PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 4, April 2009, Pages 1179–1184
S 0002-9939(08)09646-9
Article electronically published on September 25, 2008

CONGRUENCE PROPERTIES OF HERMITIAN MODULAR FORMS

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(Communicated by Wen-Ching Winnie Li)

In celebration of Tomoyoshi Ibukiyama's 60th birthday

Abstract. We study the existence of a modular form satisfying a certain congruence relation. The existence of such modular forms plays an important role in the determination of the structure of a ring of modular forms modulo p. We give a criterion for the existence of such a modular form in the case of Hermitian modular forms.

1. Introduction

In [7], H. P. F. Swinnerton-Dyer determined the structure of a ring of modular forms mod p in the elliptic modular case. In his argument, the existence of a certain modular form plays an important role. Namely, he used the fact that there exists a modular form f of weight $p-1$ with p-integral Fourier coefficients such that

$$f \equiv 1 \pmod{p}.$$

(Also cf. Serre [6].) In the elliptic modular case, such a form can be constructed easily. In fact, we may take $f = E_{p-1}$ (the normalized Eisenstein series of weight $p-1$). However, the problem of existence in the case of Siegel modular forms turns out to be difficult. For example, the Siegel–Eisenstein series $E_{p-1}^{(n)}$ of weight $p-1$ is no longer a solution in general. In [2], S. Boecherer and the second author studied this problem and gave some criteria for the existence problem in the case of Siegel modular forms.

In this paper, we give a criterion of the existence problem in the case of Hermitian modular forms over the imaginary quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$.

2. Hermitian modular forms

We start by recalling the definition of Hermitian modular forms. For details, please refer to [3]. The Hermitian half-space $\mathbb{H}_n$ of degree n is defined by

$$\mathbb{H}_n := \left\{ Z \in M_n(\mathbb{C}) \mid \frac{1}{2\sqrt{-1}}(Z - {}^t\overline{Z}) > 0 \right\}.$$

Received by the editors April 1, 2008.
2000 *Mathematics Subject Classification.* Primary 11F33; Secondary 11F55.
Key words and phrases. Congruences for modular and p-adic modular forms.
The second author was supported in part by Grant-in-Aid for Scientific Research 19540061.

Let $\mathbb{K}$ be an imaginary quadratic field with discriminant $d_{\mathbb{K}}$. We denote by $\mathcal{O} = \mathcal{O}_{\mathbb{K}}$ the ring of integers and by $\mathcal{O}^{\times}$ the group of units in $\mathcal{O}$.

The Hermitian modular group of degree n over $\mathbb{K}$,

$$U_n(\mathcal{O}) := \{M \in M_{2n}(\mathcal{O}) \mid {}^t\overline{M} J_n M = J_n\}, \quad J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

acts on $\mathbb{H}_n$ by

$$Z \longmapsto M < Z >:= (AZ+B)(CZ+D)^{-1}$$

for all $Z \in \mathbb{H}_n$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in U_n(\mathcal{O})$.

Let $\Gamma \subset U_n(\mathcal{O})$ be a subgroup of $U_n(\mathcal{O})$. A holomorphic function $F(Z)$ on $\mathbb{H}_n$ is called a Hermitian modular form of weight k for Γ if it satisfies the functional equations:

$$F(M < Z >) = \det(CZ+D)^k F(Z)$$

for all $Z \in \mathbb{H}_n$ and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$. (We assume the holomorphy at the cusps in the case $n = 1$.) We denote by $M_k(\Gamma)$ the space of Hermitian modular forms of weight k for Γ. Later we mainly deal with the case $\Gamma = U_n(\mathcal{O})$ or $SU_n(\mathcal{O}) := U_n(\mathcal{O}) \cap SL_{2n}(\mathcal{O})$. In both cases, $F \in M_k(\Gamma)$ has a Fourier expansion of the form

$$F(Z) = \sum_{0 \leq T \in \Lambda_n} a_F(T) \exp\{2\pi\sqrt{-1}\mathrm{tr}(TZ)\},$$

where T runs over the lattice

$$\Lambda_n = \Lambda_n(\mathbb{K}) := \{\, T = (t_{ij}) \in Her_n(\mathbb{K}) \,|\, t_{ii} \in \mathbb{Z},\ \sqrt{d_{\mathbb{K}}}\, t_{ij} \in \mathcal{O} \,\}$$

(cf. [3]).

3. Main result

In this section, we state the main result of this paper, which gives a criterion on the existence of a modular form satisfying a certain congruence relation.

