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PRODUCING SET-THEORETIC COMPLETE INTERSECTION MONOMIAL CURVES IN $\mathbb{P}^n$

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Abstract. In this paper we describe an algorithm for producing infinitely many examples of set-theoretic complete intersection monomial curves in $\mathbb{P}^{n+1}$, starting with a single set-theoretic complete intersection monomial curve in $\mathbb{P}^n$. Moreover we investigate the numerical criteria to decide when these monomial curves can or cannot be obtained via semigroup gluing.

1. Introduction

It is well known that a variety in an n-space can be written as the intersection of n hypersurfaces set theoretically; see [5]. It is then natural to ask whether this number is minimal. A curve in n-space which is the intersection of $n-1$ hypersurfaces is called a set-theoretic complete intersection, s.t.c.i. for short. If moreover its defining ideal is generated by $n-1$ polynomials, then it is called an ideal theoretic complete intersection, abbreviated i.t.c.i. Determining set-theoretic or ideal-theoretic complete intersection curves is a classical and long-standing problem in algebraic geometry. An associated problem is to give explicitly the equations of the hypersurfaces involved. When the characteristic of the field K is positive, it is known that all monomial curves are s.t.c.i.'s in $\mathbb{P}^n$; see [8]. However the question is still open in the characteristic zero case despite the tremendous progress in this direction; see for example [6, 7, 16] and the references there for some recent activity.

The purpose of the present paper is to describe a method to produce *infinitely many* s.t.c.i. monomial curves starting from one single s.t.c.i. monomial curve; see section 4. Our approach has the side novelty of describing explicitly the equations of hypersurfaces on which these new monomial curves lie as an s.t.c.i. On the other hand, semigroup gluing being one of the most popular techniques of recent research, we develop numerical criteria to determine when these new curves can or cannot be obtained via gluing; see section 3. In the last section we discuss several consequences and variations of these results.

2. Preliminaries

Throughout the paper, K will be assumed to be an algebraically closed field of characteristic zero. By an *affine monomial curve* $C(m_1, \dots, m_n)$, for some positive

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integers $m_1 < \cdots < m_n$ with $gcd(m_1, \ldots, m_n) = 1$, we mean a curve with generic zero $(v^{m_1}, \ldots, v^{m_n})$ in the affine n-space $\mathbb{A}^n$, over K. By a *projective monomial curve* $\overline{C}(m_1, \ldots, m_n)$ we mean a curve with generic zero

$$(u^{m_n}, u^{m_n - m_1} v^{m_1}, \ldots, u^{m_n - m_{n-1}} v^{m_{n-1}}, v^{m_n})$$

in the projective n-space $\mathbb{P}^n$, over K. Note that $\overline{C}(m_1, \ldots, m_n)$ is the projective closure of $C(m_1, \ldots, m_n)$.

Whenever we write $\overline{C} \subset \mathbb{P}^n$ to simplify the notation, we always mean a monomial curve $\overline{C}(m_1, \ldots, m_n)$ for some fixed positive integers $m_1 < \cdots < m_n$ with $gcd(m_1, \ldots, m_n) = 1$.

Let m be a positive integer in the numerical semigroup generated by $m_1, \ldots, m_n$; i.e. $m = s_1 m_1 + \cdots + s_n m_n$, where $s_1, \ldots, s_n$ are some non-negative integers. Note that in general there is no unique choice for $s_1, \ldots, s_n$ to represent m in terms of $m_1, \ldots, m_n$. We define the degree $\delta(m)$ of m to be the minimum of all possible sums $s_1 + \cdots + s_n$. If ℓ is a positive integer with $gcd(\ell, m) = 1$, then we say that the monomial curve $\overline{C}(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{P}^{n+1}$ is an *extension* of $\overline{C}$. We similarly define $C(\ell m_1, \ldots, \ell m_n, m)$ to be an *extension* of $C = C(m_1, \ldots, m_n)$. We say that an extension is *nice* if $\delta(m) > \ell$ and *bad* otherwise, adopting the terminology of [1].

When the integers $m_1, \ldots, m_n$ are fixed and understood in a discussion, we will use $\overline{C}_{\ell,m}$ to denote the extensions $\overline{C}(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{P}^{n+1}$ and use $C_{\ell,m}$ to denote the extensions $C(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{A}^{n+1}$.

2.1. **Extensions of monomial curves in $\mathbb{A}^n$.** Let $C = C(m_1, \ldots, m_n)$ be an s.t.c.i. monomial curve in $\mathbb{A}^n$. In this section, we show that all extensions of C, in the sense defined above, are s.t.c.i. For this we first define, for any ideal $I \subset K[x_1, \ldots, x_{n+1}]$, $\Gamma_\ell(I)$ to be the ideal which is generated by all polynomials of the form $\Gamma_\ell(g)$, where $\Gamma_\ell(g(x_1, \ldots, x_{n+1})) = g(x_1, \ldots, x_n, x_{n+1}^\ell)$, for all $g \in I$. We use the following trick of M. Morales:

Lemma 2.1 ([9, Lemma 3.2])**.** *Let Y_ℓ be the monomial curve $C(\ell m_1, \ldots, \ell m_n, m_{n+1})$ in $\mathbb{A}^{n+1}$. Then $I(Y_\ell) = \Gamma_\ell(I(Y_1))$.*

For any extension of C of the form $C_{\ell,m}$, we obviously have $I(C) \subset I(C_{\ell,m})$ and $I(C_{\ell,m}) \cap K[x_1, \ldots, x_n] = I(C)$. The exact relation between the ideals of C and $C_{\ell,m}$ are given by the following lemma.

Lemma 2.2. *Let $m = s_1 m_1 + \cdots + s_n m_n$. For any positive integer ℓ with $gcd(\ell, m) = 1$ we have $I(C_{\ell,m}) = I(C) + (G)$, where $G = {x_1}^{s_1} \cdots {x_n}^{s_n} - x_{n+1}^\ell$.*

Proof.

