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ON ENDOMORPHISM RINGS AND DIMENSIONS OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let $(R, \mathfrak{m})$ denote an n-dimensional complete local Gorenstein ring. For an ideal I of R let $H^i_I(R), i \in \mathbb{Z}$, denote the local cohomology modules of R with respect to I. If $H^i_I(R) = 0$ for all $i \neq c = \operatorname{height} I$, then the endomorphism ring of $H^c_I(R)$ is isomorphic to R. Here we prove that this is true if and only if $H^i_I(R) = 0$ for $i = n, n-1$, provided $c \geq 2$ and R/I has an isolated singularity, resp. if I is set-theoretically a complete intersection in codimension at most one. Moreover, there is a vanishing result of $H^i_I(R)$ for all $i > m, m$ a given integer, and an estimate of the dimension of $H^i_I(R)$.

1. MAIN RESULTS

Let $(R, \mathfrak{m})$ denote a local Noetherian ring with $n = \dim R$. For the ideal $I \subset R$ let $H^i_I(\cdot), i \in \mathbb{Z}$, denote the local cohomology functor with respect to I; see [2] for its definition and basic results. It is a difficult question to describe $\sup\{i \in \mathbb{Z} | H^i_I(R) \neq 0\}$, the cohomological dimension $\operatorname{cd} I$ of I with respect to R. Recall that $\operatorname{height} I \leq \operatorname{cd} I$. Recently some interesting results for ideals with $c = \operatorname{height} I = \operatorname{cd} I$, the so-called cohomologically complete intersections have been proved. If $(R, \mathfrak{m})$ is a complete local ring, Hellus and Stückrad [5] have shown that the endomorphism ring $\operatorname{Hom}_R(H^c_I(R), H^c_I(R))$ is isomorphic to R. See also [4, Lemma 2.8] for a more functorial proof and a slight extension in the case where $(R, \mathfrak{m})$ is a Gorenstein ring.

The first aim of the consideration here is a characterization of when the endomorphism ring of $H^c_I(R)$ is isomorphic to R. To this end we call I locally a cohomologically complete intersection provided $\operatorname{cd} IR_{\mathfrak{p}} = \operatorname{height} I$ for all prime ideals $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$. For instance, if I has an isolated singularity, it is locally a cohomologically complete intersection.

Theorem 1.1. *Let $(R, \mathfrak{m})$ denote a complete local Gorenstein ring with $n = \dim R$. Let I be an ideal of* $\operatorname{height} I = c$. *Suppose that I is locally a cohomologically complete intersection. Then the following conditions are equivalent:*

(i) *The natural homomorphism $R \to \operatorname{Hom}_R(H^c_I(R), H^c_I(R))$ is an isomorphism.*
(ii) $H^i_I(R) = 0$ *for $i = n - 1, n$.*

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In a certain sense, condition (ii) of Theorem 1.1 provides a numerical condition for the property that the endomorphism ring of $H^c_I(R)$ is R. In the case of R being a regular local ring containing a field, Huneke and Lyubeznik [6, Theorem 2.9] have given a topological characterization of the above condition (ii).

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a local Gorenstein ring. Let $J \subset I$ denote two ideals of height c.*

(a) *There is a natural homomorphism*
$$\operatorname{Hom}_R(H^c_J(R), H^c_J(R)) \to \operatorname{Hom}_R(H^c_I(R), H^c_I(R)).$$

(b) *Suppose that* $\operatorname{Rad} JR_\mathfrak{p} = \operatorname{Rad} IR_\mathfrak{p}$ *for all* $\mathfrak{p} \in V(I)$ *with* $\dim R_\mathfrak{p} \le c+1$. *Then the homomorphism in (a) is an isomorphism.*

(c) *Let R be in addition complete. Let J denote a cohomologically complete intersection contained in I and satisfying the assumptions of (b). Then* $R \to \operatorname{Hom}_R(H^c_I(R), H^c_I(R))$ *is an isomorphism.*

Our results are based on a certain estimate of $\dim H^i_I(R), i > c$; see Theorem 3.1. In the case of a regular local ring, partial results of this type have been used by Ken-Ichiroh Kawasaki [7] for the study of Lyubeznik numbers (see [8] for their definition). Here we use the truncation complex as invented in [4, Section 2] (see Definition 2.1). Moreover it provides some technical statements about the endomorphism ring of $H^c_I(R), c = \operatorname{height} I$ (see Lemma 2.2).

