{x_i}s_{x_{i-1}}\cdots s_{x_1}}$ (as opposed to $-\alpha_{s_{x_1}\cdots s_{x_{i-1}}s_{x_i}s_{x_{i-1}}\cdots s_{x_1}}$). Also, if $s_{x_2}\cdots s_{x_N}$ is reduced, then $s_{x_1}s_{x_2}\cdots s_{x_N}$ is reduced if and only if s_{x_1} is *not* an inversion of $s_{x_2}\cdots s_{x_N}$.

The previous three paragraphs are very well known; a good reference for this material and far more concerning Coxeter groups is [1], particularly Chapters 1 and 4. We now describe one additional combinatorial tool and one geometric tool. For i between 1 and n, define the map $\pi_i : W \to W$ by $\pi_i(w) = s_i w$ if $\ell(s_i w) > \ell(w)$ and $\pi_i(w) = w$ otherwise. This is sometimes known as the degenerate Hecke action. The condition that $\ell(s_i w) > \ell(w)$ is equivalent to the condition that s_i is *not* an inversion of w. If $s_{x_1}\cdots s_{x_N}$ is reduced, then $\pi_{x_1}\cdots\pi_{x_N}e = s_{x_1}\cdots s_{x_N}$. Also, if s_i and s_j commute, so do π_i and π_j. We call $\pi_{x_1}\cdots\pi_{x_N}e$ the ***Demazure product*** of $x_1\cdots x_n$. For a quick introduction to the properties of the Demazure product, see Section 3 of [7].

Let $c = s_{x_1}\dots s_{x_n}$ be a Coxeter element of W. A simple reflection s is called ***initial in*** c if it is the first letter of some reduced word for c and is called ***final in*** c if it is the last letter of some reduced word for c. So s_{x_1} is initial in c and s_{x_n} is final in c. We define a skew symmetric bilinear form ω_c on V by $\omega_c(\alpha_{x_i}, \alpha_{x_j}) = B(\alpha_{x_i}, \alpha_{x_j})$ for $i < j$. (By skew-symmetry, $\omega_c(\alpha_{x_i}, \alpha_{x_j}) = -B(\alpha_{x_i}, \alpha_{x_j})$ for $i > j$ and $\omega_c(\alpha_i, \alpha_i) = 0$.) It is easy to check that ω_c does not depend on the choice of a reduced word for c.

Proposition 1.1. *With the above notation:*

(1) *For all v and $w \in V$, we have $\omega_{s_{x_1}cs_{x_1}}(s_{x_1}v, s_{x_1}w) = \omega_c(v, w)$.*
(2) *For all positive roots α_t, $\omega_c(\alpha_{s_{x_1}}, \alpha_t) \le 0$, with equality only if s_{x_1} and t commute.*
(3) *For all positive roots α_t, $\omega_c(\alpha_t, \alpha_{s_{x_n}}) \le 0$, with equality only if s_{x_n} and t commute.*

Proof. We first check property (1). Let $c = s_1\cdots s_n$ with $s = s_1$. We recall the formula $sv = v - B(\alpha_s, v)\alpha_s$. It is enough to check property (1) in the case that v and w are simple roots, say $v = \alpha_{s_i}$ and $w = \alpha_{s_j}$ with $i < j$. We consider two cases.

[3] Those interested in Kac-Moody algebras and quiver theory would call these vectors real roots; those from a Coxeter-theoretic background would simply call them roots. We follow the latter convention.

Case 1 ($i = 1$). Then

$$\begin{aligned}\omega_{scs}(s\alpha_s, s\alpha_{s_j}) &= \omega_{scs}(-\alpha_s, \alpha_{s_j} - B(\alpha_s, \alpha_{s_j})\alpha_s) \\ &= -\omega_{scs}(\alpha_s, \alpha_{s_j}) = B(\alpha_s, \alpha_{s_j}) = \omega_c(\alpha_s, \alpha_{s_j}).\end{aligned}$$

We used that s is final in scs and initial in c to deduce the signs in the last two equalities.