Case $\ell = 1$**:** We show that $I(C_{1,m}) = I(C) + ({x_1}^{s_1} \cdots {x_n}^{s_n} - x_{n+1})$.
For any polynomial $f \in K[x_1, \ldots, x_{n+1}]$, there are polynomials $g \in K[x_1, \ldots, x_n]$ and $h \in K[x_1, \ldots, x_{n+1}]$ such that

$$\begin{aligned} f(x_1, \ldots, x_{n+1}) &= f(x_1, \ldots, x_n, x_{n+1} - x_1^{s_1} \cdots x_n^{s_n} + x_1^{s_1} \cdots x_n^{s_n}) \\ &= g(x_1, \ldots, x_n) + (x_1^{s_1} \cdots x_n^{s_n} - x_{n+1}) h(x_1, \ldots, x_{n+1}). \end{aligned}$$

This identity implies that $f \in I(C_{1,m})$ if and only if $g \in I(C)$.

Case $\ell > 1$**:** Applying Lemma 2.1 with $Y_1 = C_{1,m}$ we have

$$\begin{aligned} I(C_{\ell,m}) &= \Gamma_\ell(I(C_{1,m})) \text{ by Lemma 2.1} \\ &= \Gamma_\ell(I(C) + (x_1^{s_1} \cdots x_n^{s_n} - x_{n+1})) \text{ by the first part of this lemma} \\ &= I(C) + (G). \end{aligned}$$

□

This lemma provides an alternate proof to the following theorem, which is a special case of [16, Theorem 2].

Theorem 2.3. *If $C \subset \mathbb{A}^n$ is an s.t.c.i. monomial curve, then all extensions of the form $C_{\ell,m} \subset \mathbb{A}^{n+1}$ are also s.t.c.i. monomial curves.*

Proof. Since $I(C_{\ell,m}) = I(C) + (G)$ by Lemma 2.2, it follows that

$$\begin{aligned} Z(I(C_{\ell,m})) &= Z(I(C) + (G)), \\ C_{\ell,m} &= Z(I(C)) \cap Z(G), \end{aligned}$$

where $Z(\cdot)$ denotes the zero set as usual. Hence $C_{\ell,m}$ is an s.t.c.i. if C is also. □

3. Extensions that cannot be obtained by gluing

If $\overline{C}(m_1, \ldots, m_{n+1})$ is a monomial curve in $\mathbb{P}^{n+1}$, then there is a corresponding semigroup $\mathbb{N}T$, where

$$T = \{(m_{n+1}, 0), (m_{n+1} - m_1, m_1), \ldots, (m_{n+1} - m_n, m_n), (0, m_{n+1})\} \subset \mathbb{N}^2.$$

Let $T = T_1 \sqcup T_2$ be a decomposition of T into two disjoint proper subsets. Without loss of generality assume that the cardinality of T_1 is less than or equal to the cardinality of T_2. $\mathbb{N}T$ is called a *gluing* of $\mathbb{N}T_1$ and $\mathbb{N}T_2$ if there exists a non-zero $\alpha \in \mathbb{N}T_1 \cap \mathbb{N}T_2$ such that $\mathbb{Z}\alpha = \mathbb{Z}T_1 \cap \mathbb{Z}T_2$. Following the literature we write $I(T)$ for the ideal of the toric variety corresponding to the affine semigroup $\mathbb{N}T$. Note that if $\mathbb{N}T$ is a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$, then we have $I(T) = I(T_1) + I(T_2) + (G_\alpha)$, where G_α is the relation polynomial; see [16].

We note that the condition $\mathbb{Z}\alpha = \mathbb{Z}T_1 \cap \mathbb{Z}T_2$ is not fulfilled when T_1 is not a singleton. Hence we formulate this observation to be the following:

Proposition 3.1. *If T_1 is not a singleton, then $\mathbb{N}T$ is not a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$.*

Proof. If T_1 is not a singleton, then neither is T_2 by the assumption on the cardinalities of these sets. Thus $\mathbb{Z}T_1$ and $\mathbb{Z}T_2$ are submodules of $\mathbb{Z}^2$ of rank two each. It is elementary to show that their intersection has rank two. For instance, let r and t be generators of $\mathbb{Z}T_1$. Then the images of r and t have finite order in the finite group $\mathbb{Z}^2/\mathbb{Z}T_2$, meaning that ar and bt are in $\mathbb{Z}T_2$ for some positive integers a and b. Then the rank two $\mathbb{Z}$-module generated by ar and bt is contained in the intersection $\mathbb{Z}T_1 \cap \mathbb{Z}T_2$, which must be of rank two itself, being a submodule of $\mathbb{Z}^2$.

Hence the intersection cannot be generated by a single element. Thus $\mathbb{N}T$ is not a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$. □

This proposition means that the only way to show that an extension in $\mathbb{P}^{n+1}$ is an s.t.c.i. via gluing is to apply the technique to a projective monomial curve in $\mathbb{P}^n$. Thus we discuss the case where T_1 is a singleton. But if T_1 is $\{(m_{n+1}, 0)\}$ or $\{(0, m_{n+1})\}$, then $\mathbb{N}T_1 \cap \mathbb{N}T_2 = \{(0,0)\}$. So it is sufficient to deal with the case where T_1 is of the form $\{(m_{n+1} - m_i, m_i)\}$, for some $i \in \{1, \ldots, n\}$.

From now on, Δ_i denotes the greatest common divisor of the positive integers $m_1, \ldots, \widehat{m_i}, \ldots, m_{n+1}$ (m_i is omitted), for $i = 1, \ldots, n$. Note that we have $gcd(\Delta_i, m_i) = 1$, for all $i = 1, \ldots, n$, since $gcd(m_1, \ldots, m_{n+1}) = 1$.

