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SMOOTHNESS OF RADIAL SOLUTIONS TO MONGE-AMPÈRE EQUATIONS

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Abstract. We prove that generalized convex radial solutions to the generalized Monge-Ampère equation $\det D^2u = f(|x|^2/2, u, |\nabla u|^2/2)$ with f smooth are always smooth away from the origin. Moreover, we characterize the global smoothness of these solutions in terms of the order of vanishing of f at the origin.

1. Introduction

It is well known that the radial homogeneous functions $u = c_{m,n}\,|x|^{2+\frac{2m}{n}}$ provide *nonsmooth* solutions to the Monge-Ampère equation $\det D^2u = |x|^{2m}$ with smooth right-hand side when $m \in \mathbb{N} \setminus n\mathbb{N}$. This raises the question of when radial solutions u to the generalized equation

$$\det D^2 u = k\left(x, u, Du\right), \qquad x \in \mathbb{B}_n, \tag{1.1}$$

are smooth, given that k is smooth and nonnegative. When u is radial, (1.1) reduces to a nonlinear ODE on $[0, 1)$ that is singular at the endpoint 0. It is thus easy to prove that u is always smooth away from the origin, even where k vanishes, but smoothness at the origin is more complicated and determined by the order of vanishing of k there.

In fact, Monn [9] proves that if $k = k\left(x\right)$ is independent of u and Du, then a radial solution u to (1.1) is smooth if $k^{\frac{1}{n}}$ is smooth, and Derridj [4] has extended this criterion to the case when $k\left(x, u, Du\right) = f\left(\frac{|x|^2}{2}, u, \frac{|\nabla u|^2}{2}\right)$ factors as

$$f\left(t, \xi, \zeta\right) = \kappa\left(t\right)\phi\left(t, \xi, \zeta\right) \tag{1.2}$$

with κ smooth and nonnegative on $[0, 1)$, $\kappa\left(0\right) = 0$, and ϕ smooth and positive on $[0, 1) \times \mathbb{R} \times [0, \infty)$. Moreover, Monn also shows that u is smooth if $k = k\left(x\right)$ vanishes to *infinite* order at the origin.

These results leave open the case when k has the general form $k\left(x, u, Du\right)$ and vanishes to infinite order at the origin. The purpose of this paper is to show that radial solutions u are smooth in this remaining case as well. The following theorem encompasses all of the aforementioned results and applies to generalized convex solutions u and also with $f = \kappa\phi$ as in (1.2) but where ϕ is only assumed positive and bounded, not smooth.

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Theorem 1.1. *Suppose that u is a generalized convex radial solution (in the sense of Alexandrov) to the generalized Monge-Ampère equation* (1.1) *with*

$$k(x,u,Du)=f\left(\frac{|x|^2}{2},u,\frac{|\nabla u|^2}{2}\right),$$

where f is smooth and nonnegative on $[0,1)\times\mathbb{R}\times[0,\infty)$. Then u is smooth in the deleted ball $\mathbb{B}_n\setminus\{0\}$.

Suppose moreover that there are positive constants c,C such that

$$cf(t,0,0)\le f(t,\xi,\zeta)\le Cf(t,0,0) \tag{1.3}$$

for (ξ,ζ) near $(0,0)$. Let $\tau\in\mathbb{Z}_+\cup\{\infty\}$ be the order of vanishing of $f(t,0,0)$ at 0. Then u is smooth at the origin if and only if $\tau\in n\mathbb{Z}_+\cup\{\infty\}$.

The case when $k=k(x)$ is independent of u and Du is handled by Monn in [9] using an explicit formula for u in terms of k:

$$g(t)=C+\left(\frac{n}{2}\right)^{\frac{1}{n}}\int_0^t\frac{\left(\int_0^s w^{\frac{n}{2}}f(w)\frac{dw}{w}\right)^{\frac{1}{n}}}{\sqrt{s}}ds, \tag{1.4}$$

where $u(x)=g\left(\frac{r^2}{2}\right)$ and $k(x)=f\left(\frac{r^2}{2}\right)\ge 0$ with $r=|x|$, $x\in\mathbb{R}^n$. In the case that k vanishes to infinite order at the origin, an inequality of Hadamard is used as well. The following scale-invariant version follows from Corollary 5.2 in [9]:

$$\max_{0\le t\le x}\left|F^{(\ell)}(t)\right|\le C_{k,\ell}F(x)^{\frac{k-\ell}{k}}\max_{0\le t\le x}\left|F^{(k)}(t)\right|^{\frac{\ell}{k}},\qquad 0\le x\le 1, \tag{1.5}$$

for all $1\le\ell\le k-1$ and $k\in\mathbb{N}$ provided F is smooth, nondecreasing on $[0,1)$ and vanishes to infinite order at 0.