Case 2 ($i > 1$). Then

$$\begin{aligned}\omega_{scs}(s\alpha_{s_i}, s\alpha_{s_j}) &= \omega_{scs}(\alpha_{s_i} - B(\alpha_s, \alpha_{s_i})\alpha_s, \alpha_{s_j} - B(\alpha_s, \alpha_{s_j})\alpha_s) \\ &= \omega_{scs}(\alpha_{s_i}, \alpha_{s_j}) - B(\alpha_s, \alpha_{s_i})\omega_{scs}(\alpha_s, \alpha_{s_j}) - B(\alpha_s, \alpha_{s_j})\omega_{scs}(\alpha_{s_i}, \alpha_s).\end{aligned}$$

Now, s is final in scs, so $\omega_{scs}(\alpha_s, \alpha_{s_j}) = -B(\alpha_s, \alpha_{s_j})$ and $\omega_{scs}(\alpha_{s_i}, \alpha_s) = B(\alpha_{s_i}, \alpha_s)$. Thus,

$$\begin{aligned}&- B(\alpha_s, \alpha_{s_i})\omega_{scs}(\alpha_s, \alpha_{s_j}) - B(\alpha_s, \alpha_{s_j})\omega_{scs}(\alpha_{s_i}, \alpha_s) \\ &\qquad = B(\alpha_s, \alpha_{s_i})B(\alpha_s, \alpha_{s_j}) - B(\alpha_s, \alpha_{s_j})B(\alpha_{s_i}, \alpha_s) = 0\end{aligned}$$

and we deduce that

$$\omega_{scs}(s\alpha_{s_i}, s\alpha_{s_j}) = \omega_{scs}(\alpha_{s_i}, \alpha_{s_j}) = B(\alpha_{s_i}, \alpha_{s_j}) = \omega_c(\alpha_{s_i}, \alpha_{s_j}).$$

We have used that s_i comes before s_j in a reduced word for scs, as well as in a reduced word for c, to deduce the signs of the last two equalities. This concludes the proof of property (1).

We now prove property (2). Let $\alpha_t = \sum_{r \in S} a_r \alpha_r$. Since α_t is a positive root, all of the coefficients a_r are nonnegative. Then, as s is initial in c, we have $\omega_c(\alpha_s, \alpha_t) = \sum_{r \in S \setminus \{s\}} a_r B(\alpha_s, \alpha_r)$. Every term in this sum is nonpositive, so $\omega_c(\alpha_s, \alpha_t)$ is nonpositive, which is the first half of the claim. Now, suppose that $\omega_c(\alpha_s, \alpha_t)$ is zero. Then a_r is zero whenever r and s don't commute. Let J be the set of simple reflections which commute with s. So the root α_t is in the span of the set $\{\alpha_r\}_{r \in J}$. This implies that the reflection t is in W_J, which is the subgroup of W generated by J. (Proof: Let p be a point in V^*, the dual vector space to V, such that $\langle p, \alpha_s \rangle = 0$ for $s \in J$ and $\langle p, \alpha_s \rangle \geq 0$ for $s \notin J$. Then the contragredient action of t on V^* fixes p. By [1, Lemma 4.5.1], the stabilizer of p is W_J.) Since every generator of W_J commutes with s, so does t. This completes the proof of property (2); the proof of property (3) is precisely analogous. □

2. Admissible sequences

This section essentially recapitulates (part of) Section 2 of [5], and we will try to repeat the terminology from [5] as much as possible. Let c be a Coxeter element of W. A sequence $x_1, x_2, \ldots, x_N$ of elements of $\{1, 2, \ldots, n\}$ is called ***c-admissible*** if s_{x_1} is initial in c, s_{x_2} is initial in $s_{x_1} c s_{x_1}$, s_{x_3} is initial in $s_{x_2} s_{x_1} c s_{x_1} s_{x_2}$ and so forth. This definition can be understood in a purely graph-theoretic mannner, viewing c as an orientation of Γ. The condition that s_x is initial in c means that x is a sink of (Γ, c) and changing c to $s_x c s_x$ means reversing all edges incident to x, so that x changes from a sink to a source.

We put an equivalence relation on the set of admissible sequences by setting two sequences to be equivalent if they differ only by interchanging the order of non-adjacent vertices. Write $[u]$ for the equivalence class of u. Let $\mathfrak{S}$ denote the set of admissible sequences modulo this equivalence relation. When it is necessary to emphasize the dependence on c, we will write $\mathfrak{S}_c$ and say that elements of $\mathfrak{S}_c$ are c-admissible. The following obvious observation will be of repeated importance:

Proposition 2.1. *If s and t are two vertices of Γ, connected by an edge which is c-oriented from t to s, then the occurrences of s and t in any c-admissible sequence must alternate, with s coming first.*

We can now state a result which immediately implies Theorem 1.

Theorem 2. *Let W be an infinite, irreducible Coxeter group and let c be a Coxeter element. Let $x_1x_2\dots x_N$ be any c-admissible sequence. Then $s_{x_1}\cdots s_{x_N}$ is reduced.*

The rest of the paper is devoted to the proof of Theorem 2.