Proposition 3.2. *If $T_1 = \{(m_{n+1} - m_{i_0}, m_{i_0})\}$ for some fixed $i_0 \in \{1, \ldots, n\}$, then $\mathbb{N}T$ is a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$ if and only if there exist non-negative integers d_j, for $j = 1, \ldots, \widehat{i_0}, \ldots, n+1$, satisfying the following two conditions:*

(I) $\Delta_{i_0} m_{i_0} = \sum_{\substack{j=1 \\ j \neq i_0}}^{n+1} d_j m_j$ *and*

(II) $\Delta_{i_0} \geq \sum_{\substack{j=1 \\ j \neq i_0}}^{n+1} d_j$.

Proof. Let $\alpha = \Delta_{i_0}(m_{n+1} - m_{i_0}, m_{i_0})$. We first show that $\mathbb{Z}T_1 \cap \mathbb{Z}T_2 = \mathbb{Z}\alpha$. Since $\Delta_{i_0} = gcd(m_1, \ldots, \widehat{m_{i_0}}, \ldots, m_{n+1})$, there are $z_j \in \mathbb{Z}$, for $j = 1, \ldots, \widehat{i_0}, \ldots, n+1$, such that $\Delta_{i_0} = \sum_{j \neq i_0} z_j m_j$. So $\Delta_{i_0} m_{i_0} = \sum_{j \neq i_0} m_{i_0} z_j m_j$, which implies that

$$\Delta_{i_0}(m_{n+1} - m_{i_0}, m_{i_0}) = \sum_{j \neq i_0} m_{i_0} z_j (m_{n+1} - m_j, m_j) + (\Delta_{i_0} - \sum_{j \neq i_0} m_{i_0} z_j)(m_{n+1}, 0).$$

Thus $\alpha = \Delta_{i_0}(m_{n+1} - m_{i_0}, m_{i_0}) \in \mathbb{Z}T_1 \cap \mathbb{Z}T_2$, implying $\mathbb{Z}\alpha \subseteq \mathbb{Z}T_1 \cap \mathbb{Z}T_2$.

For the converse inclusion, take $c(m_{n+1} - m_{i_0}, m_{i_0}) \in \mathbb{Z}T_1 \cap \mathbb{Z}T_2$, for some $c \in \mathbb{Z}$. Then, obviously we have $c(m_{n+1} - m_{i_0}, m_{i_0}) \in \mathbb{Z}T_2$, which implies that $cm_{i_0} \in \mathbb{Z}(\{m_1, \ldots, \widehat{m_{i_0}}, \ldots, m_{n+1}\}) = \mathbb{Z}\Delta_{i_0}$. So Δ_{i_0} divides cm_{i_0}. If $\Delta_{i_0} > 1$, then Δ_{i_0} divides c, since it does not divide m_{i_0} (remember that $gcd(\Delta_{i_0}, m_{i_0}) = 1$). If $\Delta_{i_0} = 1$, obviously Δ_{i_0} divides c. Thus, $c(m_{n+1} - m_{i_0}, m_{i_0})$ is a multiple of α and $\mathbb{Z}T_1 \cap \mathbb{Z}T_2 \subseteq \mathbb{Z}\alpha$.

Since $\mathbb{Z}T_1 \cap \mathbb{Z}T_2 = \mathbb{Z}\alpha$, it will follow by definition that $\mathbb{N}T$ is a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$ if and only if $\alpha \in \mathbb{N}T_1 \cap \mathbb{N}T_2$. But, if $\alpha \in \mathbb{N}T_1 \cap \mathbb{N}T_2$, then there exist non-negative integers d_j and d for which we have

$$\begin{aligned} \Delta_{i_0}(m_{n+1} - m_{i_0}, m_{i_0}) &= \sum_{j \neq i_0} d_j(m_{n+1} - m_j, m_j) + d(m_{n+1}, 0), \\ (\Delta_{i_0} m_{n+1} - \Delta_{i_0} m_{i_0}, \Delta_{i_0} m_{i_0}) &= ([d + \sum_{j \neq i_0} d_j] m_{n+1} - \sum_{j \neq i_0} d_j m_j, \sum_{j \neq i_0} d_j m_j). \end{aligned}$$

Thus, $\Delta_{i_0} m_{i_0} = \sum_{j \neq i_0} d_j m_j$ and $d = \Delta_{i_0} - \sum_{j \neq i_0} d_j$. Since $d \geq 0$, we see that the conditions (I) and (II) hold. On the other hand, if (I) and (II) hold, then we observe that $\alpha \in \mathbb{N}T_1 \cap \mathbb{N}T_2$, by the equalities above. Thus, the condition $\alpha \in \mathbb{N}T_1 \cap \mathbb{N}T_2$ is equivalent to the existence of the non-negative integers d_j satisfying (I) and (II). □

As a direct consequence of Proposition 3.2, we get the following:

Corollary 3.3. *If $\Delta_{i_0} = 1$, for some fixed $i_0 \in \{1, \ldots, n\}$, then $\mathbb{N}T$ cannot be obtained as a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$, where $T_1 = \{(m_{n+1} - m_{i_0}, m_{i_0})\}$ and $T_2 = T - T_1$.*

Proof. We apply Proposition 3.2. If (I) does not hold, we are done. If (I) does hold, then we have two cases: either $\sum_{\substack{j=1 \\ j \neq i_0}}^{n+1} d_j = 1$ or $\sum_{\substack{j=1 \\ j \neq i_0}}^{n+1} d_j > 1$. The first case forces $m_{i_0} = m_j$ for some $j \neq i_0$, from (I), but this contradicts the way we choose m_i's. The second case causes (II) to fail, as $\Delta_{i_0} = 1$. □

Example 3.4. If we consider the curve $\overline{C}(2, 3, 4, 8) \subset \mathbb{P}^4$ and take $i_0 = 2$, then the conditions (I) and (II) of the above proposition hold. Thus this curve can be obtained by gluing.