In terminology the author follows the paper [4].

2. The truncation complex

Let $(R, \mathfrak{m}, k)$ denote a local Gorenstein ring with $n = \dim R$. First of all we will recall the truncation complex as it was introduced in [4, Section 2] and in a different context in [9, §4]. Let $R \xrightarrow{\sim} E^\cdot$ denote a minimal injective resolution of R as an R-module. It is a well-known fact that
$$E^i \simeq \bigoplus_{\mathfrak{p} \in \operatorname{Spec} R, \operatorname{height} \mathfrak{p} = i} E_R(R/\mathfrak{p}),$$
where $E_R(R/\mathfrak{p})$ denotes the injective hull of $R/\mathfrak{p}$ (see [1] for these and related results about Gorenstein rings).

Now let $I \subset R$ denote an ideal and let $c = \operatorname{height} I$. Then $d = \dim R/I = n - c$. The local cohomology modules $H^i_I(R), i \in \mathbb{Z}$, are, by definition, the cohomology modules of the complex $\Gamma_I(E^\cdot)$. Because $\Gamma_I(E_R(R/\mathfrak{p})) = 0$ for all $\mathfrak{p} \notin V(I)$, it follows that $\Gamma_I(E^\cdot)^i = 0$ for all $i < c$. Therefore $H^c_I(R) = \operatorname{Ker}(\Gamma_I(E^\cdot)^c \to \Gamma_I(E^\cdot)^{c+1})$. This observation provides an embedding $H^c_I(R)[-c] \to \Gamma_I(E^\cdot)$ of complexes of R-modules.

Definition 2.1. The cokernel of the embedding $H^c_I(R)[-c] \to \Gamma_I(E^\cdot)$ is defined as $C^\cdot_R(I)$, the truncation complex with respect to I. So there is a short exact sequence of complexes of R-modules
$$0 \to H^c_I(R)[-c] \to \Gamma_I(E^\cdot) \to C^\cdot_R(I) \to 0.$$
In particular it follows that $H^i(C^\cdot_R(I)) = 0$ for $i \le c$ or $i > n$ and $H^i(C^\cdot_R(I)) \simeq H^i_I(R)$ for $c < i \le n$.

First we need to establish some basic results about the truncation complex. For more details we refer to the exposition in [4, Section 2].

Lemma 2.2. *With the previous notation, the following results hold:*

(a) *There exist an exact sequence*

$$0 \to H^{n-1}_{\mathfrak{m}}(C^{\cdot}_R(I)) \to H^d_{\mathfrak{m}}(H^c_I(R)) \to E \to H^n_{\mathfrak{m}}(C^{\cdot}_R(I)) \to 0$$

and isomorphisms $H^{i-c}_{\mathfrak{m}}(H^c_I(R)) \simeq H^{i-1}_{\mathfrak{m}}(C^{\cdot}_R(I))$ *for* $i < n$.

(b) $H^d_{\mathfrak{m}}(H^c_I(R)) \neq 0$ *and* $H^{i-c}_{\mathfrak{m}}(H^c_I(R)) = 0$ *for* $i > n$.

(c) *Let* $\mathfrak{p} \in V(I)$ *denote a prime ideal. Then there is an isomorphism*

$$C^{\cdot}_R(I) \otimes_R R_{\mathfrak{p}} \simeq C^{\cdot}_{R_{\mathfrak{p}}}(IR_{\mathfrak{p}})$$

provided $\operatorname{height} I = \operatorname{height} IR_{\mathfrak{p}}$.

(d) *There is a natural isomorphism* $\operatorname{Hom}_R(H^c_I(R), H^c_I(R)) \simeq \operatorname{Ext}^c_R(H^c_I(R), R)$.

