

A CHARACTERIZATION OF FINITE PREHOMOGENEOUS VECTOR SPACES ASSOCIATED WITH PRODUCTS OF SPECIAL LINEAR GROUPS AND DYNKIN QUIVERS

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ABSTRACT. For a given finite-type quiver Γ , we will consider scalar-removed representations $(S_d, R_d(\Gamma))$, where S_d is a direct product of special linear algebraic groups and $R_d(\Gamma)$ is the representation defined naturally by Γ and a dimension vector d . In this paper, we give a necessary and sufficient condition on d that $R_d(\Gamma)$ has only finitely many S_d -orbits. This condition can be paraphrased as a condition concerning lattices of small rank spanned by positive roots of Γ . To determine such scalar-removed representations having only finitely many orbits is very fundamental to the open problem of classification of the so-called semisimple finite prehomogeneous vector spaces. We consider everything over an algebraically closed field of characteristic zero.

1. INTRODUCTION

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver with r vertices (here Γ_0 , respectively Γ_1 , is the set of vertices, respectively arrows). Then for an r -tuple of non-negative integers $d = (d^{(i)})_{i \in \Gamma_0}$ (we call it a dimension vector), the group $G_d = \prod_{i \in \Gamma_0} GL(d^{(i)})$ acts naturally on $R_d(\Gamma) = \bigoplus_{\alpha \in \Gamma_1} M(d^{(e\alpha)}, d^{(s\alpha)})$, where we consider everything over an algebraically closed field of characteristic zero, and we denote by $M(d^{(e\alpha)}, d^{(s\alpha)})$ the vector space consisting of $d^{(e\alpha)} \times d^{(s\alpha)}$ matrices and by $s\alpha$ (resp. $e\alpha$) the starting (resp. ending) point for an arrow $\alpha \in \Gamma_1$. We will call $(G_d, R_d(\Gamma))$ a representation associated with Γ .

In general, let $\rho : G \rightarrow GL(V)$ be a rational representation of a connected linear algebraic group G on a finite-dimensional vector space V . If V is decomposed into a finite union of G -orbits, it must have a unique Zariski dense orbit; hence (G, V) is a prehomogeneous vector space (abbreviated PV). Such a PV is called a *finite PV* (abbreviated FP). If G is semisimple, we call (G, V) semisimple. Some classes of FPs have already been classified, for example, by Sato–Kimura [9, §8] in the case of irreducible ρ , by Kimura–Kasai–Yasukura [6] in the case where each irreducible component has sufficient scalar multiplication, and by Kimura–Kamiyoshi–Maki–Ouchi–Takano [5] in the case of type $(G \times GL_n, \rho \otimes A_1)$.

In the case where Γ is finite-type (i.e., its underlying graph is one of the Dynkin diagrams of type A_n , D_n , E_6 , E_7 , or E_8), it is well-known that $(G_d, R_d(\Gamma))$ is an

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FP for arbitrary d . However, the scalar-removed representation $(S_d, R_d(\Gamma))$, where $S_d = \prod_{i \in \Gamma_0} SL(d^{(i)}) \subset G_d$, may not be a PV, and not even an FP. Note that the condition whether $(S_d, R_d(\Gamma))$ is an FP does not depend on the choice of an orientation of Γ , but the condition whether it is a PV does. In the case where Γ is of type A_n , it is known that such a condition can be characterized by existence of a certain relative invariant (see §5 of [7]). Such a characterization via relative invariants, however, fails even for D_4 -type Γ , and it seems more complicated, in general, to write down concretely the condition whether $(S_d, R_d(\Gamma))$ for a given dimension vector d is an FP or not (see Theorem 4.1).

In this paper, we classify all scalar-removed FPs *associated with* finite-type quivers. According to the result of [6], to classify such scalar-removed FPs is fundamental to the classification of all semisimple (i.e., without any scalar multiplication) FPs, because (G, ρ, V) is also necessarily an FP if $(H, \rho|_H, V)$ for a subgroup $H \subset G$ is an FP. Our theorem (Theorem 3.4) gives a necessary and sufficient condition, for a given dimension vector d , whether $(S_d, R_d(\Gamma))$ is an FP or not. As mentioned in §3, this condition can be paraphrased as a condition whether a certain lattice of small rank spanned by positive roots of Γ contains d or not. This viewpoint gives us a lucid explanation for conditions on d (look again at the twenty conditions listed in Theorem 4.1); that is, to determine FPs, $(S_d, R_d(\Gamma))$ is nothing but determining lattices of small rank spanned by positive roots. Thus, for an arbitrary finite-type quiver Γ and a dimension vector d , we can *mechanically* determine whether a given representation $(S_d, R_d(\Gamma))$ is an FP or not.