However, if we consider the monomial curve $\overline{C}(2,4,7,8) \subset \mathbb{P}^4$, then for every choice of i_0, either $\Delta_{i_0} = 1$ or else condition (II) of the above proposition fails. Hence this curve cannot be obtained by gluing.

Corollary 3.5. *Let $\overline{C}_{\ell,m} \subset \mathbb{P}^{n+1}$ be a bad extension of $\overline{C} = \overline{C}(m_1,\ldots,m_n)$, i.e. $\ell \geq \delta(m)$. If $\overline{C}$ is an s.t.c.i. on the hypersurfaces $f_1 = \cdots = f_{n-1} = 0$, then $\overline{C}_{\ell,m}$ can be shown to be an s.t.c.i. on the hypersurfaces $f_1 = \cdots = f_{n-1} = 0$ and $F = x_{n+1}^{\ell} - x_0^{\ell-\delta(m)} x_1^{s_1} \cdots x_n^{s_n} = 0$ by the technique of gluing, where $m = s_1 m_1 + \cdots + s_n m_n$ and $s_1 + \cdots + s_n = \delta(m)$.*

Proof. Since $m_1 < \cdots < m_n$ and $m = s_1 m_1 + \cdots + s_n m_n \leq \delta(m) m_n \leq \ell m_n$, it follows that ℓm_n is the biggest number among $\{\ell m_1, \ldots, \ell m_n, m\}$. The extension $\overline{C}_{\ell,m}$ corresponds to the semigroup $\mathbb{N}T$, where $T = T_1 \cup T_2$, $T_1 = \{(\ell m_n - m, m)\}$ and $T_2 = \{(\ell m_n, 0), (\ell m_n - \ell m_1, \ell m_1), \ldots, (\ell m_n - \ell m_{n-1}, \ell m_{n-1}), (0, \ell m_n)\}$. Since $gcd(\ell m_1, \ldots, \ell m_n) = \ell$, $\ell m = s_1(\ell m_1) + \cdots + s_n(\ell m_n)$ and $\ell \geq \delta(m)$, $\mathbb{N}T$ is a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$, by Proposition 3.2. Since $I(T) = I(T_1) + I(T_2) + (F)$, the claim follows from [16, Theorem 2]. □

4. The main results

Since *bad* extensions are shown to be an s.t.c.i. by the technique of gluing (see Corollary 3.5 above), we study *nice* extensions of monomial curves in this section. By using the theory developed in the previous section, one can check which of these extensions can be obtained by the technique of gluing semigroups.

Throughout this section we will assume that

- $\overline{C} = \overline{C}(m_1, \ldots, m_n) \subset \mathbb{P}^n$ is an s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$.
- $m = s_1 m_1 + \cdots + s_n m_n$ for some nonnegative integers $s_1, \ldots, s_n$ such that $s_1 + \cdots + s_n = \delta(m)$.
- ℓ is a positive integer with $gcd(\ell, m) = 1$.
- $\delta(m) > \ell$.

Remark 4.1. Since $\overline{C}$ is an s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$, its affine part C is an s.t.c.i. on $g_1 = \cdots = g_{n-1} = 0$, where $g_i(x_1, \ldots, x_n) = f_i(1, x_1, \ldots, x_n)$ is the dehomogenization of f_i, $i = 1, \ldots, n-1$. It follows from Theorem 2.3 that $C_{\ell,m}$ is an s.t.c.i. on the hypersurfaces $g_i = 0$ and $G = {x_1}^{s_1} \cdots \ {x_n}^{s_n} - x_{n+1}^{\ell} = 0$. So the ideal of the affine curve $C_{\ell,m}$ contains g_i's and G. Hence the ideal of the projective closure of $C_{\ell,m}$ must contain (at least) f_i's and F, where F is the homogenization of G. Now, since $f_1, \ldots, f_{n-1}, F \in I(\overline{C}_{\ell,m})$, we always have $\overline{C}_{\ell,m} \subseteq Z(f_1, \ldots, f_{n-1}, F)$.

4.1. The case where f_i's are general, but m is special.

In this section we assume that m is a multiple of m_n; i.e. $m = s_n m_n$, where s_n is a positive integer. Note that $(s_1, \ldots, s_{n-1}) = (0, \ldots, 0)$ and $\delta(m) = s_n$ in this case.

Theorem 4.2. *Let $\overline{C} \subset \mathbb{P}^n$ be an s.t.c.i. on the hypersurfaces $f_1 = \cdots = f_{n-1} = 0$, $gcd(\ell, s_n m_n) = 1$ and $s_n > \ell$. Then the nice extensions $\overline{C}_{\ell, s_n m_n}$ in $\mathbb{P}^{n+1}$ are s.t.c.i.'s on $f_1 = \cdots = f_{n-1} = F = 0$, where $F = x_n^{s_n} - x_0^{s_n - \ell} x_{n+1}^{\ell}$.*

Proof. The fact that these nice extensions are s.t.c.i.'s can be seen easily by [14, Theorem 3.4] taking $b_1 = m_1, \ldots, b_{n-1} = m_{n-1}$, $d = m_n$ and $k = (s_n - \ell) m_n$. In addition to this, we provide here the equation of the binomial hypersurface $F = 0$ on which these extensions lie as s.t.c.i. monomial curves.

