

ON ENDOMORPHISM RINGS AND DIMENSIONS OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let (R, \mathfrak{m}) denote an n -dimensional complete local Gorenstein ring. For an ideal I of R let $H_I^i(R), i \in \mathbb{Z}$, denote the local cohomology modules of R with respect to I . If $H_I^i(R) = 0$ for all $i \neq c = \text{height } I$, then the endomorphism ring of $H_I^c(R)$ is isomorphic to R . Here we prove that this is true if and only if $H_I^i(R) = 0$ for $i = n, n - 1$, provided $c \geq 2$ and R/I has an isolated singularity, resp. if I is set-theoretically a complete intersection in codimension at most one. Moreover, there is a vanishing result of $H_I^i(R)$ for all $i > m$, m a given integer, and an estimate of the dimension of $H_I^i(R)$.

1. MAIN RESULTS

Let (R, \mathfrak{m}) denote a local Noetherian ring with $n = \dim R$. For the ideal $I \subset R$ let $H_I^i(\cdot), i \in \mathbb{Z}$, denote the local cohomology functor with respect to I ; see [2] for its definition and basic results. It is a difficult question to describe $\sup\{i \in \mathbb{Z} \mid H_I^i(R) \neq 0\}$, the cohomological dimension $\text{cd } I$ of I with respect to R . Recall that $\text{height } I \leq \text{cd } I$. Recently some interesting results for ideals with $c = \text{height } I = \text{cd } I$, the so-called cohomologically complete intersections have been proved. If (R, \mathfrak{m}) is a complete local ring, Hellus and Stückrad [5] have shown that the endomorphism ring $\text{Hom}_R(H_I^c(R), H_I^c(R))$ is isomorphic to R . See also [4, Lemma 2.8] for a more functorial proof and a slight extension in the case where (R, \mathfrak{m}) is a Gorenstein ring.

The first aim of the consideration here is a characterization of when the endomorphism ring of $H_I^c(R)$ is isomorphic to R . To this end we call I locally a cohomologically complete intersection provided $\text{cd } IR_{\mathfrak{p}} = \text{height } I$ for all prime ideals $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$. For instance, if I has an isolated singularity, it is locally a cohomologically complete intersection.

Theorem 1.1. *Let (R, \mathfrak{m}) denote a complete local Gorenstein ring with $n = \dim R$. Let I be an ideal of height $I = c$. Suppose that I is locally a cohomologically complete intersection. Then the following conditions are equivalent:*

- (i) *The natural homomorphism $R \rightarrow \text{Hom}_R(H_I^c(R), H_I^c(R))$ is an isomorphism.*
- (ii) *$H_I^i(R) = 0$ for $i = n - 1, n$.*

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In a certain sense, condition (ii) of Theorem 1.1 provides a numerical condition for the property that the endomorphism ring of $H_I^c(R)$ is R . In the case of R being a regular local ring containing a field, Huneke and Lyubeznik [6, Theorem 2.9] have given a topological characterization of the above condition (ii).

Theorem 1.2. *Let (R, \mathfrak{m}) be a local Gorenstein ring. Let $J \subset I$ denote two ideals of height c .*

(a) *There is a natural homomorphism*

$$\text{Hom}_R(H_J^c(R), H_J^c(R)) \rightarrow \text{Hom}_R(H_I^c(R), H_I^c(R)).$$

(b) *Suppose that $\text{Rad } JR_{\mathfrak{p}} = \text{Rad } IR_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq c + 1$. Then the homomorphism in (a) is an isomorphism.*

(c) *Let R be in addition complete. Let J denote a cohomologically complete intersection contained in I and satisfying the assumptions of (b). Then $R \rightarrow \text{Hom}_R(H_I^c(R), H_I^c(R))$ is an isomorphism.*

Our results are based on a certain estimate of $\dim H_I^i(R), i > c$; see Theorem 3.1. In the case of a regular local ring, partial results of this type have been used by Ken-Ichiroh Kawasaki [7] for the study of Lyubeznik numbers (see [8] for their definition). Here we use the truncation complex as invented in [4, Section 2] (see Definition 2.1). Moreover it provides some technical statements about the endomorphism ring of $H_I^c(R), c = \text{height } I$ (see Lemma 2.2).