Let p be a rational prime and $\mathbb{Z}_{(p)} = \{\, \frac{a}{b} \in \mathbb{Q} \,|\, p \nmid b \,\}$ denote the localization of $\mathbb{Z}$ at p. We denote by $M_k(\Gamma)_{\mathbb{Z}_{(p)}}$ the subset of $M_k(\Gamma)$ consisting of $F \in M_k(\Gamma)$ such that all of its Fourier coefficients belong to $\mathbb{Z}_{(p)}$.

Our main result can be stated as follows:

Theorem 3.1. *(1) Assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$. There exists a Hermitian modular form $F_{p-1} \in M_{p-1}(SU_n(\mathcal{O}))_{\mathbb{Z}_{(p)}}$ such that*

$$F_{p-1} \equiv 1 \pmod{p}$$

if

$$p \equiv 1 \pmod{4}.$$

(2) Assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$. There exists a Hermitian modular form $F_{p-1} \in M_{p-1}(U_n(\mathcal{O}))_{\mathbb{Z}_{(p)}}$ such that

$$F_{p-1} \equiv 1 \pmod{p}$$

if and only if

$$p \equiv 1 \pmod{4}.$$

4. p-Special Hermitian matrix

In order to prove our theorem, we need to consider the existence of a p-special Hermitian matrix.

A Hermitian matrix $H = {}^t\overline{H} \in Her_m(\mathbb{K})$ is called p-special if H satisfies the following four conditions:

(i) H is positive definite,

(ii) H is even integral, namely, $H \in 2\Lambda_m$,

(iii) $\det H = \left(\frac{2}{\sqrt{|d_{\mathbb{K}}|}}\right)^m$,

(iv) (the main condition) there exists a p-group C_p in the finite unitary group

$$U_m(H;\mathcal{O}) := \{U \in M_m(\mathcal{O}) \mid {}^t\overline{U}HU = H\}$$

such that the group C_p acts freely on $\mathcal{O}^m \backslash \{\mathbf{0}\}$.

Our proof of the main result mainly depends on the existence of a p-integral Hermitian matrix. To demonstrate the existence of such a matrix we use the result of Bayer-Fluckiger, which guarantees the existence of a suitable even unimodular lattice over $\mathbb{Z}$.

Theorem 4.1 (Bayer-Fluckiger [1]). *Let m be a positive integer such that m is not a power of 2. Then there exists a definite unimodular lattice having an automorphism with characteristic polynomial Φ_m if and only if m is mixed and $\varphi(m)$ is divisible by 8. Here Φ_m is the m-th cyclotomic polynomial.*

Our result in this section is as follows:

Proposition 4.2. *Assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$. If p is a prime number such that $p \equiv 1 \pmod 4$, then there exists a p-special Hermitian matrix H of rank $p-1$.*

Proof. First we assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ and $p \equiv 1 \pmod 4$. If we put $m = 4p$, then m is not a prime power and $\varphi(m) = 2(p-1)$ is divisible by 8. Hence, by Theorem 4.1, there exists an even unimodular positive definite lattice (L, S) having an automorphism with characteristic polynomial Φ_m (m-th cyclotomic polynomial), where S is the associated bilinear form. We denote by t such an automorphism. The order of the automorphism t^p is 4, and L becomes a $\mathbb{Z}[\sqrt{-1}]$-module by identifying $\sqrt{-1}$ with t^p. Since $\mathbb{Z}[\sqrt{-1}]$ is principal, one may construct a $\mathbb{Z}[\sqrt{-1}]$-basis of L. Hence L becomes a $\mathbb{Z}[\sqrt{-1}]$-lattice of rank $p-1$. (The $\mathbb{Z}$-rank of L is $\varphi(m) = 2(p-1)$.) The corresponding Gram matrix H is the desired matrix. Indeed, one can confirm that the Hermitian matrix H satisfies the conditions (i)-(iv) of the p-special Hermitian matrix.

(i) The positivity of H comes from that of L.

(ii) The bilinear form $S : L \times L \longrightarrow \mathbb{Z}$ satisfies

$$S(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2}(h(\boldsymbol{x},\boldsymbol{y}) + h(\boldsymbol{y},\boldsymbol{x})),$$

where $h : L \times L \longrightarrow \mathbb{Q}(\sqrt{-1})$ is the Hermitian form associated with H. Since $h(\boldsymbol{x},\boldsymbol{x}) = S(\boldsymbol{x},\boldsymbol{x}) \in 2\mathbb{Z}$, H is even integral.