2. PRELIMINARIES

We consider everything over an algebraically closed field \mathbb{K} of characteristic zero.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver with r vertices, where $\Gamma_0 = \{1, 2, \dots, r\}$ is the set of vertices and Γ_1 is the set of arrows. For each arrow $\alpha \in \Gamma_1$, we denote its starting point, respectively ending point, by $s\alpha$, respectively $e\alpha$; for example, if $i \xrightarrow{\alpha} j$ for an arrow $\alpha \in \Gamma_1$, we have $s\alpha = i$ and $e\alpha = j$.

For an r -tuple of non-negative integers $\mathbf{d} = (d^{(i)})_{i \in \Gamma_0}$ (we will call such an r -tuple a dimension vector), the direct product of general linear algebraic groups $G_{\mathbf{d}} = \prod_{i \in \Gamma_0} GL(d^{(i)})$ acts on the vector space $R_{\mathbf{d}}(\Gamma) = \bigoplus_{\alpha \in \Gamma_1} M(d^{(e\alpha)}, d^{(s\alpha)})$ by $g \cdot X = (g^{(e\alpha)} X^{(\alpha)} (g^{(s\alpha)})^{-1})_{\alpha \in \Gamma_1}$ for $g = (g^{(i)})_{i \in \Gamma_0} \in G_{\mathbf{d}}$ and $X = (X^{(\alpha)})_{\alpha \in \Gamma_1} \in R_{\mathbf{d}}(\Gamma)$, where we denote by $M(d^{(i)}, d^{(j)})$ the set of $d^{(i)} \times d^{(j)}$ matrices. In the case of $d^{(i)} = 0$, we will consider corresponding things to be trivial. We call $(G_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ a *representation associated with* Γ .

On the other hand, each element of the vector space $R_{\mathbf{d}}(\Gamma)$ is sometimes called a *representation of* Γ . In such a context we call \mathbf{d} the dimension of $X \in R_{\mathbf{d}}(\Gamma)$ and denote it by $\dim X = \mathbf{d}$. For two representations X and Y of Γ with the same dimension \mathbf{d} (that is, $X, Y \in R_{\mathbf{d}}(\Gamma)$), we say that they are isomorphic if X and Y belong to the same $G_{\mathbf{d}}$ -orbit. We will express such representations as $X \cong Y$.

Let X and Y be representations of Γ with dimensions \mathbf{d} and \mathbf{d}' , respectively. We define their direct sum $X \oplus Y$ by

$$X \oplus Y = \left(\begin{bmatrix} X^{(\alpha)} & 0 \\ 0 & Y^{(\alpha)} \end{bmatrix} \right)_{\alpha \in \Gamma_1}.$$

This is a representation of Γ with dimension $\mathbf{d} + \mathbf{d}'$, that is, an element of the vector space $R_{\mathbf{d}+\mathbf{d}'}(\Gamma)$. If a representation X cannot be expressed as the direct

sum of two non-zero representations, then we say that X is indecomposable. It is known that any representation X can be uniquely decomposed (up to order) into a direct sum of indecomposable representations; that is, there exist indecomposable representations X_1, X_2, \dots, X_s such that

$$X \cong m_1 X_1 \oplus m_2 X_2 \oplus \dots \oplus m_s X_s,$$

where $m_k X_k = X_k \oplus \dots \oplus X_k$ is the direct sum of m_k copies of X_k .

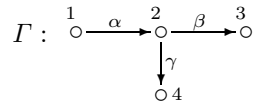
In fact, if Γ is a finite-type quiver, there are only finitely many isomorphic classes of representations of Γ , and the correspondence $X \mapsto \dim X$ gives a bijection between the isomorphic classes of representations and the positive roots of Γ .

Next we define homomorphisms between two representations X and Y , where we put $\dim X = (d^{(i)})_{i \in \Gamma_0}$ and $\dim Y = (d'^{(i)})_{i \in \Gamma_0}$ respectively. A homomorphism from X to Y is an element $g = (g^{(i)})_{i \in \Gamma_0} \in \bigoplus_{i \in \Gamma_0} M(d'^{(i)}, d^{(i)})$ satisfying $g^{(e\alpha)} X^{(\alpha)} = Y^{(\alpha)} g^{(s\alpha)}$ for any arrow $\alpha \in \Gamma_1$. In other words, if we regard each matrix $X^{(\alpha)}$ as a linear map between numerical vector spaces, a homomorphism $g = (g^{(i)})_{i \in \Gamma_0}$ makes the following diagram commutative for each $\alpha \in \Gamma_1$:

$$\begin{array}{ccc} \mathbb{K}^{d^{(s\alpha)}} & \xrightarrow{X^{(\alpha)}} & \mathbb{K}^{d^{(e\alpha)}} \\ g^{(s\alpha)} \downarrow & & \downarrow g^{(e\alpha)} \\ \mathbb{K}^{d'^{(s\alpha)}} & \xrightarrow{Y^{(\alpha)}} & \mathbb{K}^{d'^{(e\alpha)}} \end{array}$$