In terminology the author follows the paper [4].

2. THE TRUNCATION COMPLEX

Let (R, \mathfrak{m}, k) denote a local Gorenstein ring with $n = \dim R$. First of all we will recall the truncation complex as it was introduced in [4, Section 2] and in a different context in [9, §4]. Let $R \xrightarrow{\sim} E^\cdot$ denote a minimal injective resolution of R as an R -module. It is a well-known fact that

$$E^i \simeq \bigoplus_{\mathfrak{p} \in \text{Spec } R, \text{height } \mathfrak{p}=i} E_R(R/\mathfrak{p}),$$

where $E_R(R/\mathfrak{p})$ denotes the injective hull of R/\mathfrak{p} (see [1] for these and related results about Gorenstein rings).

Now let $I \subset R$ denote an ideal and let $c = \text{height } I$. Then $d = \dim R/I = n - c$. The local cohomology modules $H_I^i(R), i \in \mathbb{Z}$, are, by definition, the cohomology modules of the complex $\Gamma_I(E^\cdot)$. Because $\Gamma_I(E_R(R/\mathfrak{p})) = 0$ for all $\mathfrak{p} \notin V(I)$, it follows that $\Gamma_I(E^\cdot)^i = 0$ for all $i < c$. Therefore $H_I^c(R) = \text{Ker}(\Gamma_I(E^\cdot)^c \rightarrow \Gamma_I(E^\cdot)^{c+1})$. This observation provides an embedding $H_I^c(R)[-c] \rightarrow \Gamma_I(E^\cdot)$ of complexes of R -modules.

Definition 2.1. The cokernel of the embedding $H_I^c(R)[-c] \rightarrow \Gamma_I(E^\cdot)$ is defined as $C_R(I)$, the truncation complex with respect to I . So there is a short exact sequence of complexes of R -modules

$$0 \rightarrow H_I^c(R)[-c] \rightarrow \Gamma_I(E^\cdot) \rightarrow C_R(I) \rightarrow 0.$$

In particular it follows that $H^i(C_R(I)) = 0$ for $i \leq c$ or $i > n$ and $H^i(C_R(I)) \simeq H_I^i(R)$ for $c < i \leq n$.

First we need to establish some basic results about the truncation complex. For more details we refer to the exposition in [4, Section 2].

Lemma 2.2. *With the previous notation, the following results hold:*

(a) *There exist an exact sequence*

$$0 \rightarrow H_m^{n-1}(C_R(I)) \rightarrow H_m^d(H_I^c(R)) \rightarrow E \rightarrow H_m^n(C_R(I)) \rightarrow 0$$

and isomorphisms $H_m^{i-c}(H_I^c(R)) \simeq H_m^{i-1}(C_R(I))$ for $i < n$.

(b) *$H_m^d(H_I^c(R)) \neq 0$ and $H_m^{i-c}(H_I^c(R)) = 0$ for $i > n$.*

(c) *Let $\mathfrak{p} \in V(I)$ denote a prime ideal. Then there is an isomorphism*

$$C_R(I) \otimes_R R_{\mathfrak{p}} \simeq C_{R_{\mathfrak{p}}}(IR_{\mathfrak{p}})$$

provided $\text{height } I = \text{height } IR_{\mathfrak{p}}$.

(d) *There is a natural isomorphism $\text{Hom}_R(H_I^c(R), H_I^c(R)) \simeq \text{Ext}_R^c(H_I^c(R), R)$.*

Proof. For the proof of (a) apply the derived functor $R\Gamma_m(\cdot)$ to the short exact sequence given in Definition 2.1. Then $R\Gamma_m(\Gamma_I(E^\cdot)) \simeq E[-n]$. So the long exact cohomology sequence of the corresponding exact sequence of complexes provides what is claimed (see [4, Lemma 2.2] for the details). Statement (b) is shown in [4, Corollary 2.9] and [4, Lemma 1.2].