Proof. For the proof of (a) apply the derived functor $\mathrm{R}\Gamma_{\mathfrak{m}}(\cdot)$ to the short exact sequence given in Definition 2.1. Then $\mathrm{R}\Gamma_{\mathfrak{m}}(\Gamma_I(E^{\cdot})) \simeq E[-n]$. So the long exact cohomology sequence of the corresponding exact sequence of complexes provides what is claimed (see [4, Lemma 2.2] for the details). Statement (b) is shown in [4, Corollary 2.9] and [4, Lemma 1.2].

For the proof of (c) localize the exact sequence in Definition 2.1 at $\mathfrak{p}$. Then there is a short exact sequence of complexes

$$0 \to H^c_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})[-c] \to \Gamma_{IR_{\mathfrak{p}}}(E^{\cdot}_{R_{\mathfrak{p}}}) \to C^{\cdot}_R(I) \otimes_R R_{\mathfrak{p}} \to 0.$$

To this end recall first that $c = \operatorname{height} IR_{\mathfrak{p}} = \operatorname{height} I$ and that the local cohomology commutes with localization. Furthermore $E^{\cdot} \otimes R_{\mathfrak{p}}$ is isomorphic to the minimal injective resolution $E^{\cdot}_{R_{\mathfrak{p}}}$ of $R_{\mathfrak{p}}$. Then the definition of the truncation complex proves the claim.

Finally, we prove (d). As shown at the beginning of this section, there is an exact sequence $0 \to H^c_I(R) \to \Gamma_I(E^{\cdot})^c \to \Gamma_I(E^{\cdot})^{c+1}$. This induces a natural commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(H^c_I(R), H^c_I(R)) \arrow[r] \arrow[d] & \operatorname{Hom}_R(H^c_I(R), \Gamma_I(E^{\cdot}))^c \arrow[r] \arrow[d] & \operatorname{Hom}_R(H^c_I(R, \Gamma_I(E^{\cdot}))^{c+1} \arrow[d] \\
0 \arrow[r] & \operatorname{Ext}^c_R(H^c_I(R), R) \arrow[r] & \operatorname{Hom}_R(H^c_I(R), E^{\cdot})^c \arrow[r] & \operatorname{Hom}_R(H^c_I(R), E^{\cdot})^{c+1}
\end{tikzcd}$$

because $\Gamma_I(E^{\cdot})$ is a subcomplex of $E^{\cdot}$. The last two vertical homomorphisms are isomorphisms because $\operatorname{Hom}_R(X, E_R(R/\mathfrak{p})) = 0$ for an R-module X with $\operatorname{Supp}_R X \subset V(I)$ and $\mathfrak{p} \notin V(I)$. Therefore the first vertical map is also an isomorphism. $\square$

In order to compute the local cohomology of the truncation complex $C^{\cdot}_R(I)$, there is the following spectral sequence for computation of the hyper cohomology of a complex.

Proposition 2.3. *With the notation of Definition* 2.1, *there is the spectral sequence*

$$E^{p,q}_2 = H^p_{\mathfrak{m}}(H^q(C^{\cdot}_R(I))) \Longrightarrow E^{p+q}_{\infty} = H^{p+q}_{\mathfrak{m}}(C^{\cdot}_R(I)),$$

where $H^q(C^{\cdot}_R(I)) = 0$ *for* $i \leq c$ *and* $i > n$, *and* $H^q(C^{\cdot}_R(I)) \simeq H^q_I(R)$ *for* $c < i \leq n$.

Proof. The spectral sequence is a particular case of the spectral sequence of hyper cohomology (cf. [10]). For the initial terms check the definition of the truncation complex. $\square$

In the following we shall use the notion of the dimension $\dim X$ for R-modules X which are not necessarily finitely generated. This is defined by $\dim X = \dim \operatorname{Supp}_R X$, where the dimension of the support is understood in the Zariski topology of $\operatorname{Spec} R$. In particular, $\dim X < 0$ means $X = 0$.