We need a small combinatorial lemma. For $u=[x_1\dots x_N]\in\mathfrak{S}$, let $\phi(u)_x$ be the number of occurrences of x in $x_1x_2\cdots x_N$. So $\phi(u)$ is an integer-valued function on the vertices of Γ. We put the structure of a poset on $\mathfrak{S}$ by setting $u\preccurlyeq v$ if one can choose representatives $u_1\dots u_M$ and $v_1\dots v_N$ for the equivalence classes u and v such that $M\leq N$ and $u_i=v_i$ for $i\leq M$.

Proposition 2.2. *We have $u_1\dots u_M\preccurlyeq v_1\dots v_N$ if and only if $\phi(u_1\dots u_M)_x\leq \phi(v_1\dots v_N)_x$ for every vertex x of Γ.*

This is part of [5, Proposition 3.2]; we provide a short proof.

Proof. The "only if" direction is obvious; we prove the "if" direction by induction on M. The base case $M=0$ is obvious. Note that u_1 is necessarily a sink of Γ. Since $\phi(u_1\dots u_M)_{u_1}\leq\phi(v_1\dots v_N)_{u_1}$, the vertex u_1 must occur somewhere in $v_1\dots v_N$. Let v_r be the first appearance of u_1. Let w be any vertex neighboring u_1; we claim that w does not occur among $v_1, v_2, \dots, v_{r-1}$. This is because, as noted above, the occurrences of u_1 and w in $v_1\dots v_N$ must alternate, with u_1 appearing first. So $v_1\dots v_N$ is equivalent to $v_rv_1v_2\dots v_{r-1}v_{r+1}\dots v_N$. By induction, $u_2u_3\dots u_M\preccurlyeq v_1v_2\dots v_{r-1}v_{r+1}\dots v_N$ in $\mathfrak{S}_{s_{u_1}cs_{u_1}}$, so $u_1\dots u_M\preccurlyeq v_1\dots v_N$ in $\mathfrak{S}_c$. □

Remark. Kleiner and Pelley characterize the image of ϕ in $\mathbb{Z}^n$ and use it to show that the poset $\mathfrak{S}$ is a distributive semi-lattice. Hohlweg, Lange, and Thomas, in [3], study the lower interval of reduced words in $\mathfrak{S}$, in the case that W is finite, and show that it is a distributive lattice as well. Hopefully, these lattices are related to the appearance of lattice theory in Nathan Reading's and the author's work. (See [10], [11], [12].)

3. The crucial lemmas

Now, let W be a Coxeter group and Γ its Dynkin diagram. As discussed above, there is a bijection between Coxeter elements of W and acyclic orientations of Γ, and we will feel free to use the same symbol to refer both to an orientation and the corresponding Coxeter element. In this section, we will establish the following.

Proposition 3.1. *Let $x_1\cdots x_N$ be of minimal length among all c-admissible sequences with Demazure product w. Then the word $s_{x_1}\cdots s_{x_N}$ is reduced and $w=s_{x_1}\cdots s_{x_N}$.*

Note that, at this point, we have not made any assumptions about W being infinite or irreducible. That will come later, when we apply this result to prove that particular words are reduced. The key technical trick of this note is contained in the following lemma, which will be essential in the proof of Proposition 3.1.

Lemma 3.2. *Suppose that $x_1 \ldots x_N$ is c-admissible and $s_{x_1} \ldots s_{x_N}$ is a reduced word of W. Let t_i be the reflection $s_{x_1} \cdots s_{x_{i-1}} s_{x_i} s_{x_{i-1}} \cdots s_{x_1}$. Then $\omega_c(\alpha_{t_i}, \alpha_{t_j}) \leq 0$ for $i < j$, and equality implies that t_i and t_j commute.*

Proof. Our proof is by induction on i. If $i = 1$, then x_1 is a sink of c and the result is part (2) of Proposition 1.1. If $i > 1$, then, by induction, we have $\omega_{s_{x_1} c s_{x_1}}(\alpha_{s_{x_1} t_i s_{x_1}}, \alpha_{s_{x_1} t_j s_{x_1}}) \leq 0$, with equality implying that $s_{x_1} t_i s_{x_1}$ and $s_{x_2} t_j s_{x_2}$ commute. But, since $s_{x_1} \cdots s_{x_N}$ is reduced, we know that $\alpha_{s_{x_1} t_i s_{x_1}} = s_{x_1} \alpha_{t_i}$ and $\alpha_{s_{x_1} t_j s_{x_1}} = s_{x_1} \alpha_{t_j}$. By part (1) of Proposition 1.1, we have $\omega_c(\alpha_{t_i}, \alpha_{t_j}) = \omega_{s_{x_1} c s_{x_1}}(s_{x_1} \alpha_{t_i}, s_{x_1} \alpha_{t_j}) \leq 0$ as desired. Moreover, t_i and t_j commute if and only if $s_{x_1} t_i s_{x_1}$ and $s_{x_2} t_j s_{x_2}$ do. □