For the proof of (c) localize the exact sequence in Definition 2.1 at \mathfrak{p} . Then there is a short exact sequence of complexes

$$0 \rightarrow H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})[-c] \rightarrow \Gamma_{IR_{\mathfrak{p}}}(E_{R_{\mathfrak{p}}}^\cdot) \rightarrow C_R(I) \otimes_R R_{\mathfrak{p}} \rightarrow 0.$$

To this end recall first that $c = \text{height } IR_{\mathfrak{p}} = \text{height } I$ and that the local cohomology commutes with localization. Furthermore $E^\cdot \otimes R_{\mathfrak{p}}$ is isomorphic to the minimal injective resolution $E_{R_{\mathfrak{p}}}^\cdot$ of $R_{\mathfrak{p}}$. Then the definition of the truncation complex proves the claim.

Finally, we prove (d). As shown at the beginning of this section, there is an exact sequence $0 \rightarrow H_I^c(R) \rightarrow \Gamma_I(E^\cdot)^c \rightarrow \Gamma_I(E^\cdot)^{c+1}$. This induces a natural commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(H_I^c(R), H_I^c(R)) & \longrightarrow & \text{Hom}_R(H_I^c(R), \Gamma_I(E^\cdot)^c) & \longrightarrow & \text{Hom}_R(H_I^c(R), \Gamma_I(E^\cdot)^{c+1}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ext}_R^c(H_I^c(R), R) & \longrightarrow & \text{Hom}_R(H_I^c(R), E^\cdot)^c & \longrightarrow & \text{Hom}_R(H_I^c(R), E^\cdot)^{c+1} \end{array}$$

because $\Gamma_I(E^\cdot)$ is a subcomplex of E^\cdot . The last two vertical homomorphisms are isomorphisms because $\text{Hom}_R(X, E_R(R/\mathfrak{p})) = 0$ for an R -module X with $\text{Supp}_R X \subset V(I)$ and $\mathfrak{p} \notin V(I)$. Therefore the first vertical map is also an isomorphism. \square

In order to compute the local cohomology of the truncation complex $C_R(I)$, there is the following spectral sequence for computation of the hyper cohomology of a complex.

Proposition 2.3. *With the notation of Definition 2.1, there is the spectral sequence*

$$E_2^{p,q} = H_m^p(H^q(C_R(I))) \implies E_\infty^{p+q} = H_m^{p+q}(C_R(I)),$$

where $H^q(C_R(I)) = 0$ for $i \leq c$ and $i > n$, and $H^q(C_R(I)) \simeq H_I^q(R)$ for $c < i \leq n$.

Proof. The spectral sequence is a particular case of the spectral sequence of hyper cohomology (cf. [10]). For the initial terms check the definition of the truncation complex. \square

In the following we shall use the notion of the dimension $\dim X$ for R -modules X which are not necessarily finitely generated. This is defined by $\dim X = \dim \text{Supp}_R X$, where the dimension of the support is understood in the Zariski topology of $\text{Spec } R$. In particular, $\dim X < 0$ means $X = 0$.

Lemma 2.4. *With the notation above, we have the following results:*

- (a) $\dim H_I^i(R) \leq n - i$ for all $i \geq c = \text{height } I$.
- (b) $\dim H_I^c(R) = \dim R/I$.
- (c) If $\dim H_I^i(R) < n - i$ for all $i > c$, then R/I is unmixed, i.e. $c = \text{height } IR_{\mathfrak{p}}$ for all minimal $\mathfrak{p} \in V(I)$.

Proof. (a): This result is well-known (see for instance [7]).

