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INVARIANT SUBSPACES OF SUPER LEFT-COMMUTANTS

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ABSTRACT. For a positive operator Q on a Banach lattice, one defines $\langle Q] = \{T \geq 0 : TQ \leq QT\}$ and $[Q\rangle = \{T \geq 0 : TQ \geq QT\}$. There have been several recent results asserting that, under certain assumptions on Q, $[Q\rangle$ has a common invariant subspace. In this paper, we use the technique of minimal vectors to establish similar results for $\langle Q]$.

Throughout this paper, we assume that X is a real Banach lattice with positive cone X_+; $\mathcal{L}(X)$ stands for the space of all (bounded linear) operators on X. Let Q be a positive operator on X. By an ***invariant subspace of*** Q we mean a closed subspace V of X such that $V \neq \{0\}$, $V \neq X$ and $QV \subseteq V$. The ***super left-commutant*** $\langle Q]$ and the ***super right-commutant*** of $[Q\rangle$ of Q are defined as follows:

$$\langle Q] = \{T \geq 0 : TQ \leq QT\}, \qquad [Q\rangle = \{T \geq 0 : TQ \geq QT\}.$$

The symbol $B(x, r)$ stands for the closed ball of radius r centered at x. If $a < b$ in X, we write $[a, b] = \{x \in X : a \leq x \leq b\}$. A subspace $Y \subseteq X$ is an (order) ***ideal*** if $|y| \leq |x|$ and $x \in Y$ imply $y \in Y$. For $K \in \mathcal{L}(X)$ we say that K is ***dominated*** by Q if $|Kx| \leq Q|x|$ for every $x \in X$. Obviously, every operator in $[0, Q] = \{K \in \mathcal{L}(X) : 0 \leq K \leq Q\}$ is dominated by Q. For more details on positive operators, we refer the reader to [AA02].

Suppose that Q is compact-friendly (see the definition below) and quasinilpotent. It was shown in [AA02] that every sequence in $[Q\rangle$ has a (common) invariant subspace, which is also invariant under Q. Furthermore, if X is order complete, then the entire $[Q\rangle$ has an invariant subspace. Using the technique of minimal vectors (see [AE98, Tr04, AT05, GT]) we prove in this paper that the same results hold for $\langle Q]$. First we prove an extension of a fact in [AT05].

Definition 1. A collection of operators $\mathcal{F} \subseteq \mathcal{L}(X)$ ***localizes*** a set $A \subseteq X$ if for every sequence (x_n) in A there exists a subsequence (x_{n_i}) and a sequence (K_i) in $\mathcal{F}$ such that $K_i x_{n_i}$ converges to a non-zero vector.

Theorem 2 ([AT05])**.** *Suppose that Q is a positive quasinilpotent one-to-one operator with dense range and $x_0 \in X_+$ with $\|x_0\| > 1$. If the set of all operators dominated by Q localizes $B(x_0, 1) \cap [0, x_0]$, then there exists an invariant subspace for $\langle Q]$. Moreover, if $[0, Q]$ localizes $B(x_0, 1) \cap [0, x_0]$, then $\langle Q]$ has an invariant closed ideal.*

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We extend Theorem 2 as follows.

Theorem 3. *Suppose that Q is a positive quasinilpotent operator and $x_0 \in X_+$ with $\|x_0\| > 1$. If there exists R in $\langle Q]$ such that the set of all operators dominated by R localizes $B(x_0,1)\cap[0,x_0]$, then there exists an invariant subspace for $\langle Q]$. Moreover, if $[0,R]$ localizes $B(x_0,1)\cap[0,x_0]$, then $\langle Q]$ has an invariant closed ideal.*

Proof. Suppose $R \in \langle Q]$ such that the set of all operators dominated by R localizes $B(x_0,1)\cap[0,x_0]$. Since the ideal generated by Range Q is invariant under $\langle Q]$ by [GT, Lemma 0.5], we assume that this ideal is dense in X. As in the proof of [AT05, Theorem 8] and [GT, Theorem 5.5], we find a sequence (K_i) of operators dominated by R and an increasing sequence of integers (n_i) such that $K_i(x_0 \wedge Q^{n_i-1}y_{n_i-1})$ converges to some vector $w \neq 0$, and (f_{n_i}) w*-converges to a positive functional $g \neq 0$, where (y_n) is a sequence of 2-*minimal* vectors and (f_n) is a sequence of 2-minimal functionals for Q and $B(x_0,1)+X_+$.