Lemma 2.4. *With the notation above, we have the following results:*

(a) $\dim H^i_I(R) \leq n - i$ *for all* $i \geq c = \operatorname{height} I$.
(b) $\dim H^c_I(R) = \dim R/I$.
(c) *If* $\dim H^i_I(R) < n-i$ *for all* $i > c$*, then* R/I *is unmixed, i.e.* $c = \operatorname{height} IR_{\mathfrak{p}}$ *for all minimal* $\mathfrak{p} \in V(I)$.

Proof. (a): This result is well-known (see for instance [7]).

(b): Let $\mathfrak{p} \in V(I)$ denote a minimal prime ideal in $V(I)$ such that $\dim R_{\mathfrak{p}} = c$. Then $H^c_I(R) \otimes_R R_{\mathfrak{p}} \simeq H^c_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0$ by the Grothendieck non-vanishing result. So, $\mathfrak{p} \in \operatorname{Supp} H^c_I(R)$ and $\dim R/\mathfrak{p} = d$. Together with (a) this proves the claim.

(c): Let $\mathfrak{p} \in V(I)$ be minimal with $h := \operatorname{height} IR_{\mathfrak{p}} > c$. Then $h = \dim R_{\mathfrak{p}}$ and

$$0 \neq H^h_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \simeq H^h_I(R) \otimes_R R_{\mathfrak{p}}.$$

This implies that $\mathfrak{p} \in \operatorname{Supp} H^h_I(R)$ with $\dim R/\mathfrak{p} + h = n$, a contradiction. □

Proof of Theorem 1.2. Let $\alpha \geq 1$ denote an integer. The inclusion $J \subset I$ induces a short exact sequence $0 \to I^\alpha/J^\alpha \to R/J^\alpha \to R/I^\alpha \to 0$. By applying the long exact cohomology sequence with respect to $\operatorname{Ext}^{\cdot}_R(\cdot, R)$ and passing to the direct limit, we get the following exact sequence:

$$0 \to H^c_I(R) \to H^c_J(R) \xrightarrow{\phi} \varinjlim \operatorname{Ext}^c_R(I^\alpha/J^\alpha, R).$$

Recall that $\operatorname{grade} I^\alpha/J^\alpha \geq c$ for all α. Let $X = \operatorname{Im} \phi$. The short exact sequence $0 \to H^c_I(R) \to H^c_J(R) \to X \to 0$ provides (after applying $\operatorname{Ext}^{\cdot}_R(\cdot, R)$) a natural homomorphism

$$\operatorname{Ext}^c_R(H^c_J(R), R) \to \operatorname{Ext}^c_R(H^c_I(R), R).$$

By Lemma 2.2 (d) this proves the statement in (a).

In order to prove (b) we may assume that $JR_{\mathfrak{p}} = IR_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$ with $\dim R_{\mathfrak{p}} \leq c+1$. This is possible because local cohomology does not change by passing to the radical. Next we claim that $\dim X \leq d-2$. This follows because $\dim_R I^\alpha/J^\alpha \leq d-2$ for all $\alpha \in \mathbb{N}$ under the additional assumption of $J \subset I$. Moreover, $\dim X \leq d-2$ is true by a localization argument and the embedding $X \to \varinjlim \operatorname{Ext}^c_R(I^\alpha/J^\alpha, R)$.

By passing to the completion and because of the Matlis duality (see [4, Lemma 1.2]), it will be enough to show that the natural homomorphism $H^d_{\mathfrak{m}}(H^c_I(R)) \to H^d_{\mathfrak{m}}(H^c_J(R))$ is an isomorphism. Now this is true by virtue of the local cohomology with respect to the maximal ideal applied to the short exact sequence $0 \to H^c_I(R) \to H^c_J(R) \to X \to 0$ and the fact that $\dim X \leq d-2$.