Since $\overline{C}_{\ell,s_n m_n} \subseteq Z(f_1,\ldots,f_{n-1},F)$, we need to get the converse inclusion. Take a point $P=(p_0,\ldots,p_n,p_{n+1}) \in Z(f_1,\ldots,f_{n-1},F)$. Then, since $f_i \in K[x_0,\ldots,x_n]$, we have $f_i(P)=f_i(p_0,\ldots,p_n)=0$, for all $i=1,\ldots,n-1$. Since $Z(f_1,\ldots,f_{n-1})=\overline{C}$ in $\mathbb{P}^n$ by assumption, the last observation implies that

$$(p_0,\ldots,p_n)=(u^{m_n},u^{m_n-m_1}v^{m_1},\ldots,u^{m_n-m_{n-1}}v^{m_{n-1}},v^{m_n}).$$

If $p_0=0$, then $u=0$, yielding that $(p_0,\ldots,p_{n-1},p_n)=(0,\ldots,0,p_n)$. Since $s_n>\ell$, we also have $p_n=0$ by $F(0,\ldots,0,p_n,p_{n+1})={p_n}^{s_n}-p_0^{s_n-\ell}p_{n+1}^{\ell}=0$. So we observe that $(p_0,\ldots,p_n,p_{n+1})=(0,\ldots,0,1)$, which is on the curve $\overline{C}_{\ell,s_n m_n}$. If $p_0=1$, then $(1,p_1,\ldots,p_n,p_{n+1})\in Z(g_1,\ldots,g_{n-1},G)$ by the assumption, where g_i and G are polynomials defined in Remark 4.1. Since $C_{\ell,s_n m_n}$ is an s.t.c.i. on the hypersurfaces $g_1=\cdots=g_{n-1}=0$ and $G=0$, it follows that $(1,p_1,\ldots,p_n,p_{n+1})\in C_{\ell,s_n m_n}\subset \overline{C}_{\ell,s_n m_n}$. □

4.2. **The case where f_i's are special and m is general.** Assume now that m is not a multiple of m_n, i.e. $(s_1,\ldots,s_{n-1})\neq(0,\ldots,0)$. Recall that we choose $s_1,\ldots,s_n$ in the representation of $m=s_1m_1+\cdots+s_nm_n$ in such a way that $s_1+\cdots+s_n$ is minimum, i.e. $s_1+\cdots+s_n=\delta(m)$. First we prove a lemma where no restriction on the f_i is required.

Lemma 4.3. *Let $\overline{C}\subset\mathbb{P}^n$ be an s.t.c.i. on $f_1=\cdots=f_{n-1}=0$ and $\delta(m)>\ell$. Then $Z(f_1,\ldots,f_{n-1},F)=\overline{C}_{\ell,m}\cup L\subset\mathbb{P}^{n+1}$, where $F={x_1}^{s_1}\cdots\ {x_n}^{s_n}-x_0^{\delta(m)-\ell}x_{n+1}^{\ell}$ and L is the line $x_0=\cdots=x_{n-1}=0$.*

Proof. We first prove $\overline{C}_{\ell,m}\cup L\subseteq Z(f_1,\ldots,f_{n-1},F)$. By Remark 4.1, it is sufficient to see that $L\subseteq Z(f_1,\ldots,f_{n-1},F)$. For this, we take a point $P=(p_0,\ldots,p_{n+1})$ on the line L, i.e., $P=(0,\ldots,0,p_n,p_{n+1})$. Since $(s_1,\ldots,s_{n-1})\neq(0,\ldots,0)$ and $\delta(m)>\ell$, we see that $F(P)=0$. Letting $v\in K$ be any m_n-th root of p_n, we get $(0,\ldots,0,p_n)=(0,\ldots,0,v^{m_n})\in\overline{C}=Z(f_1,\ldots,f_{n-1})$. Since the polynomials f_i are in $K[x_0,\ldots,x_n]$, it follows that $f_i(P)=f_i(0,\ldots,0,p_n)=0$, for all $i=1,\ldots,n-1$. Thus $P\in Z(f_1,\ldots,f_{n-1},F)$.

For the converse inclusion, take $P=(p_0,\ldots,p_n,p_{n+1})\in Z(f_1,\ldots,f_{n-1},F)$. Then, for all $i=0,\ldots,n-1$, we get $f_i(p_0,\ldots,p_n)=f_i(P)=0$, implying that

$$(p_0,\ldots,p_n)=(u^{m_n},u^{m_n-m_1}v^{m_1},\ldots,u^{m_n-m_{n-1}}v^{m_{n-1}},v^{m_n}).$$

If $p_0=0$, then $u=0$, yielding that $(p_0,\ldots,p_n)=(0,\ldots,0,p_n)$. Thus, we get $P=(p_0,\ldots,p_n,p_{n+1})=(0,\ldots,0,p_n,p_{n+1})\in L$. If $p_0=1$, then by assumption we know that $P=(1,p_1,\ldots,p_n,p_{n+1})\in Z(g_1,\ldots,g_{n-1},G)$. Since $C_{\ell,m}$ is an s.t.c.i. on the hypersurfaces $g_1=\cdots=g_{n-1}=0$ and $G=0$, it follows that $P=(1,p_1,\ldots,p_n,p_{n+1})\in C_{\ell,m}\subset\overline{C}_{\ell,m}$. □

To get rid of L in the intersection of the hypersurfaces $f_1=\cdots=f_{n-1}=0$ and $F=0$, we modify the $F={x_1}^{s_1}\cdots\ {x_n}^{s_n}-x_0^{\delta(m)-\ell}x_{n+1}^{\ell}$ of Lemma 4.3, as in the work of Bresinsky (see [4]), for some special choice of $f_1,\ldots,f_{n-1}$. In this way we construct a new polynomial F^* from F such that $Z(f_1,\ldots,f_{n-1},F^*)=\overline{C}_{\ell,m}$, where F^* is a polynomial of the form

$$F^*=x_n^{\alpha}+x_0^{\beta}H(x_0,\ldots,x_{n+1}),$$

where β is a positive integer.