(b): Let $\mathfrak{p} \in V(I)$ denote a minimal prime ideal in $V(I)$ such that $\dim R_{\mathfrak{p}} = c$. Then $H_I^c(R) \otimes_R R_{\mathfrak{p}} \simeq H_{\mathfrak{p}R_{\mathfrak{p}}}^c(R_{\mathfrak{p}}) \neq 0$ by the Grothendieck non-vanishing result. So, $\mathfrak{p} \in \text{Supp } H_I^c(R)$ and $\dim R/\mathfrak{p} = d$. Together with (a) this proves the claim.

(c): Let $\mathfrak{p} \in V(I)$ be minimal with $h := \text{height } IR_{\mathfrak{p}} > c$. Then $h = \dim R_{\mathfrak{p}}$ and

$$0 \neq H_{\mathfrak{p}R_{\mathfrak{p}}}^h(R_{\mathfrak{p}}) \simeq H_I^h(R) \otimes_R R_{\mathfrak{p}}.$$

This implies that $\mathfrak{p} \in \text{Supp } H_I^h(R)$ with $\dim R/\mathfrak{p} + h = n$, a contradiction. □

Proof of Theorem 1.2. Let $\alpha \geq 1$ denote an integer. The inclusion $J \subset I$ induces a short exact sequence $0 \rightarrow I^\alpha/J^\alpha \rightarrow R/J^\alpha \rightarrow R/I^\alpha \rightarrow 0$. By applying the long exact cohomology sequence with respect to $\text{Ext}_R^c(\cdot, R)$ and passing to the direct limit, we get the following exact sequence:

$$0 \rightarrow H_I^c(R) \rightarrow H_J^c(R) \xrightarrow{\phi} \varinjlim \text{Ext}_R^c(I^\alpha/J^\alpha, R).$$

Recall that $\text{grade } I^\alpha/J^\alpha \geq c$ for all α . Let $X = \text{Im } \phi$. The short exact sequence $0 \rightarrow H_I^c(R) \rightarrow H_J^c(R) \rightarrow X \rightarrow 0$ provides (after applying $\text{Ext}_R^c(\cdot, R)$) a natural homomorphism

$$\text{Ext}_R^c(H_J^c(R), R) \rightarrow \text{Ext}_R^c(H_I^c(R), R).$$

By Lemma 2.2 (d) this proves the statement in (a).

In order to prove (b) we may assume that $JR_{\mathfrak{p}} = IR_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq c + 1$. This is possible because local cohomology does not change by passing to the radical. Next we claim that $\dim X \leq d - 2$. This follows because $\dim_R I^\alpha/J^\alpha \leq d - 2$ for all $\alpha \in \mathbb{N}$ under the additional assumption of $J \subset I$. Moreover, $\dim X \leq d - 2$ is true by a localization argument and the embedding $X \rightarrow \varinjlim \text{Ext}_R^c(I^\alpha/J^\alpha, R)$.

By passing to the completion and because of the Matlis duality (see [4, Lemma 1.2]), it will be enough to show that the natural homomorphism $H_m^d(H_I^c(R)) \rightarrow H_m^d(H_J^c(R))$ is an isomorphism. Now this is true by virtue of the local cohomology with respect to the maximal ideal applied to the short exact sequence $0 \rightarrow H_I^c(R) \rightarrow H_J^c(R) \rightarrow X \rightarrow 0$ and the fact that $\dim X \leq d - 2$.

For the proof of (c) recall that for a cohomologically complete intersection J it is known that the endomorphism ring of $H_J^c(R)$ is isomorphic to R (see [5] or [4, Lemma 3.3]). □

3. DIMENSIONS OF LOCAL COHOMOLOGY

As before let (R, \mathfrak{m}) denote a n -dimensional Gorenstein ring. Let $I \subset R$ be an ideal with $c = \text{height } I$ and $\dim R/I = n - c$. We prove the following theorem in order to estimate the dimension of local cohomology modules. To this end let us fix the abbreviation $h(\mathfrak{p}) = \dim R_{\mathfrak{p}} - c$ for a prime ideal $\mathfrak{p} \in V(I)$.