2. Proof of Theorem 1.1

We begin by considering Theorem 1.1 in the case that u is a classical C^2 solution to (1.1) and f satisfies (1.2) where $f(t,0,0)$ vanishes to *finite* order ℓ at 0. If k is independent of u and Du, Monn uses formula (1.4) in [9] to show that u is smooth when $f(w)^{\frac{1}{n}}$ is smooth. In particular this applies when $\ell\in n\mathbb{Z}_+$. In the general case, we note that (1.3) implies (1.2), the assumption made in [4]. Indeed, using $f^{(k)}(0,\xi,\zeta)=0$ for $0\le k\le\ell-1$ we can write

$$f(s,\xi,\zeta)=\int_0^1\frac{(1-t)^{\ell-1}}{(\ell-1)!}\frac{d^\ell}{dt^\ell}f(ts,\xi,\zeta)\,dt=s^\ell\psi(s,\xi,\zeta),$$

where $\psi(s,\xi,\zeta)$ is smooth and $\psi(0,\xi,\zeta)=\frac{f^{(\ell)}(0,\xi,\zeta)}{\ell!}>0$. Thus the results of Derridj [4] apply to show that u is smooth for general k when $\ell\in n\mathbb{Z}_+$.

2.1. **Generalized Monge-Ampère equations.** We now consider radial *generalized convex* solutions u to the generalized Monge-Ampère equation (1.1), where we assume $k(\cdot,u,q)$ and $k(x,u,\cdot)$ are radial. We first establish that $u\in C^2(\mathbb{B}_n)\cap C^\infty(\mathbb{B}_n\setminus\{0\})$. We note that results of Guan, Trudinger and Wang in [6] and [8] yield $u\in C^{1,1}(\mathbb{B}_n)$ for many k in (1.1), but not in the generality possible in the radial case here. In order to deal with general k it would be helpful to have a formula for u in terms of k, but this is problematic. Instead we prove Theorem 1.1

for general k *without* solving for the solution explicitly, but using an inductive argument that is based on estimates (1.5) when k vanishes to infinite order at the origin.

Assume that u is a generalized convex solution of (1.1) in the sense of Alexandrov (see [1] and [3]) and define $\varphi(t)$ by

$$\varphi\left(\frac{r^2}{2}\right) = k\left(x, u(x), Du(x)\right) = f\left(\frac{|x|^2}{2}, u(x), \frac{|\nabla u(x)|^2}{2}\right). \tag{2.1}$$

Then φ is bounded since u is Lipschitz continuous. It follows that the *convex radial* function u is continuously differentiable at the origin, since otherwise it would have a conical singularity there and its representing measure μ_u would have a Dirac component at the origin. Let g be given by formula (1.4) with φ in place of f, i.e.

$$g(t) = C_u + \left(\frac{n}{2}\right)^{\frac{1}{n}} \int_0^t \frac{\left(\int_0^s w^{\frac{n}{2}} \varphi(w) \frac{dw}{w}\right)^{\frac{1}{n}}}{\sqrt{s}} ds, \tag{2.2}$$

and with constant C_u chosen so that u and $\widetilde{u}$ agree on the unit sphere where

$$\widetilde{u}(x) = g\left(\frac{r^2}{2}\right), \qquad 0 \le r < 1. \tag{2.3}$$

We claim that $\widetilde{u}$ is a generalized convex solution to (1.1) in the sense of Alexandrov. To see this we first note that

$$D^2\widetilde{u}(r\mathbf{e}_1) = \begin{bmatrix} g'' r^2 + g' & 0 & \cdots & 0 \\ 0 & g' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g' \end{bmatrix}$$

is positive semidefinite; hence $\widetilde{u}$ is convex. To prove that the representing measure $\mu_{\widetilde{u}}$ of $\widetilde{u}$ is kdx it suffices to show, since both g and f are radial, that

$$\mu_{\widetilde{u}}(E) = |B_{\widetilde{u}}(E)| = \int_E k$$

for all annuli $E = \{x \in \mathbb{B}_n : r_1 < |x| < r_2\}$, $0 < r_1 < r_2 < 1$ where

$$B_{\widetilde{u}}(E) = \bigcup_{r_1 < |x| < r_2} \{\nabla \widetilde{u}_1(x)\} = \left\{a \in \mathbb{B}_n : \frac{\partial}{\partial r}\widetilde{u}(r_1\mathbf{e}_1) < |a| < \frac{\partial}{\partial r}\widetilde{u}(r_2\mathbf{e}_1)\right\}.$$