Note that when $x_0=0$, the vanishing of F^* implies that $x_n=0$. It follows from the last part of the proof of Lemma 4.3 that this property of F^* ensures that we have a point at infinity in the intersection of $f_1 = \cdots = f_{n-1} = 0$ and $F^* = 0$ instead of a line.

The construction of F^* can be described as follows. We first assume that $f_i = x_i^{a_i} - x_0^{a_i-b_i}x_n^{b_i} = 0$, where $a_i > b_i$ are positive integers, for all $i = 1, \dots, n-1$. Let $p = a_1 \cdots a_{n-1}$ and $p_i = \frac{b_i}{a_i}p$, for $i = 1, \dots, n-1$. Take the p-th power of F, and for every occurrence of $x_i^{a_i}$ substitute $x_0^{a_i-b_i}x_n^{b_i}$ for all $i = 1, \dots, n-1$. Then we have

$$\begin{aligned} F^p &= x_0^\gamma x_n^\alpha + x_0^{\delta(m)-\ell} H(x_0, \dots, x_{n+1}) \ \ mod(f_1, \dots, f_{n-1}) \\ &= x_0^\gamma [x_n^\alpha + x_0^{\delta(m)-\ell-\gamma} H(x_0, \dots, x_{n+1})] \ \ mod(f_1, \dots, f_{n-1}), \end{aligned}$$

where $\gamma = \sum_{i=1}^{n-1}(p-p_i)s_i$, $\alpha = ps_n + \sum_{i=1}^{n-1} p_i s_i$ and H is a polynomial. Letting

$$F^*(x_0, \dots, x_{n+1}) = x_n^\alpha + x_0^{\delta(m)-\ell-\gamma} H(x_0, \dots, x_{n+1}),$$

we observe that

$$F^p(x_0, \dots, x_{n+1}) = x_0^\gamma F^*(x_0, \dots, x_{n+1}) \ \ mod(f_1, \dots, f_{n-1}). \tag{4.1}$$

Recall that m is an element of the numerical semigroup generated by $m_1, \dots, m_n$; i.e. $m = s_1 m_1 + \cdots + s_n m_n$ with $s_1 + \cdots + s_n = \delta(m)$. If m is large enough that $s_n > \ell + \sum_{i=1}^{n-1}(p - p_i - 1)s_i$ (or equivalently $\delta(m) - \ell - \gamma > 0$), then F^* is the required polynomial. (Otherwise, F^* may not be a polynomial.) Hence we conclude the following:

Theorem 4.4. *Let p, p_i, f_i and F^* be as above. Assume that m is chosen so that $s_n > \ell + \sum_{i=1}^{n-1}(p - p_i - 1)s_i$. Then, for all $\ell < \delta(m)$ with $gcd(\ell, m) = 1$, the nice extensions $\overline{C}_{\ell,m} \subset \mathbb{P}^{n+1}$ are s.t.c.i.'s on $f_1 = \cdots = f_{n-1} = 0$ and $F^* = 0$.*

Proof. We will show that $\overline{C}_{\ell,m}$ is an s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$ and $F^* = 0$. To do this, take a point $P = (p_0, \dots, p_{n+1}) \in \overline{C}_{\ell,m}$. Then, $F(P) = 0$ and $f_i(P) = 0$, for all $i = 1, \dots, n-1$, since $Z(f_1, \dots, f_{n-1}, F) = \overline{C}_{\ell,m} \cup L$, by Lemma 4.3. From equation (4.1) it follows that $F^*(P) = 0$ or $p_0 = 0$. Since P is a point on the monomial curve $\overline{C}_{\ell,m}$, it can be parameterized as follows:

$$(u^m, u^{m-\ell m_1}v^{\ell m_1}, \dots, u^{m-\ell m_n}v^{\ell m_n}, v^m).$$

So if $p_0 = 0$, we get $u = 0$ and thus $p_i = 0$, for all $i = 1, \dots, n$. Therefore $P = (0, \dots, 0, 1)$, and hence $F^*(P) = 0$ in any case.

Conversely, let $P = (p_0, \dots, p_{n+1}) \in Z(f_1, \dots, f_{n-1}, F^*)$. If $p_0 = 0$, then $p_i = 0$ by $f_i(P) = 0$, for all $i = 1, \dots, n-1$. Since $\delta(m) - \ell - \gamma > 0$, we have $p_n = 0$ by $F^*(P) = 0$. Thus $P = (0, \dots, 0, 1)$, which is always on the curve $\overline{C}_{\ell,m}$. If $p_0 = 1$, then C is an s.t.c.i. on the hypersurfaces given by $g_i = x_i^{a_i} - x_{i+1}^{b_i} = 0$, for $i = 1, \dots, n-1$, by the assumption. Hence, Theorem 2.3 implies that $C_{\ell,m}$ is an s.t.c.i. on $g_1 = \cdots = g_{n-1} = 0$ and $G = {x_1}^{s_1} \cdots {x_n}^{s_n} - x_{n+1}^{\ell} = 0$. Thus $P = (1, p_1, \dots, p_{n+1}) \in C_{\ell,m} \subset \overline{C}_{\ell,m}$. □

Remark 4.5. The *nice* extensions in Theorem 4.4 can also be shown to be s.t.c.i.'s by using [14, Theorem 3.4]. But to show that the hypotheses of [14, Theorem 3.4] are satisfied by these extensions is much more difficult than the proof here. As a byproduct we also constructed here the hypersurface $F^* = 0$ on which these *nice* extensions are s.t.c.i.'s.