We denote by $\text{Hom}(X, Y)$ the set of all homomorphisms from X to Y , which can be regarded as a \mathbb{K} -vector space in the natural way.

Example 2.1. Let us consider the following D_4 -type quiver Γ :



There are twelve positive roots of type D_4 , which are given by the following:

$$\begin{aligned} \mathbf{d}_1 &= (1, 0, 0; 0), & \mathbf{d}_2 &= (1, 1, 0; 0), & \mathbf{d}_3 &= (1, 1, 1; 0), & \mathbf{d}_4 &= (1, 1, 0; 1), \\ \mathbf{d}_5 &= (0, 1, 0; 0), & \mathbf{d}_6 &= (1, 2, 1; 1), & \mathbf{d}_7 &= (0, 1, 0; 1), & \mathbf{d}_8 &= (0, 1, 1; 0), \\ \mathbf{d}_9 &= (1, 1, 1; 1), & \mathbf{d}_{10} &= (0, 1, 1; 1), & \mathbf{d}_{11} &= (0, 0, 1; 0), & \mathbf{d}_{12} &= (0, 0, 0; 1). \end{aligned}$$

Let X_k be an indecomposable representation corresponding to the positive root \mathbf{d}_k . For example, $X_6 = (X^{(\alpha)}, X^{(\beta)}, X^{(\gamma)}) \in R_{\mathbf{d}_6}(\Gamma) = M(d_6^{(2)}, d_6^{(1)}) \oplus M(d_6^{(3)}, d_6^{(2)}) \oplus M(d_6^{(4)}, d_6^{(2)})$ is given by

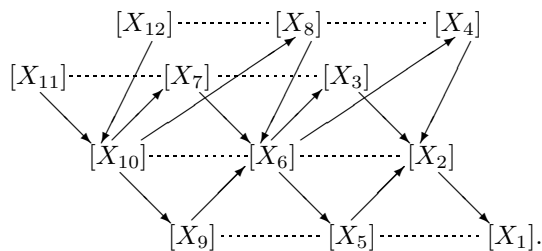
$$X^{(\alpha)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad X^{(\beta)} = [1 \quad 0], \quad X^{(\gamma)} = [1 \quad 1].$$

We see that the above representative system X_1, X_2, \dots, X_{12} satisfies the condition

$$(2.1) \quad \text{Hom}(X_i, X_j) = 0 \quad \text{if } i < j.$$

Remark 2.2. In the above example, we have numbered the positive roots of Γ (and also a complete representative system of the isomorphic classes of its indecomposable representations) to satisfy the condition (2.1). In fact, in the case of finite-type Γ , we can always do such a numbering. Recall the so-called Auslander–Reiten quiver (see, for example, Chapter VII of [2]). The vertices of the AR-quiver

of Γ are in one-to-one correspondence with the isomorphic classes of indecomposable representations of Γ , and there is an arrow $[X_i] \rightarrow [X_j]$ if and only if there exists an *irreducible* morphism $X_i \rightarrow X_j$. For example, the AR-quiver of the above Γ is given by



In the case where Γ is finite-type, it is known that the AR-quiver of Γ consists of a single component and that it is acyclic (see, for example, Proposition 5.13 of [1]). Hence, if Γ is finite-type, we may assume that a complete representative system which is numbered by an appropriate order (that is, we number from one of the tips of the component) satisfies condition (2.1).

We will denote by $\text{End } X = \text{Hom}(X, X)$ the endomorphism ring of X and by $H_X = (\text{End } X)^\times$ its multiplicative group. In other words, H_X is nothing but the isotropy subgroup at $X \in R_{\mathbf{d}}(\Gamma)$; that is, $H_X = \{g \in G_{\mathbf{d}} \mid g \cdot X = X\}$. In the case of $X \cong Y$, we see that H_X and H_Y are conjugate to each other.