Since $\frac{\partial}{\partial r}\widetilde{u}(r_i\mathbf{e}_i) = g'\left(\frac{r_i^2}{2}\right) r_i = g'(t_i)\sqrt{2t_i}$ with $t_i = \frac{r_i^2}{2}$, we thus have

$$\begin{aligned} |B_{\widetilde{u}}(E)| &= \left|\left\{a \in \mathbb{B}_n : g'(t_1)\sqrt{2t_1} < |a| < g'(t_2)\sqrt{2t_2}\right\}\right| \\ &= \frac{\omega_n}{n}\left\{g'(t_2)^n (2t_2)^{\frac{n}{2}} - g'(t_1)^n (2t_1)^{\frac{n}{2}}\right\} \\ &= \frac{\omega_n}{n}\frac{n}{2} 2^{\frac{n}{2}} \int_{t_1}^{t_2} w^{\frac{n}{2}} \varphi(w) \frac{dw}{w} \\ &= \omega_n \int_{r_1}^{r_2} r^{n-1} \varphi\left(\frac{r^2}{2}\right) dr = \int_E k. \end{aligned}$$

In particular the *convex radial* function $\widetilde{u}$ must be continuously differentiable, since otherwise there is a jump discontinuity in the radial derivative of $\widetilde{u}$ at some distance r from the origin that results in a singular component in $\mu_{\widetilde{u}}$ supported on the sphere of radius r.

Now the uniqueness of Alexandrov solutions to the Dirichlet problem (see e.g. [3]) yields $u=\widetilde{u}$, and hence $u\in C^1(\mathbb{B}_n)$. Thus $\varphi\in C[0,1)$, and from (2.3) we have $u(x)=g\left(\frac{|x|^2}{2}\right)$ and

$$\varphi(t)=f\left(t,g(t),tg'(t)^2\right), \tag{2.4}$$

where using (2.2) we compute that

$$g'(t)=\left\{\frac{n}{2}t^{-\frac{n}{2}}\int_0^t s^{\frac{n}{2}-1}\varphi(s)\,ds\right\}^{\frac{1}{n}}. \tag{2.5}$$

In particular $g'\in C[0,1)$. We now obtain by induction that $g\in C^\infty(0,1)$; hence $u\in C^\infty(\mathbb{B}_n\setminus\{0\})$. Indeed, if $g\in C^\ell(0,1)$, then (2.4) implies $\varphi\in C^{\ell-1}(0,1)$ and then (2.2) implies $g\in C^{\ell+1}(0,1)$.

It will be convenient to use fractional integral operators at this point. For $\beta>0$ and f continuous define

$$\begin{aligned} T_\beta f(s) &= \int_0^s \left(\frac{w}{s}\right)^\beta f(w)\frac{dw}{w}, \qquad s\neq 0,\\ T_\beta f(0) &= \frac{1}{\beta}f(0) \end{aligned}$$

so that

$$g(t)=C+\left(\frac{n}{2}\right)^{\frac{1}{n}}\int_0^t\left(T_{\frac{n}{2}}f(s)\right)^{\frac{1}{n}}ds. \tag{2.6}$$

We claim that for f smooth, nonnegative and of finite type ℓ, $\ell\in\mathbb{Z}_+$, the same is true of $T_\beta f$ for all $\beta>0$. This follows immediately from the identity

$$\frac{d^k}{ds^k}T_\beta f(s)=T_{\beta+k}f^{(k)}(s), \qquad k\in\mathbb{N}, \tag{2.7}$$

and the estimate

$$T_{\beta+k}f^{(k)}(s)=\frac{1}{\beta+k}f^{(k)}(0)+O(|s|).$$

When $k=1$, (2.7) follows from differentiating and then integrating by parts, and the general case is then obtained by iteration.