Suppose $T \in \langle Q]$. Using the facts that K_i is dominated by R for each i, $TQ \leq QT$, $RQ \leq QR$, and by Propositions 5.3(v) and 5.4 of [GT], we have

$$\begin{aligned}\Big|f_{n_i}\big(QTK_i(x_0\wedge Q^{n_i-1}y_{n_i-1})\big)\Big| &\leq f_{n_i}\Big(QT\big|K_i(x_0\wedge Q^{n_i-1}y_{n_i-1})\big|\Big)\\ &\leq f_{n_i}\big(QTR(x_0\wedge Q^{n_i-1}y_{n_i-1})\big) \leq f_{n_i}\big(QTRQ^{n_i-1}y_{n_i-1}\big)\\ \leq f_{n_i}\big(Q^{n_i}TRy_{n_i-1}\big) \leq \big\|Q^{*n_i}f_{n_i}\big\|\cdot\|TR\|\cdot\|y_{n_i-1}\| &\leq \frac{4\|x_0\|\|TR\|\|y_{n_i-1}\|}{\|y_{n_i}\|} \to 0.\end{aligned}$$

Thus,

$$f_{n_i}\big(QTK_i(x_0\wedge Q^{n_i-1}y_{n_i-1})\big) \to 0.$$

On the other hand,

$$QTK_i(x_0\wedge Q^{n_i-1}y_{n_i-1}) \to QTw$$

in norm. Since $f_{n_i} \xrightarrow{w^*} g$, we conclude that $g(QTw)=0$; hence $(Q^*g)(Tw)=0$.

Since the ideal generated by Range Q is dense and $g \neq 0$ is positive, we have $Q^*g \neq 0$. Let Y be the linear span of $\langle Q]w$, that is, $Y = \operatorname{lin}\{Tw : T \in \langle Q]\}$. Since $\langle Q]$ is a multiplicative semigroup, Y is invariant under every $T \in \langle Q]$. It follows from $0 \neq w \in Y$ that Y is non-zero. Finally, $\overline{Y} \neq X$ because Q^*g vanishes on Y.

Suppose that $[0,R]$ localizes $B(x_0,1)\cap[0,x_0]$ for some $x_0 \geq 0$ and $\|x_0\| > 1$. Then the vector w constructed in the previous argument is positive. Let E be the ideal generated by $\langle Q]w$, that is,

$$E = \big\{y \in X \,:\, |y| \leq Tw \text{ for some } T \in \langle Q]\big\}.$$

The ideal E is non-trivial since $w \in E$, and it is easy to see that E is invariant under $\langle Q]$. Since the positive functional Q^*g vanishes on Tw it must also vanish on E; consequently $\overline{E} \neq X$ since $Q^*g \neq 0$. □

Remark 4. It was shown in [GT] that with some minor adjustments, Theorem 2 can be extended from Banach lattices to ordered Banach spaces with generating cones. In a similar fashion, Theorem 3 can be extended to such spaces as well.

Next we present several applications of Theorem 3. Recall that an operator on a Banach lattice is *AM-**compact*** if it maps order bounded sets to relatively compact sets. In [FTT08], the authors proved the following extension of earlier results by R. Drnovšek (see [AA02, Theorems 10.44 and 10.50]): if Q is a quasinilpotent positive operator on a Banach lattice with a quasiinterior point such that some operator in $[Q\rangle$ dominates a non-zero AM***-compact*** operator, then $[Q\rangle$ has an invariant closed ideal. Our next theorem provides a similar result for $\langle Q]$.

Theorem 5. *If Q is a positive quasinilpotent operator and there exists a non-zero AM-compact operator K dominated by an operator in $\langle Q]$, then $\langle Q]$ has an invariant subspace. Furthermore, if $K \geq 0$, then $\langle Q]$ has a closed invariant ideal.*

Proof. Let K be a non-zero AM-compact operator dominated by an operator $R \in \langle Q]$. We can find $x_0 \geq 0$ with $\|x_0\| > 1$ such that $0 \notin \overline{K(B(x_0, 1) \cap [0, x_0])}$. Therefore, the set of operators dominated by R localizes $B(x_0, 1) \cap [0, x_0]$. Theorem 3 completes the proof. □

Since every compact operator is an AM-compact operator, we have the following simple consequence of Theorem 5.

Corollary 6. *If Q is a positive quasinilpotent operator and there exists a non-zero compact operator K dominated by an operator in $\langle Q]$, then $\langle Q]$ has an invariant subspace. Furthermore, if $K \geq 0$, then $\langle Q]$ has a closed invariant ideal.*

Following [AA02] we give the following definition.

Definition 7. A positive operator $Q : X \to X$ is ***compact-friendly*** if there exist three operators R, K, and $C \neq 0$ such that $RQ = QR$, K is compact, and C is dominated by both R and K.

Remark 8. *If Q is a quasinilpotent compact-friendly operator and $C^3 \neq 0$, where C is as in Definition 7, then $\langle Q]$ has a common invariant subspace.* Indeed, by Theorem 16.14 of [AB85], C^3 is compact and C^3 is dominated by R^3 which is in $\langle Q]$. Then we use Corollary 6. Furthermore, if C is positive, then by Theorem 3 and Corollary 6, $\langle Q]$ has a common invariant ideal.