We are interested in the restriction map φ_X between rational character groups $\mathcal{X}(G_{\mathbf{d}})$ and $\mathcal{X}(H_X)$, where we denote by $\mathcal{X}(G)$ the group consisting of all rational characters of G . It is known that rational character groups of linear algebraic groups are finitely generated abelian groups.

According to Proposition 1.2 of [8], the rank of $\text{Im } \varphi_X$ describes the condition whether the $G_{\mathbf{d}}$ -orbit $G_{\mathbf{d}}X$ decomposes into infinitely many $S_{\mathbf{d}}$ -orbits, where we put $S_{\mathbf{d}} = \prod_{i \in \Gamma_0} SL(d^{(i)})$. Now we note the following fact:

Lemma 2.3. *Let X be a point of $R_{\mathbf{d}}(\Gamma)$ and $\varphi_X : \mathcal{X}(G_{\mathbf{d}}) \rightarrow \mathcal{X}(H_X)$ the restriction map which is induced by the canonical injection $H_X \hookrightarrow G_{\mathbf{d}}$. Then, the $G_{\mathbf{d}}$ -orbit $G_{\mathbf{d}}X$ is decomposed into infinitely many $S_{\mathbf{d}}$ -orbits if and only if $\text{rank Im } \varphi_X < r$, where r is the number of vertices of Γ .*

Proof. We put $H'_X = H_X \cap S_{\mathbf{d}}$, which is a normal subgroup of H_X . Let us consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & S_{\mathbf{d}} & \longrightarrow & G_{\mathbf{d}} & \longrightarrow & G_{\mathbf{d}}/S_{\mathbf{d}} & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 1 & \longrightarrow & H'_X & \longrightarrow & H_X & \longrightarrow & H_X/H'_X & \longrightarrow & 1,
 \end{array}$$

where each vertical map is the canonical injection. Then this induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \text{Ker } \psi_X & \longrightarrow & \text{Ker } \varphi_X & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & \mathcal{X}(G_{\mathbf{d}}/S_{\mathbf{d}}) & \longrightarrow & \mathcal{X}(G_{\mathbf{d}}) & \longrightarrow & 1 \\
 & & \downarrow \psi_X & & \downarrow \varphi_X & & \\
 1 & \longrightarrow & \mathcal{X}(H_X/H'_X) & \longrightarrow & \mathcal{X}(H_X) & \longrightarrow & \mathcal{X}(H'_X).
 \end{array}$$

Note that ψ_X is surjective since both $G_{\mathbf{d}}/S_{\mathbf{d}}$ and H_X/H'_X are tori. On the other hand, we see that $\mathcal{X}(G_{\mathbf{d}}/S_{\mathbf{d}})$ is isomorphic to $\mathcal{X}(G_{\mathbf{d}})$ since any character of $S_{\mathbf{d}}$ is trivial; hence we have $\text{Ker } \psi_X \simeq \text{Ker } \varphi_X$. According to Proposition 1.2 of [8], the $G_{\mathbf{d}}$ -orbit $G_{\mathbf{d}}X$ is decomposed into infinitely many $S_{\mathbf{d}}$ -orbits if and only if $\dim G_{\mathbf{d}}X > \dim S_{\mathbf{d}}X$, which is equivalent to the condition that $r = \text{rank } \mathcal{X}(G_{\mathbf{d}}) = \text{rank } \mathcal{X}(G_{\mathbf{d}}/S_{\mathbf{d}}) > \text{rank } \mathcal{X}(H_X/H'_X)$. Since these character groups are finitely generated abelian groups, we see that this condition is equivalent to the condition $r - \text{rank Im } \psi_X = \text{rank Ker } \psi_X > 0$; that is, $r - \text{rank Im } \varphi_X = \text{rank Ker } \varphi_X > 0$. Thus we obtain our assertion. \square

Lemma 2.4. *Let X be an indecomposable representation of a finite-type quiver Γ and $\dim X = (d^{(i)})_{i \in \Gamma_0}$ its dimension. Then we have*

$$\text{End } X = \{(\alpha \cdot I_{d^{(i)}})_{i \in \Gamma_0} \mid \alpha \in \mathbb{K}\},$$

where I_u means the identity matrix of degree u . That is to say, $\text{End } X$ is isomorphic to the base field \mathbb{K} and each component is a scalar matrix.