We now begin the proof of Proposition 3.1. Our proof is by induction on N; if $N = 1$ the result is trivial. Let w and $x_1 \cdots x_N$ be as in the statement of Proposition 3.1 with $N > 1$ and assume that the result is known, for all c, for all smaller values of N. Abbreviate $s = s_{x_1}$ and $w' = \pi_{x_2} \cdots \pi_{x_N} e$. We note that $x_2 \cdots x_N$ is of minimal length among scs-admissible sequences with Demazure product w'; if $y_2 \cdots y_M$ were a shorter such sequence, then $x_1 y_2 \cdots y_M$ would be a shorter c-admissible sequence with Demazure product w. So, by induction, $s_{x_2} \cdots s_{x_N}$ is reduced and is equal to w'. The only way that $s_{x_1} s_{x_2} \cdots s_{x_N}$ might not be reduced then is if s is an inversion of w' and $w = w'$. We adopt the notation u_i for $s_{x_2} \cdots s_{x_{i-1}} s_{x_i} s_{x_{i-1}} \cdots s_{x_2}$, where $2 \leq i \leq N$, so the u_i are the inversions of $s_{x_2} \cdots s_{x_N}$. Suppose, for the sake of contradiction, that $s = u_a$ and, thus, $w = w'$. In this case, $s s_{x_2} \cdots s_{x_{a-1}} s_{x_{a+1}} \cdots s_{x_N} = w$ and the word $s s_{x_2} \cdots s_{x_{a-1}} s_{x_{a+1}} \cdots s_{x_N}$ is reduced.

Consider any b between $a+1$ and N. On the one hand, $\omega_{s_{x_1} c s_{x_1}}(\alpha_{u_a}, \alpha_{u_b}) \leq 0$ by Lemma 3.2. (Recall that $x_2 \cdots x_N$ is reduced.) On the other hand, $u_a = s_{x_1}$ and s_{x_1} is the final letter in $s_{x_1} c s_{x_1}$, so $\omega_{s_{x_1} c s_{x_1}}(\alpha_{u_a}, \alpha_{u_b}) \geq 0$ by part (3) of Proposition 1.1. We deduce that, for all b with $a + 1 \leq b \leq N$, we have $\omega_{s_{x_1} c s_{x_1}}(\alpha_{u_a}, \alpha_{u_b}) = 0$ and, by Lemma 3.2, $u_a u_b = u_b u_a$. Thus, we deduce that u_a commutes with u_b for all b with $a + 1 \leq b \leq N$.

Write $v_b = s_{x_a} \cdots s_{x_{b-1}} s_{x_b} s_{x_{b-1}} \cdots s_{x_a}$. Then $s_{x_a} = v_a$ commutes with v_b for all b with $a + 1 \leq b \leq N$. From the identity $s_{x_b} = v_a v_{a+1} \cdots v_b \cdots v_{a+1} v_a$, we conclude that s_{x_a} commutes with s_{x_b} for all b between a and N. But then $x_1 x_2 \cdots x_{a-1} x_{a+1} \cdots x_N$ is c-admissible. We saw above that $s s_{x_2} \cdots s_{x_{a-1}} s_{x_{a+1}} \cdots s_{x_N} = w$ and the word $s s_{x_2} \cdots s_{x_{a-1}} s_{x_{a+1}} \cdots s_{x_N}$ is reduced, so $x_1 x_2 \cdots x_{a-1} x_{a+1} \cdots x_N$ has Demazure product w. This contradicts our choice of $x_1 x_2 \cdots x_{a-1} x_a x_{a+1} \cdots x_N$ as the shortest c-admissible sequence with Demazure product w. This contradiction concludes the proof of Proposition 3.1.