Now suppose that f satisfies (1.3) and let

$$\kappa(t)=f(t,0,0)$$

vanish to infinite order at 0. If κ vanishes in a neighbourhood of 0, then so does g, and we have $g\in C^\infty[0,1)$ and $u\in C^\infty(\mathbb{B}_n)$. Thus we will assume $\int_0^t\kappa>0$ for $t>0$ in what follows. Note that (2.7) then implies that $T_{\frac{n}{2}}\kappa(t)$ is smooth and positive on $(0,1)$ and vanishes to infinite order at 0. Since $g'\in C[0,1)$, it follows that $\varphi(t)\le C\kappa(t)$. Thus we have the inequality $T_{\frac{n}{2}}\varphi(t)\le CT_{\frac{n}{2}}\kappa(t)$, and from (2.5) we now conclude that $g'(t)$ also vanishes to infinite order at 0. Now $\varphi(t)\approx\kappa(t)$ from (1.3), and so also $T_{\frac{n}{2}}\varphi(t)\approx T_{\frac{n}{2}}\kappa(t)$. From

$$g''(t)=\frac{\varphi(t)}{2t\left(\frac{n}{2}T_{\frac{n}{2}}\varphi(t)\right)^{1-\frac{1}{n}}}-\frac{1}{2t}\left(\frac{n}{2}T_{\frac{n}{2}}\varphi(t)\right)^{\frac{1}{n}}, \tag{2.8}$$

we then have

$$|g''(t)| \leq C\frac{\kappa(t)}{2t\left(\frac{n}{2}T_{\frac{n}{2}}\kappa(t)\right)^{1-\frac{1}{n}}} + C\frac{1}{2t}\left(\frac{n}{2}T_{\frac{n}{2}}\kappa(t)\right)^{\frac{1}{n}}, \qquad 0<t<1. \tag{2.9}$$

An application of (1.5) with $\ell = 1$, $k > n$ and $F(t) = \int_0^t s^{\frac{n}{2}-1}\kappa(s)\,ds$ yields $t^{\frac{n}{2}}\kappa(t) = F'(t) \leq CF(t)^{1-\frac{1}{k}}$, and so the first term on the right side of (2.9) is bounded by a multiple of $t^{-\frac{1}{2}}F(t)^{\frac{1}{n}-\frac{1}{k}}$. Thus the right side of (2.9), and hence also $g''(t)$, vanishes to infinite order at 0. In particular $g'' \in C[0,1)$, and we conclude $u \in C^2(\mathbb{B}_n)$ in this case as well.

Summarizing, we have $u \in C^\infty(\mathbb{B}_n \setminus \{0\})$, and in the case that f satisfies (1.3), we also have $u \in C^2(\mathbb{B}_n)$. Thus from the above we have that

$$\varphi(t) = f\left(t, g(t), tg'(t)^2\right) = \kappa(t)\,\phi\left(t, g(t), tg'(t)^2\right),$$

where $u(x) = g\left(\frac{|x|^2}{2}\right) \in C^2(\mathbb{B}_n)$, g is given by (2.2) and $\varphi \in C^1[0,1)$ by (2.1). Note that we cannot use (1.5) on the function $\int_0^t s^{\frac{n}{2}-1}\varphi(s)\,ds$ here since we have no *a priori* control on higher derivatives of $\varphi(s) = f\left(s, g(s), sg'(s)^2\right)$. Instead we will use (1.5) on the function $\int_0^t s^{\frac{n}{2}-1}\kappa(s)\,ds$ together with an inductive argument to control derivatives of g.

From the above we have that $g'' \in C[0,1) \cap C^\infty(0,1)$. Now differentiate (2.8) for $t > 0$ using (2.7) to obtain

$$\begin{aligned} g'''(t) = {} & \frac{1}{2}\left(\frac{n}{2}\right)^{\frac{1}{n}-1}\left\{\frac{\varphi'(t)}{tT_{\frac{n}{2}}\varphi(t)^{1-\frac{1}{n}}} - \left(\frac{1}{n}-1\right)\frac{\varphi(t)\,T_{\frac{n}{2}+1}\varphi'(t)}{tT_{\frac{n}{2}}\varphi(t)^{2-\frac{1}{n}}}\right\} \\ & -\frac{1}{2}\left(\frac{n}{2}\right)^{\frac{1}{n}-1}\frac{\varphi(t)}{t^2T_{\frac{n}{2}}\varphi(t)^{1-\frac{1}{n}}} - \frac{1}{2}\left(\frac{n}{2}\right)^{\frac{1}{n}}\left\{\frac{1}{n}\frac{T_{\frac{n}{2}+1}\varphi'(t)}{tT_{\frac{n}{2}}\varphi(t)^{1-\frac{1}{n}}} - \frac{T_{\frac{n}{2}}\varphi(t)^{\frac{1}{n}}}{t^2}\right\}, \end{aligned} \tag{2.10}$$

and then compute that

$$\begin{aligned} \varphi'(t) \quad = \quad & \kappa'(t)\,\phi\left(t, g(t), tg'(t)^2\right) \\ & +\kappa(t)\,\phi_1\left(t, g(t), tg'(t)^2\right) \\ & +\kappa(t)\,\phi_2\left(t, g(t), tg'(t)^2\right)g'(t) \\ & +\kappa(t)\,\phi_3\left(t, g(t), tg'(t)^2\right)\left\{g'(t)^2 + 2tg'(t)\,g''(t)\right\}. \end{aligned} \tag{2.11}$$