For the proof of (c) recall that for a cohomologically complete intersection J it is known that the endomorphism ring of $H^c_J(R)$ is isomorphic to R (see [5] or [4, Lemma 3.3]). □

3. Dimensions of local cohomology

As before let $(R, \mathfrak{m})$ denote a n-dimensional Gorenstein ring. Let $I \subset R$ be an ideal with $c = \operatorname{height} I$ and $\dim R/I = n - c$. We prove the following theorem in order to estimate the dimension of local cohomology modules. To this end let us fix the abbreviation $h(\mathfrak{p}) = \dim R_{\mathfrak{p}} - c$ for a prime ideal $\mathfrak{p} \in V(I)$.

Theorem 3.1. *Let $l \geq 1$ denote an integer. With the previous notation the following conditions are equivalent:*

(i) $\dim H^i_I(R) \leq n - l - i$ *for all* $i > c$.

(ii) *For all* $\mathfrak{p} \in V(I)$ *the natural map*

$$H^{h(\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(H^c_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})) \to E(k(\mathfrak{p}))$$

is bijective (resp. surjective if $l = 1$) *and*

$$H^i_{\mathfrak{p}R_{\mathfrak{p}}}(H^c_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})) = 0$$

for all $h(\mathfrak{p}) - l + 1 < i < h(\mathfrak{p})$.

Proof. (i) $\Longrightarrow$ (ii): From Lemma 2.4 it follows that R/I is unmixed; i.e. $c = \operatorname{height} I = \operatorname{height} IR_{\mathfrak{p}}$ for all minimal prime ideals $\mathfrak{p} \in V(I)$. In particular this implies that $h(\mathfrak{p}) = \dim R_{\mathfrak{p}}/IR_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \in V(I)$. Moreover

$$\dim R/\mathfrak{p} + \dim H^i_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \leq \dim H^i_I(R)$$

because the localization commutes with local cohomology. So our assumption (i) implies that $\dim H^i_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \leq \dim R_{\mathfrak{p}} - l - i$ for all $i > \operatorname{height} IR_{\mathfrak{p}} = c$. Therefore it will be enough to prove the statement in (ii) for $\mathfrak{p} = \mathfrak{m}$, the maximal ideal of $(R, \mathfrak{m})$.

From Lemma 2.2 (a) it will be enough to show the vanishing of $H^i_{\mathfrak{m}}(C^{\cdot}_R(I)$ for all $i > n - l$. To this end consider the spectral sequence of Proposition 2.3. By our assumption we have for the initial terms $E_2^{p,q} = H^p_{\mathfrak{m}}(H^q_I(R)) = 0$ for all $p + q > n - l$, where $q \neq c$. This provides the vanishing of the limit terms $H^i_{\mathfrak{m}}(C^{\cdot}_R(I)) = 0$ for all $i > n - l$, as required.

(ii) $\Longrightarrow$ (i): Because $l \geq 1$ the first statement in (ii) provides that $H^{h(\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(H^c_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}))$ does not vanish. By virtue of Lemma 2.2 (b) it follows that $\dim R_{\mathfrak{p}}/IR_{\mathfrak{p}} \geq h(\mathfrak{p})$ for all $\mathfrak{p} \in V(I)$, whence $c = \operatorname{height} IR_{\mathfrak{p}}$ for all $\mathfrak{p} \in V(I)$. As a consequence (cf. Lemma 2.2 (c)) we see that $C^{\cdot}_R(I) \otimes_R R_{\mathfrak{p}} \simeq C^{\cdot}_{R_{\mathfrak{p}}}(IR_{\mathfrak{p}})$ for all $\mathfrak{p} \in V(I)$.

Now we proceed by induction on $d = \dim R/I$. In the case of $d = 0$ the ideal I is $\mathfrak{m}$-primary. Therefore the statement is true because R is a Gorenstein ring. So let $d > 0$. First we show that the inductive hypothesis implies