Example 4.6. We start with $\overline{C} = \overline{C}(3,4,6) \subset \mathbb{P}^3$. Let $\ell = 1$ and $m = 6s+7$, for some positive integer s. Then $\delta(m) = s+2$, $s_1 = s_2 = 1$ and $s_3 = s$. Thus we get the nice extensions $\overline{C}_{1,6s+7} = \overline{C}(3,4,6,6s+7) \subset \mathbb{P}^4$. Since $\Delta_1 = gcd(4,6,6s+7) = 1$, $\Delta_2 = gcd(3,6,6s+7) = 1$ and $\Delta_3 = gcd(3,4,6s+7) = 1$, it follows from Corollary 3.3 that these curves cannot be obtained by gluing. Using the software Macaulay, it is easy to see that the ideal of $\overline{C}_{1,6s+7}$ is minimally generated by the polynomials

$$\begin{aligned} f_1 &= x_1^2 - x_0x_3, \\ f_2 &= x_2^3 - x_0x_3^2, \\ f_3 &= x_3^{s+3} - x_0^{s-1}x_1x_2^2x_4, \\ f_4 &= x_2x_3^{s+1} - x_0^sx_1x_4, \\ f_5 &= x_1x_3^{s+2} - x_0^sx_2^2x_4, \\ F &= x_1x_2x_3^s - x_0^{s+1}x_4. \end{aligned}$$

Since $\overline{C}(3,4,6) \subset \mathbb{P}^3$ is an s.t.c.i. on the surfaces $f_1 = 0$ and $f_2 = 0$, it follows from Theorem 4.4 that $\overline{C}_{1,6s+7}$ is an s.t.c.i. on $f_1 = 0$, $f_2 = 0$ and

$$\begin{aligned} F^* = x_3^{6s+7} - 6x_0^{s-1}x_1x_2^2x_3^{5s+4}x_4 + 15x_0^{2s}x_2x_3^{4s+4}x_4^2 - 20x_0^{3s}x_1x_3^{3s+3}x_4^3 \\ +15x_0^{4s}x_2^2x_3^{2s+1}x_4^4 - 6x_0^{5s}x_1x_2x_3^sx_4^5 + x_0^{6s+1}x_4^6 = 0, \end{aligned}$$

provided that $s > 2$.

5. Variations and consequences of the main results

In this section, we give some consequences of Theorem 4.2, and hence all the notation is as in that theorem. We also include some theorems about *nice* extensions of projective monomial curves that are variations of Theorem 4.4.

5.1. **Consequences of Theorem 4.2.**

Since arithmetically Cohen-Macaulay monomial curves are s.t.c.i.'s in $\mathbb{P}^3$ (see [12]), we get the following corollary as a consequence of Theorem 4.2.

Corollary 5.1. *Let $\overline{C}(m_1,m_2,m_3)$ be an arithmetically Cohen-Macaulay monomial curve in $\mathbb{P}^3$. Let $m = s_3m_3$, $gcd(\ell,m) = 1$ and $\delta(m) = s_3 > \ell$. Then the nice extensions $\overline{C}_{\ell,s_3m_3} = \overline{C}(\ell m_1,\ell m_2,\ell m_3,s_3m_3)$ are all s.t.c.i.'s in $\mathbb{P}^4$.* □

Remark 5.2. There are very few examples of s.t.c.i. monomial curves in $\mathbb{P}^n$, where $n > 3$. We know that the rational normal curve $\overline{C}(1,2,\ldots,n)$ is an s.t.c.i. in $\mathbb{P}^n$, for any $n > 0$ (see [11, 14]). Applying Theorem 4.2 to $\overline{C}(1,2,\ldots,n) \subset \mathbb{P}^n$, we can produce infinitely many new examples of s.t.c.i. monomial curves in $\mathbb{P}^{n+1}$:

Corollary 5.3. *For all positive integers ℓ, n and s with $gcd(\ell,sn) = 1$, the monomial curves $\overline{C}(\ell,2\ell,\ldots,n\ell,sn) \subset \mathbb{P}^{n+1}$ are s.t.c.i.'s.*

Proof. Let $m = sn$. Clearly $\delta(m) = s$. If $s \leq \ell$, then the curves $\overline{C}_{\ell,m} = \overline{C}(\ell,2\ell,\ldots,n\ell,sn) \subset \mathbb{P}^{n+1}$ are bad extensions of $\overline{C}(1,2,\ldots,n) \subset \mathbb{P}^n$. Hence they are s.t.c.i.'s by Corollary 3.5. If $s > \ell$, then these curves are nice extensions of $\overline{C}(1,2,\ldots,n) \subset \mathbb{P}^n$. Therefore they are s.t.c.i.'s by Theorem 4.2. □

In [10], all complete intersection (i.t.c.i.) lattice ideals are characterized by gluing semigroups. But, for a given projective monomial curve it is not easy to find two subsemigroups whose ideals are complete intersections. So, as another application of Theorem 4.2 we can produce infinitely many i.t.c.i. monomial curves:

Proposition 5.4. *If $\overline{C} \subset \mathbb{P}^n$ is an i.t.c.i., then the nice extensions $\overline{C}_{\ell,s_nm_n} \subset \mathbb{P}^{n+1}$ are i.t.c.i.'s for all positive integers ℓ and s_n with $s_n > \ell$, $gcd(\ell, s_nm_n) = 1$.*

Proof. Since $\overline{C}$ is an s.t.c.i. on the binomial hypersurfaces $f_1 = \cdots = f_{n-1} = 0$, it follows from Theorem 4.2 that $\overline{C}_{\ell,s_nm_n}$ is an s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$ and $F(x_0,\dots,x_{n+1}) = x_n^{s_n} - x_0^{s_n-\ell}x_{n+1}^{\ell} = 0$. Since these are all binomial, the monomial curves $\overline{C}_{\ell,s_nm_n}$ are i.t.c.i.'s on the same hypersurfaces, by [2, Theorem 4]. □