(iii) If we denote by $B \in Sym_{2(p-1)}(\mathbb{Z})$ the Gram matrix associated with the bilinear form S, then we have

$$\det B = (\det H)^2 \cdot \left(\frac{\sqrt{|d_{\mathbb{K}}|}}{2} \right)^{2(p-1)}.$$

Since $\det B = 1$ and $|d_{\mathbb{K}}| = 4$, we have $\det H = 1$. This shows that the matrix H satisfies the condition (iii).

(iv) We recall the definition of the automorphism t mentioned above. In this case, the p-group $C_p := < t^4 >$ acts freely on $\mathcal{O}^{p-1} \backslash \{\mathbf{0}\}$ because the characteristic polynomial of t is Φ_m.

Next we assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$ and $p \equiv 1 \pmod 4$. If we put $m = 3p$, then, by a similar argument to that stated above, there exists an even unimodular positive definite lattice (L, S) having an automorphism with characteristic polynomial Φ_m. We denote by s such an automorphism. The order of the automorphism s^p is 3, and L becomes an $\mathcal{O}_{\mathbb{K}}$-module by identifying $\omega = \frac{-1+\sqrt{-3}}{2}$ with s^p. Since $\mathcal{O}_{\mathbb{K}} = \mathbb{Z} + \omega\mathbb{Z}$ is principal, one may construct an $\mathcal{O}_{\mathbb{K}}$-basis of L. One can prove that the corresponding Gram matrix satisfies the conditions of the p-special Hermitian matrix in a way similar to the case $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$. This completes the proof of Proposition 4.2. □

Example 4.3. We give examples of H in the case $p = 5$:

$$\text{The case} \quad \mathbb{K} = \mathbb{Q}(\sqrt{-1}), \quad H = \begin{pmatrix} 2 & 0 & 1+\sqrt{-1} & \sqrt{-1} \\ 0 & 2 & \sqrt{-1} & 1-\sqrt{-1} \\ 1-\sqrt{-1} & -\sqrt{-1} & 2 & 0 \\ -\sqrt{-1} & 1+\sqrt{-1} & 0 & 2 \end{pmatrix}.$$

$$\text{The case} \quad \mathbb{K} = \mathbb{Q}(\sqrt{-3}), \quad H = \begin{pmatrix} 2 & 0 & \frac{2}{\sqrt{-3}} & \frac{2}{\sqrt{-3}} \\ 0 & 2 & \frac{2}{\sqrt{-3}} & \frac{-2}{\sqrt{-3}} \\ \frac{-2}{\sqrt{-3}} & \frac{-2}{\sqrt{-3}} & 2 & 0 \\ \frac{-2}{\sqrt{-3}} & \frac{2}{\sqrt{-3}} & 0 & 2 \end{pmatrix}.$$

5. Proof of the main theorem

In this section, we prove our main theorem.

Proof. (1) We assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ or $\mathbb{K} = \mathbb{Q}(\sqrt{-3})$ and that $p \equiv 1 \pmod 4$. By Proposition 4.2, there exists a p-special Hermitian matrix H of rank $p-1$. We denote by C_p the corresponding p-group (cf. section 4, the definition of a p-special Hermitian matrix (iv)). We associate the theta series

$$\vartheta_H(Z) := \sum_{X \in M_{p-1,n}(\mathcal{O})} \exp\{\pi\sqrt{-1}\,\mathrm{tr}(H[X]Z)\}, \quad Z \in \mathbb{H}_n,$$

where $H[X] := {}^t\overline{X}HX$. The modularity of ϑ_H for $SU_n(\mathcal{O})$ comes from the conditions (i), (ii), and (iii) of the p-special matrix $H \in 2\Lambda_{p-1}$ (e.g. cf. Cohen and Resnikoff [4], p. 332); namely, we have $\vartheta_H(Z) \in M_{p-1}(SU_n(\mathcal{O}))$. In particular, we have $\vartheta_H(Z) \in M_{p-1}(U_n(\mathcal{O}))$ in the case that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ because

$\sharp\mathcal{O}^\times = \sharp\mathbb{Z}[\sqrt{-1}]^\times = 4$ and the weight $p-1$ is divisible by 4. The Fourier expansion is given as follows:

$$\vartheta_H(Z) = \sum_T A(H,T)\exp\{2\pi\sqrt{-1}\,\mathrm{tr}(TZ)\},$$
$$A(H,T) = \sharp\mathcal{A}(H,T), \quad \mathcal{A}(H,T) = \{X \in M_{p-1,n}(\mathcal{O}) | H[X] = 2T\}.$$

If $T \neq O_n$, then the p-group C_p acts freely on the set $\mathcal{A}(H,T)$. Therefore, the number $A(H,T)$ is divisible by p. Since $A(H,O_n) = 1$, we have

$$\vartheta_H(Z) \equiv 1 \pmod{p}.$$

This proves (1) of Theorem 3.1.