$$\dim H^i_I(R) \leq \max\{n - l - i, 0\} \text{ for all } c < i \leq n.$$

To this end assume that $\dim H^i_I(R) > 0$ for a certain $i \geq n - l$. Choose a prime ideal $\mathfrak{p} \in \operatorname{Supp} H^i_I(R) \setminus \{\mathfrak{m}\}$. Therefore $H^i_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0$ and $i + l \leq \dim R_{\mathfrak{p}}$ by the induction hypothesis. On the other hand, $l + i \leq \dim R_{\mathfrak{p}} < n \leq l + i$, a contradiction. Second, suppose that $\dim H^i_I(R) > n - l - i$ for a certain $c < i < n - l$. Choose a prime ideal $\mathfrak{p} \in \operatorname{Supp} H^i_I(R)$ such that $\dim R/\mathfrak{p} = \dim H^i_I(R)$. Therefore $\dim R/\mathfrak{p} > n - l - i$ and $l + i > \dim R_{\mathfrak{p}}$. Moreover $H^i_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \neq 0$ and $i + l \leq \dim R_{\mathfrak{p}}$, which is again a contradiction.

With this information in mind, the spectral sequence (as given in Proposition 2.3) degenerates to isomorphisms $H^i_{\mathfrak{m}}(C^{\cdot}_R(I)) \simeq H^i(C^{\cdot}_R(I))$ for all $i > n-l$. Finally the assumption in (ii) for $\mathfrak{p} = \mathfrak{m}$ implies that $H^i_{\mathfrak{m}}(C^{\cdot}_R(I)) = 0$ for all $i > n-l$ (by Lemma 2.2). This finishes the proof because $H^i(C^{\cdot}_R(I)) \simeq H^i_I(R)$ for $i > c$. □

For $l \geq \dim R/I$ the previous result yields, as a particular case, the equivalence of the conditions (i) and (ii) of [4, Theorem 3.1]. Another corollary is the following:

Corollary 3.2. *Suppose that $c \geq 2$. With the above notation suppose that*

$$\widehat{R_{\mathfrak{p}}} \to \operatorname{Hom}_{\widehat{R_{\mathfrak{p}}}}(H^c_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}), H^c_{I\widehat{R_{\mathfrak{p}}}}(\widehat{R_{\mathfrak{p}}}))$$

is an isomorphism for all $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$ (e.g. this is satisfied in the case where I is locally a cohomologically complete intersection). Then the following conditions are equivalent:

(i) $H^i_I(R) = 0$ *for* $i = n-1, n$.
(ii) *The natural homomorphism* $\hat{R} \to \operatorname{Hom}_{\hat{R}}(H^c_{I\hat{R}}(\hat{R}), H^c_{I\hat{R}}(\hat{R}))$ *is an isomorphism.*

Proof. By the Local Duality Theorem the assumption is equivalent to the isomorphisms

$$H^{h(\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(H^c_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}})) \to E(k(\mathfrak{p}))$$

for all $\mathfrak{p} \in V(I)\setminus\{\mathfrak{m}\}$. By a localization argument and Theorem 3.1 this is equivalent to $\dim H^i_I(R) \leq \max\{n-2-i, 0\}$ for all $i > c$. Therefore, by Theorem 3.1 the statement in (ii) holds if and only if $H^i_I(R) = 0$ for $i = n-1, n$. □

Note that Corollary 3.2 proves Theorem 1.1 of the Introduction. Another corollary of Theorem 3.1 is the following vanishing result.

Corollary 3.3. *Fix the notation as above. Suppose that I is locally a cohomologically complete intersection. For an integer $l \geq 1$ the following conditions are equivalent:*

(i) $\operatorname{cd} I \leq \max\{n-l, c\}$, *i.e.* $H^i_I(R) = 0$ *for all* $i > \max\{n-l, c\}$.
(ii) *The natural homomorphism* $H^d_{\mathfrak{m}}(H^c_I(R)) \to E$ *is bijective (resp. surjective if $l = 1$) and* $H^i_{\mathfrak{m}}(H^c_I(R)) = 0$ *for* $d-l+1 < i < d$.

Proof. Note that the ideal I is locally a cohomologically complete intersection if and only if $\dim H^i_I(R) \leq 0$ for all $i > c$. This follows from localization and the fact that $H^i_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) = 0$ for all $i \neq c$ and all $\mathfrak{p} \in V(I) \setminus \{\mathfrak{m}\}$. Therefore, as a consequence of Theorem 3.1, the statement is true. □

4. Problems and examples

The first example shows that the assumptions in Corollary 3.2 are not necessary for the equivalence of the two statements. Moreover, it shows that the isomorphism $R \simeq \operatorname{Hom}_R(H^c_I(R), H^c_I(R))$ does not localize.