We will now use $\varphi \approx \kappa$, (2.10), (2.11) and (1.5) applied with $F(t) = \int_0^t s^{\frac{n}{2}-1}\kappa(s)\,ds$, to show that g''' vanishes to infinite order at 0 and $g''' \in C[0,1)$.

To see this, we first note that F is smooth, nonnegative and vanishes to infinite order at 0 since the same is true of κ. Next, for any $\ell \geq 1$ and $\varepsilon > 0$, (1.5) with k large enough yields

$$\sup_{0<s\leq t}\left|F^{(\ell)}(s)\right| \leq C_{\varepsilon,\ell}F(t)^{1-\varepsilon}. \tag{2.12}$$

Moreover we have

$$\begin{aligned}
&|\beta T_\beta h(t)| \le \sup_{0<s\le t} |h(s)|, \\
&F(t) = t^{\frac{n}{2}} T_{\frac{n}{2}} \kappa(t), \\
&T_{\frac{n}{2}} \varphi(t) \approx T_{\frac{n}{2}} \kappa(t).
\end{aligned} \tag{2.13}$$

Now using

$$\begin{aligned}
F'(t) &= t^{\frac{n}{2}-1} \kappa(t), \\
F''(t) &= t^{\frac{n}{2}-1} \kappa'(t) + \left(\frac{n}{2}-1\right) t^{\frac{n}{2}-2} \kappa(t)
\end{aligned}$$

we obtain

$$\left|\kappa'(t)\,\phi\left(t, g(t), tg'(t)^2\right)\right| \le C\,|\kappa'(t)| = C\left|t^{1-\frac{n}{2}} F''(t) - \left(\frac{n}{2}-1\right) t^{-\frac{n}{2}} F'(t)\right|,$$

and an application of (2.12) gives

$$\left|\kappa'(t)\,\phi\left(t, g(t), tg'(t)^2\right)\right| \le C_\varepsilon t^{-\frac{n}{2}} F(t)^{1-\varepsilon}.$$

We obtain similar estimates for the remaining terms in (2.11), and all together this yields

$$|\varphi'(t)| \le C_\varepsilon t^{-\alpha} F(t)^{1-\varepsilon}, \qquad \text{for some } \alpha > 0.$$

Using the second and third lines in (2.13) we now show that the first term in braces in (2.10) satisfies

$$\left|\frac{\varphi'(t)}{tT_{\frac{n}{2}}\varphi(t)^{1-\frac{1}{n}}}\right| \le C_\varepsilon \frac{t^{-\alpha}F(t)^{1-\varepsilon}}{tT_{\frac{n}{2}}\varphi(t)^{1-\frac{1}{n}}} \approx C_\varepsilon t^{\frac{n}{2}(1-\varepsilon)-\alpha-1} T_{\frac{n}{2}}\kappa(t)^{\frac{1}{n}-\varepsilon},$$

which vanishes to infinite order at 0 if $0 < \varepsilon < \frac{1}{n}$. Similar arguments, using (2.11) and the first line in (2.13) to estimate $T_{\frac{n}{2}+1}\varphi'(t)$, apply to the remaining terms in (2.10), and this completes the proof that g''' vanishes to infinite order at 0 and $g''' \in C[0,1)$.

We now observe that we can

- continue to differentiate (2.10) to obtain a formula for $g^{(\ell)}$ involving only appropriate powers of $T_{\frac{n}{2}}\varphi(t) \approx T_{\frac{n}{2}}\kappa(t)$ in the denominator and derivatives of φ of order at most $\ell - 2$ in the numerator,
- and continue to differentiate (2.11) to obtain a formula for $\varphi^{(\ell-2)}$ involving derivatives of g of order at most $\ell - 1$.

It is now clear that the above arguments apply to prove that derivatives of $g(t)$ of all orders vanish to infinite order at 0 and are continuous on $[0,1)$. This shows that g is smooth on $[0,1)$ and thus that u is smooth on $\mathbb{B}_n$.

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