(2) Assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ and that there exists a form

$$F_{p-1} \in M_{p-1}(U_n(\mathbb{Z}(\sqrt{-1})))_{\mathbb{Z}_{(p)}}$$

such that

$$F_{p-1} \equiv 1 \pmod{p}.$$

We recall the definition of the Φ-operator defined by

$$\Phi : M_k(U_n(\mathcal{O})) \longrightarrow M_k(U_{n-1}(\mathcal{O})), \quad \Phi(F)(Z) := \lim_{\lambda\to\infty} F\left(\begin{pmatrix} Z & 0 \\ 0 & i\lambda \end{pmatrix}\right), Z \in \mathbb{H}_{n-1}.$$

If we apply the Φ-operator $n-1$ times to F_{p-1}, then

$$\Phi^{(n-1)}(F_{p-1}) \in M_{p-1}(U_1(\mathbb{Z}[\sqrt{-1}]))_{\mathbb{Z}_{(p)}}$$

still satisfies the congruence relation

$$\Phi^{(n-1)}(F_{p-1}) \equiv 1 \pmod{p}.$$

If $p \not\equiv 1 \pmod{4}$, this is impossible because

$$M_k(U_1(\mathbb{Z}[\sqrt{-1}])) = \begin{cases} M_k(SL_2(\mathbb{Z})) & \text{if} \quad k \equiv 0 \pmod{4}, \\ 0 \quad \text{otherwise.} \end{cases}$$

(This comes from the fact that $U_1(\mathcal{O}) = \mathcal{O}^\times \cdot SL_2(\mathbb{Z})$.) We have proved the statement (2), thereby completing the proof of Theorem 3.1. $\square$

6. Remark

In the case that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$ and $n = 2$, there is another construction of F_{p-1}, which is based on the theory of Hermitian Jacobi forms.

We assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-1})$. Freitag [5] constructed a set of generators of the graded ring

$$M^{sym}(U_2(\mathbb{Z}[\sqrt{-1}])) = \bigoplus M_k^{sym}(U_2(\mathbb{Z}[\sqrt{-1}])),$$

where $M_k^{sym}(U_2(\mathbb{Z}[\sqrt{-1}]))$ is the subspace consisting of the symmetric Hermitian modular forms of weight k. (In general, $F \in M_k(U_n(\mathcal{O}))$ is called symmetric if $F({}^tZ) = F(Z)$.) We recall the weight 4 generator φ_4 of $M^{sym}(U_2(\mathbb{Z}[\sqrt{-1}]))$ (cf. [5]). It is known that all the Fourier coefficients of $\mathcal{E}_4 := \frac{1}{4}\varphi_4$ are integral and the constant term is equal to 1. We expand $\mathcal{E}_4$ as a Fourier-Jacobi series and take the index 1 Jacobi form $\Phi_{4,1}$. All of the Fourier coefficients of $\Phi_{4,1}$ are divisible by 240. We put $\phi_{4,1} := \frac{1}{240}\Phi_{4,1}$. Now we assume that $p \equiv 1 \pmod{4}$. Then

$$f_{p-1,1} := E_4^{\frac{p-5}{4}} \cdot \phi_{4,1}$$

becomes a Hermitian Jacobi form of weight $p-1$ and index 1. Here $E_4 = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n$ is the ordinary Eisenstein series of weight 4 for $SL_2(\mathbb{Z})$. All of the Fourier coefficients of $f_{p-1,1}$ are integral and the constant term is equal to 1. We consider the Maass lift $\mathcal{M}_k$ from the space of Hermitian Jacobi forms of weight k and index 1 to the space $M_k(U_2(\mathbb{Z}[\sqrt{-1}]))$. Then

$$F_{p-1} := -\frac{2(p-1)}{B_{p-1}}\mathcal{M}_{p-1}(f_{p-1,1})$$

is the desired form, namely, $F_{p-1} \in M_{p-1}(U_2(\mathbb{Z}[\sqrt{-1}]))_{\mathbb{Z}_{(p)}}$ and

$$F_{p-1} \equiv 1 \pmod{p}.$$

(Since the Maass lift $\mathcal{M}_k$ is defined only for k such that $k \equiv 0 \pmod 4$ in this case, we need the assumption $p \equiv 1 \pmod 4$.)

Acknowledgement

The authors wish to thank Prof. G. Nebe for suggesting the proof of Proposition 4.2.

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