Example 4.1 (cf. [4, Example 4.1])**.** Let k be an arbitrary field. Let $R = k[|x_0, \ldots, x_4|]$ denote the formal power series ring in five variables over k. Let

$$I = (x_0, x_1) \cap (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4).$$

Then $c = \operatorname{height} I = 2$ and $H_I^i(R) = 0$ for all $i \neq 2, 3$, by use of the Mayer-Vietoris sequence for local cohomology. Moreover (see [4, Example 4.1]) it can be shown that $H_I^3(R) = E_R(R/\mathfrak{p}), \mathfrak{p} = (x_0, x_1, x_3, x_4)$. The spectral sequence

$$E_2^{p,q} = H_{\mathfrak{m}}^p(H_I^q(R)) \Longrightarrow E_\infty^{p+q} = H_{\mathfrak{m}}^{p+q}(R)$$

provides an isomorphism $H_{\mathfrak{m}}^3(H_I^2(R)) \simeq E$. Recall that $H_{\mathfrak{m}}^p(H_I^3(R)) = 0$ for all $p \in \mathbb{N}$. By Local Duality it follows that the natural homomorphism $R \to \operatorname{Hom}_R(H_I^2(R), H_I^2(R))$ is an isomorphism. On the other hand, it is easily seen that this is not true for $\widehat{R_{\mathfrak{p}}}$ because

$$H^2_{IR_{\mathfrak{p}}}(R_{\mathfrak{p}}) \simeq H^2_{I_1R_{\mathfrak{p}}}(R_{\mathfrak{p}}) \oplus H^2_{I_2R_{\mathfrak{p}}}(R_{\mathfrak{p}}), I_1 = (x_0, x_1), I_2 = (x_3, x_4)$$

decomposes into two non-zero direct summands. This is seen by using the Mayer-Vietoris sequence for local cohomology.

The following example shows that the endomorphism ring $\operatorname{Hom}_R(H_I^c(R), H_I^c(R))$, $c = \operatorname{height} I$, is in general not a finitely generated R-module.

Example 4.2 (cf. [3, §3])**.** Let k denote a field and let $R = k[|x, y, u, v|]/(xu - yv)$, where $k[|x, y, u, v|]$ denotes the power series ring in four variables over k. Let $I = (u, v)R$. Then $\dim R = 3, \dim R/I = 2$ and $c = 1$. It follows that $H_I^i(R) = 0$ for $i \neq 1, 2$. Moreover $\operatorname{Supp} H_I^2(R) \subset \{\mathfrak{m}\}$. The truncation complex with the short exact sequence

$$0 \to H_I^c(R)[-c] \to \Gamma_I(E^{\cdot}) \to C_R^{\cdot}(I) \to 0$$

(of Definition 2.1) induces a short exact sequence on local cohomology

$$0 \to H_I^2(R) \to H_{\mathfrak{m}}^2(H_I^1(R)) \to E \to 0$$

(see Lemma 2.2). Hartshorne [3, §3] has shown that the socle of $H_I^2(R)$ is not a finite dimensional k-vector space. Therefore, the socle of $H_{\mathfrak{m}}^2(H_I^1(R))$ is infinite. Moreover there are the following isomorphisms

$$\operatorname{Hom}_R(H_I^1(R), H_I^1(R)) \simeq \operatorname{Ext}_R^1(H_I^1(R), R) \simeq \operatorname{Hom}_R(H_{\mathfrak{m}}^2(H_I^1(R)), E)$$

(from Lemma 2.2 (d) and [4, Lemma 1.2]). By the Nakayama Lemma this means that $\operatorname{Hom}_R(H_I^1(R), H_I^1(R))$ is not a finitely generated R-module.

So, one might ask for a characterization of the finiteness of the endomorphism ring of $H_I^c(R), c = \operatorname{height} I$.

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