Theorem 3.1. *Let $l \geq 1$ denote an integer. With the previous notation the following conditions are equivalent:*

- (i) $\dim H_I^i(R) \leq n - l - i$ for all $i > c$.
- (ii) For all $\mathfrak{p} \in V(I)$ the natural map

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^{h(\mathfrak{p})}(H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})) \rightarrow E(k(\mathfrak{p}))$$

is bijective (resp. surjective if $l = 1$) and

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^i(H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}})) = 0$$

for all $h(\mathfrak{p}) - l + 1 < i < h(\mathfrak{p})$.

Proof. (i) \implies (ii): From Lemma 2.4 it follows that R/I is unmixed; i.e. $c = \text{height } I = \text{height } IR_{\mathfrak{p}}$ for all minimal prime ideals $\mathfrak{p} \in V(I)$. In particular this implies that $h(\mathfrak{p}) = \dim R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \in V(I)$. Moreover

$$\dim R/\mathfrak{p} + \dim H_{IR_{\mathfrak{p}}}^i(R_{\mathfrak{p}}) \leq \dim H_I^i(R)$$

because the localization commutes with local cohomology. So our assumption (i) implies that $\dim H_{\mathfrak{p}R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} - l - i$ for all $i > \text{height } IR_{\mathfrak{p}} = c$. Therefore it will be enough to prove the statement in (ii) for $\mathfrak{p} = \mathfrak{m}$, the maximal ideal of (R, \mathfrak{m}) .

From Lemma 2.2 (a) it will be enough to show the vanishing of $H_{\mathfrak{m}}^i(C_R(I))$ for all $i > n - l$. To this end consider the spectral sequence of Proposition 2.3. By our assumption we have for the initial terms $E_2^{p,q} = H_{\mathfrak{m}}^p(H_I^q(R)) = 0$ for all $p+q > n-l$, where $q \neq c$. This provides the vanishing of the limit terms $H_{\mathfrak{m}}^i(C_R(I)) = 0$ for all $i > n - l$, as required.

(ii) \implies (i): Because $l \geq 1$ the first statement in (ii) provides that $H_{\mathfrak{p}R_{\mathfrak{p}}}^{h(\mathfrak{p})}(H_{IR_{\mathfrak{p}}}^c(R_{\mathfrak{p}}))$ does not vanish. By virtue of Lemma 2.2 (b) it follows that $\dim R_{\mathfrak{p}}/IR_{\mathfrak{p}} \geq h(\mathfrak{p})$ for all $\mathfrak{p} \in V(I)$, whence $c = \text{height } IR_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$. As a consequence (cf. Lemma 2.2 (c)) we see that $C_R(I) \otimes_R R_{\mathfrak{p}} \simeq C_{R_{\mathfrak{p}}}(IR_{\mathfrak{p}})$ for all $\mathfrak{p} \in V(I)$.

Now we proceed by induction on $d = \dim R/I$. In the case of $d = 0$ the ideal I is \mathfrak{m} -primary. Therefore the statement is true because R is a Gorenstein ring. So let $d > 0$. First we show that the inductive hypothesis implies

$$\dim H_I^i(R) \leq \max\{n - l - i, 0\} \text{ for all } c < i \leq n.$$

To this end assume that $\dim H_I^i(R) > 0$ for a certain $i \geq n - l$. Choose a prime ideal $\mathfrak{p} \in \text{Supp } H_I^i(R) \setminus \{\mathfrak{m}\}$. Therefore $H_{\mathfrak{p}R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}) \neq 0$ and $i+l \leq \dim R_{\mathfrak{p}}$ by the induction hypothesis. On the other hand, $l+i \leq \dim R_{\mathfrak{p}} < n \leq l+i$, a contradiction. Second, suppose that $\dim H_I^i(R) > n - l - i$ for a certain $c < i < n - l$. Choose a prime ideal $\mathfrak{p} \in \text{Supp } H_I^i(R)$ such that $\dim R/\mathfrak{p} = \dim H_I^i(R)$. Therefore $\dim R/\mathfrak{p} > n - l - i$ and $l+i > \dim R_{\mathfrak{p}}$. Moreover $H_{\mathfrak{p}R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}) \neq 0$ and $i+l \leq \dim R_{\mathfrak{p}}$, which is again a contradiction.