Corollary 5.5. *The monomial curves $\overline{C}(\ell m_1, \ell m_2, s_2m_2)$ are i.t.c.i.'s in $\mathbb{P}^3$, for all positive integers m_1, m_2, ℓ and s_2 with $s_2 > \ell$, $gcd(\ell, s_2m_2) = 1$.*

Proof. Let $m = s_2m_2$. Then $\delta(m) = s_2$ and $\overline{C}_{\ell,m} = \overline{C}(\ell m_1, \ell m_2, s_2m_2)$ is a nice extension of $\overline{C}(m_1, m_2)$, by the assumption $s_2 > \ell$. Since $\overline{C}(m_1, m_2)$ is an i.t.c.i. on $x_1^{m_2} - x_0^{m_2-m_1}x_2^{m_1} = 0$, it follows from Proposition 5.4 that the nice extensions $\overline{C}(\ell m_1, \ell m_2, s_2m_2)$ are i.t.c.i.'s on $x_1^{m_2} - x_0^{m_2-m_1}x_2^{m_1} = 0$ and $x_2^{s_2} - x_0^{s_2-\ell}x_3^{\ell} = 0$. □

To produce infinitely many examples of i.t.c.i. curves, our method starts from just one i.t.c.i. curve, whereas the semigroup gluing method produces only one example starting from one i.t.c.i. The following example illustrates this point.

Example 5.6. From Corollary 5.5, we know that $\overline{C}(1, 2, 4)$ is an i.t.c.i. on

$$f_1 = x_1^2 - x_0x_2 = 0 \ \text{ and } \ f_2 = x_2^2 - x_0x_3 = 0.$$

Take two positive integers ℓ and s with $s > \ell$, $gcd(\ell, 4s) = 1$. Then the monomial curves $\overline{C}(\ell, 2\ell, 4\ell, 4s) \subset \mathbb{P}^4$ are nice extensions of $\overline{C}(1, 2, 4) \subset \mathbb{P}^3$. Thus, by Proposition 5.4, the monomial curves $\overline{C}(\ell, 2\ell, 4\ell, 4s)$ are i.t.c.i.'s on

$$f_1 = x_1^2 - x_0x_2 = 0,\ f_2 = x_2^2 - x_0x_3 = 0 \ \text{ and } \ F = x_3^s - x_0^{s-\ell}x_4^{\ell} = 0.$$

The nice extensions $\overline{C}(\ell, 2\ell, 4\ell, 4s)$ can also be obtained by gluing subsemigroups generated by $T_1 = \{(4s-\ell, \ell)\}$ and $T_2 = \{(4s, 0), (4s-2\ell, 2\ell), (4s-4\ell, 4\ell), (0, 4s)\}$. But in this case one has to know that $\overline{C}(\ell, 2\ell, 2s)$ is an i.t.c.i. for each ℓ and s. In other words, starting with the fact that $\overline{C}(1, 2, 4)$ is an i.t.c.i., the gluing method can only produce $\overline{C}(1, 2, 4, 8)$ as an i.t.c.i. monomial curve.

5.2. **Variations of Theorem 4.4.** Recall that our method starts with a monomial curve $\overline{C} = Z(f_1,\dots,f_{n-1})$ in $\mathbb{P}^n$ and produces infinitely many nice extensions $\overline{C}_{\ell,m} = Z(f_1,\dots,f_{n-1},F^*)$ in $\mathbb{P}^{n+1}$. Since the construction of F^* depends on the choice of $f_1,\dots,f_{n-1}$, it is possible to start with another curve $\overline{C} = Z(f_1,\dots,f_{n-1})$ in $\mathbb{P}^n$ and obtain new families of nice extensions. In this section we provide two examples of this sort. For instance, if we assume that $\overline{C}$ is an s.t.c.i. on the hypersurfaces $f_i = x_i^{a_i} - x_0^{a_i-b_i}x_{i+1}^{b_i} = 0$, where $a_i > b_i$ are positive integers, $i = 1,\dots,n-1$, then under some suitable conditions we obtain other families of s.t.c.i. nice extensions. Let $p = a_1\cdots a_{n-1}$, $q_0 = b_1\cdots b_{n-1}$ and $q_i = a_1\cdots a_ib_{i+1}\cdots b_{n-1}$, $i = 1,\dots,n-2$. The first variation is the following:

Theorem 5.7. *Let $p, q_0, \dots, q_{n-2}$ be as above. For all m which give rise to $s_n > \ell + \sum_{i=0}^{n-2}(p - q_i - 1)s_{i+1}$ and for all ℓ with $\ell < \delta(m)$ and $gcd(\ell, m) = 1$, the nice extensions $\overline{C}_{\ell,m} \subset \mathbb{P}^{n+1}$ are s.t.c.i.'s on $f_1 = \cdots = f_{n-1} = F^* = 0$.* □

Now, we give another variation where $m = s_i m_i + s_j m_j$, for $i, j \in \{1, \dots, n\}$. For notational convenience we take $i = 1$ and $j = n$.

Theorem 5.8. *Let $\overline{C} \subset \mathbb{P}^n$ be an s.t.c.i. on the hypersurfaces given by*

$$\begin{aligned} f_1 &= x_1^a - x_0^{a-b} x_n^b = 0, \\ f_i &= x_i^{a_i} + x_0^{b_i} A(x_1, \dots, x_n) + x_1^{c_i} B(x_2, \dots, x_n) = 0, \end{aligned}$$

where a, b, $a-b$, a_i, b_i, and c_i are positive integers, for $i = 2, \dots, n-1$, and where A and B are some polynomials. For all m which give rise to $s_n > \ell + (a - b - 1)s_1$ and for all ℓ with $\ell < \delta(m)$ and $gcd(\ell, m) = 1$, the nice extensions $\overline{C}_{\ell,m} \subset \mathbb{P}^{n+1}$ are s.t.c.i.'s on $f_1 = \cdots = f_{n-1} = F^* = 0$. □

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