4. Finishing the proof

Assume that W is infinite and irreducible; recall that the second assumption simply means that the Coxeter diagram Γ is connected. We now have a powerful tool (Proposition 3.1) to prove that certain words in W are reduced. In this section, we will apply this tool to prove that $s_{x_1} \cdots s_{x_N}$ is reduced for any c-admissible sequence $x_1 \cdots x_N$. Consider the sequence $u_k = (\pi_1 \pi_2 \cdots \pi_n)^k e$. Clearly, the sequence $\ell(u_1)$, $\ell(u_2)$, … is weakly increasing. We claim that in fact it is strictly increasing. If not, there is some $u = u_k = u_{k+1}$ with $\pi_1 u = \pi_2 u = \cdots = \pi_n u = u$. But then s_i is an inversion of u for every i from 1 to n. In an infinite Coxeter group, there is no element with this property. (In a finite Coxeter group, the only element with this property is the maximal element w_0.)

Therefore, $\ell(u_k) \geq k$. By Proposition 3.1, there is a c-admissible reduced word for each u_k; call this reduced word $\mathtt{w}_k$. We know that $\mathtt{w}_k$ has length at least k. Let z be the letter that occurs most often in $\mathtt{w}_k$; then $\phi(\mathtt{w})_z \geq k/n$. (Recall the map ϕ from Section 2.) Now we use that Γ is connected. Let δ be the diameter of the graph Γ. If x and y are adjacent vertices of Γ, then x and y alternate within $\mathtt{w}_k$, so $|\phi(\mathtt{w}_k)_x - \phi(\mathtt{w}_k)_y| \leq 1$ and we deduce that $\phi(\mathtt{w}_k)_x \geq k/n - \delta$ for any x. Let M be the greatest number of times any letter occurs in $x_1 \cdots x_N$. Choosing k large enough that $k/n - \delta \geq M$, we see that $\phi(\mathtt{w}_k)_x \geq \phi(x_1 \cdots x_N)_x$ for any x. So, by Proposition 2.2, $x_1 \cdots x_N$ is equivalent to a prefix of the reduced word $\mathtt{w}_k$. In particular, $s_{x_1} \cdots s_{x_N}$ is reduced. This concludes the proof of Theorem 2 and hence proves Theorem 1.

We note one variant of this argument.

Theorem 3. *Let W be a finite Coxeter group. Then there is a c-admissible sequence $x_1 x_2 \cdots x_N$ such that $s_{x_1} s_{x_2} \cdots s_{x_N}$ is a reduced word for w_0.*

Proof. Define $u_k = (\pi_1 \pi_2 \cdots \pi_n)^k e$ as before. We must have $u_k = w_0$ for k sufficiently large. So there is a c-admissible sequence with Demazure product w_0. By Proposition 3.1, the shortest such c-admissible sequence gives a reduced word for w_0. ☐

In the introduction, because we lacked the terminology of admissible sequences, we stated this result differently. The following corollary makes the connection.

Corollary 4.1. *It is possible to interchange labels of commuting reflections in the semi-infinite sequence $c^\infty := 12 \cdots n 12 \cdots n \cdots$ so that the c-admissible sequence $x_1 x_2 \cdots x_N$ from Theorem 3 becomes a prefix of the resulting semi-infinite word.*

Proof. Let M be a positive integer which is larger than the number of times any given letter occurs in the word $x_1 x_2 \cdots x_N$. Then, by Proposition 2.2, the word $x_1 x_2 \cdots x_N$ is equivalent to a prefix of $(12 \cdots n)^M$. Adding additional letters to the right of $(12 \cdots n)^M$ does not change this. ☐

Note that we can start with the semi-infinite word A of which $x_1 x_2 \cdots x_N$ is a prefix and undo all of the commutations of commuting generators, keeping track of where the first N letters go. We thus obtain a subword of c^∞ which gives a reduced word for w_0. Consider any generator $s_i \in S$. There is some integer k such that the first k occurrences of i in $\mathtt{w}$ are in the prefix $x_1 x_2 \cdots x_N$, after which none of the later occurrences of i in A lie in $x_1 x_2 \cdots x_N$. Without loss of generality, we may assume that we never interchange i with itself while transforming A into c^∞. Thus, we have obtained a subword $\mathbf{w_0}$ of c^∞ such that $\mathbf{w_0}$ is a reduced word for w_0 and, for each $s_i \in S$, there is some k such that the first k occurrences of i in c^∞ lie in $\mathbf{w_0}$, while all the later occurrences of i do not. In other words, w_0 is c-sortable in the sense of [10]. This gives an efficient proof of Corollary 4.4 of [10] without any case by case analysis. The reduced word $\mathbf{w_0}$ plays an important role in [3]. Theorem 3, and its proof, may be of use in finding better descriptions of $\mathbf{w_0}$.

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