With this information in mind, the spectral sequence (as given in Proposition 2.3) degenerates to isomorphisms $H_m^i(C_R(I)) \simeq H^i(C_R(I))$ for all $i > n - l$. Finally the assumption in (ii) for $\mathfrak{p} = \mathfrak{m}$ implies that $H_m^i(C_R(I)) = 0$ for all $i > n - l$ (by Lemma 2.2). This finishes the proof because $H^i(C_R(I)) \simeq H_I^i(R)$ for $i > c$. \square

For $l \geq \dim R/I$ the previous result yields, as a particular case, the equivalence of the conditions (i) and (ii) of [4, Theorem 3.1]. Another corollary is the following:

Corollary 3.2. *Suppose that $c \geq 2$. With the above notation suppose that*

$$\widehat{R}_{\mathfrak{p}} \rightarrow \text{Hom}_{\widehat{R}_{\mathfrak{p}}}(H_{I\widehat{R}_{\mathfrak{p}}}^c(\widehat{R}_{\mathfrak{p}}), H_{I\widehat{R}_{\mathfrak{p}}}^c(\widehat{R}_{\mathfrak{p}}))$$

is an isomorphism for all $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$ (e.g. this is satisfied in the case where I is locally a cohomologically complete intersection). Then the following conditions are equivalent:

- (i) $H_I^i(R) = 0$ for $i = n - 1, n$.
- (ii) *The natural homomorphism $\widehat{R} \rightarrow \text{Hom}_{\widehat{R}}(H_{I\widehat{R}}^c(\widehat{R}), H_{I\widehat{R}}^c(\widehat{R}))$ is an isomorphism.*

Proof. By the Local Duality Theorem the assumption is equivalent to the isomorphisms

$$H_{\mathfrak{p}R_{\mathfrak{p}}}^{h(\mathfrak{p})}(H_{I R_{\mathfrak{p}}}^c(R_{\mathfrak{p}})) \rightarrow E(k(\mathfrak{p}))$$

for all $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$. By a localization argument and Theorem 3.1 this is equivalent to $\dim H_I^i(R) \leq \max\{n - 2 - i, 0\}$ for all $i > c$. Therefore, by Theorem 3.1 the statement in (ii) holds if and only if $H_I^i(R) = 0$ for $i = n - 1, n$. \square

Note that Corollary 3.2 proves Theorem 1.1 of the Introduction. Another corollary of Theorem 3.1 is the following vanishing result.

Corollary 3.3. *Fix the notation as above. Suppose that I is locally a cohomologically complete intersection. For an integer $l \geq 1$ the following conditions are equivalent:*

- (i) $\text{cd } I \leq \max\{n - l, c\}$, i.e. $H_I^i(R) = 0$ for all $i > \max\{n - l, c\}$.
- (ii) *The natural homomorphism $H_m^d(H_I^c(R)) \rightarrow E$ is bijective (resp. surjective if $l = 1$) and $H_m^i(H_I^c(R)) = 0$ for $d - l + 1 < i < d$.*

Proof. Note that the ideal I is locally a cohomologically complete intersection if and only if $\dim H_I^i(R) \leq 0$ for all $i > c$. This follows from localization and the fact that $H_{I R_{\mathfrak{p}}}^i(R_{\mathfrak{p}}) = 0$ for all $i \neq c$ and all $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$. Therefore, as a consequence of Theorem 3.1, the statement is true. \square

4. PROBLEMS AND EXAMPLES

The first example shows that the assumptions in Corollary 3.2 are not necessary for the equivalence of the two statements. Moreover, it shows that the isomorphism $R \simeq \text{Hom}_R(H_I^c(R), H_I^c(R))$ does not localize.