In Theorems 10.55 and 10.57 of [AA02] it was shown that under certain assumptions, the super right-commutant $[Q\rangle$ of a quasinilpotent compact-friendly operator Q has an invariant subspace. The next two theorems show that under similar assumptions, $\langle Q]$ has an invariant subspace. The proofs are similar to the proofs of Theorems 10.55 and 10.57 in [AA02], but we use Corollary 6 instead of Drnovšek's theorem as we deal with $\langle Q]$ instead of $[Q\rangle$.

Theorem 9. *If Q is a quasinilpotent compact-friendly operator, then at least one of the following is true:*

(i) *for each sequence $\{T_n\}$ in $\langle Q]$ there exists a non-trivial closed ideal that is invariant under Q and each T_n, or*
(ii) *$\langle Q]$ has an invariant subspace.*

Proof. Without loss of generality we can assume that $\|Q\| < 1$ and suppose that (T_n) is a sequence in $\langle Q]$. Pick arbitrary scalars $\alpha_n > 0$ that are small enough so that the positive operator $T = \sum_{n=1}^{\infty} \alpha_n T_n$ exists and $\|Q + T\| < 1$. Since $\langle Q]$ is a norm closed additive semigroup, it follows that the positive operator $A = \sum_{n=0}^{\infty} (Q + T)^n$ belongs to $\langle Q]$.

For each $x > 0$ we denote by J_x the principal ideal generated by Ax;

$$J_x = \{y \in X : \ |y| \leq \lambda Ax \text{ for some } \ \lambda > 0\}.$$

It follows from $x \leq Ax$ that $x \in J_x$, so that $J_x \neq 0$.

Observe that J_x is $(Q+T)$-invariant. Since $0 \leq Q, T \leq Q+T$, J_x is invariant under Q and T and thus it is also T_n-invariant for each n. Therefore, if $\overline{J_x} \neq X$ for some $x > 0$, then $\overline{J_x}$ is the desired invariant ideal.

Suppose $\overline{J_x} = X$ for each $x > 0$. Then following the proof of Theorem 10.55 in [AA02], we can construct a compact operator which is dominated by some $S \in \langle Q]$. Then Corollary 6 guarantees that $\langle Q]$ has an invariant subspace. $\square$

Remark 10. If X is order complete, then we may assume that the operator C in Definition 7 is positive. Indeed, take $x \geq 0$. For each $y \in [-x, x]$, we have $|Cy| \leq K|y| \leq Kx$. Then $|C|x = \sup_{y \in [-x,x]} |Cy| \leq Kx$, so that $|C| \leq K$. Likewise, $|C| \leq R$.

Theorem 11. *If a non-zero compact-friendly operator Q on an order complete Banach lattice is quasinilpotent, then $\langle Q]$ has a non-trivial closed invariant ideal.*

Proof. For each $x > 0$ we denote by J_x the ideal generated by the orbit $\langle Q]$, that is,

$$J_x = \bigl\{ y \in X \,:\, |y| \leq Tx \text{ for some } T \in \langle Q] \bigr\}.$$

Since $x \in J_x$, we have $J_x \neq 0$. Note that J_x is invariant under each $T \in \langle Q]$. Therefore, if $\overline{J_x} \neq X$ for some $x > 0$, then $\overline{J_x}$ is a $\langle Q]$-invariant closed ideal. So, suppose $\overline{J_x} = X$ for each $x > 0$.

By Remark 10, there exist three positive non-zero operators R, K and C such that $RQ = QR$, $C \leq R$, $C \leq K$, and K is compact.

<u>Claim</u>: For every $x > 0$, there exists $A \in \langle Q]$ such that $CAx > 0$. Indeed, since $\overline{J_x} = X$ and $C \neq 0$, there exists a positive $y \in J_x$ such that $Cy > 0$. Then $y \leq Ax$ for some $A \in \langle Q]$; hence $CAx > 0$.

Fix any $x > 0$. Applying the claim three times, we find $A_1, A_2, A_3 \in \langle Q]$ such that $CA_3CA_2CA_1x > 0$. Let $S = CA_3CA_2CA_1$. Then $S \neq 0$ and $CA_i \leq KA_i$ $(i = 1, 2, 3)$; hence S is compact by Theorem 16.14 of [AB85]. Also, $0 \leq S \leq RA_3RA_2RA_1 \in \langle Q]$. Then Corollary 6 guarantees that $\langle Q]$ has a non-trivial closed invariant ideal. $\square$

The arguments in this paper are done for a real Banach lattice for simplicity. However, they work for complex Banach lattices with straightforward modifications.

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