Example 4.1 (cf. [4, Example 4.1]). Let k be an arbitrary field. Let $R = k[[x_0, \dots, x_4]]$ denote the formal power series ring in five variables over k . Let

$$I = (x_0, x_1) \cap (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4).$$

Then $c = \text{height } I = 2$ and $H_I^i(R) = 0$ for all $i \neq 2, 3$, by use of the Mayer-Vietoris sequence for local cohomology. Moreover (see [4, Example 4.1]) it can be shown that $H_I^3(R) = E_R(R/\mathfrak{p})$, $\mathfrak{p} = (x_0, x_1, x_3, x_4)$. The spectral sequence

$$E_2^{p,q} = H_m^p(H_I^q(R)) \implies E_\infty^{p+q} = H_m^{p+q}(R)$$

provides an isomorphism $H_m^3(H_I^2(R)) \simeq E$. Recall that $H_m^p(H_I^3(R)) = 0$ for all $p \in \mathbb{N}$. By Local Duality it follows that the natural homomorphism $R \rightarrow \text{Hom}_R(H_I^2(R), H_I^2(R))$ is an isomorphism. On the other hand, it is easily seen that this is not true for $\widehat{R}_{\mathfrak{p}}$ because

$$H_{I R_{\mathfrak{p}}}^2(R_{\mathfrak{p}}) \simeq H_{I_1 R_{\mathfrak{p}}}^2(R_{\mathfrak{p}}) \oplus H_{I_2 R_{\mathfrak{p}}}^2(R_{\mathfrak{p}}), I_1 = (x_0, x_1), I_2 = (x_3, x_4)$$

decomposes into two non-zero direct summands. This is seen by using the Mayer-Vietoris sequence for local cohomology.

The following example shows that the endomorphism ring $\text{Hom}_R(H_I^c(R), H_I^c(R))$, $c = \text{height } I$, is in general not a finitely generated R -module.

Example 4.2 (cf. [3, §3]). Let k denote a field and let $R = k[[x, y, u, v]]/(xu - yv)$, where $k[[x, y, u, v]]$ denotes the power series ring in four variables over k . Let $I = (u, v)R$. Then $\dim R = 3, \dim R/I = 2$ and $c = 1$. It follows that $H_I^i(R) = 0$ for $i \neq 1, 2$. Moreover $\text{Supp } H_I^2(R) \subset \{\mathfrak{m}\}$. The truncation complex with the short exact sequence

$$0 \rightarrow H_I^c(R)[-c] \rightarrow \Gamma_I(E) \rightarrow C_R(I) \rightarrow 0$$

(of Definition 2.1) induces a short exact sequence on local cohomology

$$0 \rightarrow H_I^2(R) \rightarrow H_m^2(H_I^1(R)) \rightarrow E \rightarrow 0$$

(see Lemma 2.2). Hartshorne [3, §3] has shown that the socle of $H_I^2(R)$ is not a finite dimensional k -vector space. Therefore, the socle of $H_m^2(H_I^1(R))$ is infinite. Moreover there are the following isomorphisms

$$\text{Hom}_R(H_I^1(R), H_I^1(R)) \simeq \text{Ext}_R^1(H_I^1(R), R) \simeq \text{Hom}_R(H_m^2(H_I^1(R)), E)$$

(from Lemma 2.2 (d) and [4, Lemma 1.2]). By the Nakayama Lemma this means that $\text{Hom}_R(H_I^1(R), H_I^1(R))$ is not a finitely generated R -module.

So, one might ask for a characterization of the finiteness of the endomorphism ring of $H_I^c(R)$, $c = \text{height } I$.

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