Proof. Put $T = \{(\alpha \cdot I_{d^{(i)}})_{i \in \Gamma_0} \mid \alpha \in \mathbb{K}\}$; then T is a field that is isomorphic to the field \mathbb{K} . It is clear that T is contained in $\text{End } X$, which can be regarded as a \mathbb{K} -vector space. On the other hand, it is known that, for each indecomposable representation X of a finite-type quiver, $\text{End } X$ is nothing but the base field \mathbb{K} (see [3], §7.2). Since $\text{End } X$ is finite dimensional over the field T , we have $\text{End } X = T$. \square

Lemma 2.5. *Let X be an indecomposable representation, with dimension $\dim X = (d^{(i)})_{i \in \Gamma_0}$, of a finite-type quiver. For each positive integer m , the endomorphism ring $\text{End}(mX)$ of $mX = X \oplus \cdots \oplus X$ (the direct sum of m copies of X) is given by*

$$\text{End}(mX) = \{(A \otimes I_{d^{(i)}})_{i \in \Gamma_0} \mid A \in M(m, m)\} \simeq M(m, m),$$

where \otimes denotes Kronecker's product of matrices. In particular, we have $H_{mX} \simeq GL(m)$ and the rational character group $\mathcal{X}(H_{mX})$ is of rank one.

Proof. Let $g = (g^{(i)})_{i \in \Gamma_0}$ be an element of $\text{End}(mX)$, and write each part $g^{(i)}$ as the following $m \times m$ blocks:

$$g^{(i)} = \begin{bmatrix} g_{11}^{(i)} & \cdots & g_{1m}^{(i)} \\ \vdots & \ddots & \vdots \\ g_{m1}^{(i)} & \cdots & g_{mm}^{(i)} \end{bmatrix},$$

where each block $g_{pq}^{(i)}$ is a $d^{(i)} \times d^{(i)}$ matrix. Then, for each arrow $\alpha \in \Gamma_1$, we have

$$g_{pq}^{(e\alpha)} X^{(\alpha)} = X^{(\alpha)} g_{pq}^{(s\alpha)} \quad (p, q = 1, 2, \dots, m).$$

That is to say, for each p and q , we see that $(g_{pq}^{(i)})_{i \in \Gamma_0}$ is contained in the endomorphism ring $\text{End } X$ of an indecomposable X . Therefore it follows from Lemma 2.4 that, for each p and q , there exists a scalar $\alpha_{pq} \in \mathbb{K}$ satisfying $g_{pq}^{(i)} = \alpha_{pq} \cdot I_{d^{(i)}}$. Putting $A = [\alpha_{pq}] \in M(m, m)$, we have $g^{(i)} = A \otimes I_{d^{(i)}}$. \square

Proposition 2.6. *Let $X = m_1 X_1 \oplus m_2 X_2 \oplus \cdots \oplus m_s X_s \in R_{\mathbf{d}}(\Gamma)$ be a representation, where the X_i 's are distinct indecomposable representations which are numbered to satisfy condition (2.1). Then we have $\text{rank } \mathcal{X}(H_X) = s$.*

Proof. Put $\dim X_k = \mathbf{d}_k = (d_k^{(i)})_{i \in \Gamma_0}$, and let $\tilde{X}_k = m_k X_k$ be the direct sum of m_k copies of X_k . Then we have $\dim \tilde{X}_k = m_k \mathbf{d}_k = (m_k d_k^{(i)})_{i \in \Gamma_0}$. Let $h = (h^{(i)})_{i \in \Gamma_0}$ be an element of the isotropy subgroup $H_{\tilde{X}_1 \oplus \cdots \oplus \tilde{X}_s}$, and decompose each part $h^{(i)}$ into $s \times s$ blocks:

$$h^{(i)} = \begin{bmatrix} h_{11}^{(i)} & \cdots & h_{1s}^{(i)} \\ \vdots & \ddots & \vdots \\ h_{s1}^{(i)} & \cdots & h_{ss}^{(i)} \end{bmatrix},$$

where each block $h_{pq}^{(i)}$ is an $m_p d_p^{(i)} \times m_q d_q^{(i)}$ matrix. In the case of $d_k^{(i)} = 0$, we should remove its corresponding blocks. Thus we have

$$h_{pq}^{(e\alpha)} \tilde{X}_q^{(\alpha)} = \tilde{X}_p^{(\alpha)} h_{pq}^{(s\alpha)}$$

for each arrow $\alpha \in \Gamma_1$, and hence $h_{pq} = (h_{pq}^{(i)})_{i \in \Gamma_0} \in \text{Hom}(\tilde{X}_q, \tilde{X}_p)$. Then condition (2.1) implies $h_{pq} = (h_{pq}^{(i)})_{i \in \Gamma_0} = 0$ for any p and q satisfying $q < p$. Therefore we see that each part $h^{(i)}$ is contained in a subgroup consisting of upper triangular block matrices (i.e., it is contained in the standard parabolic subgroup corresponding to the partition $m_1 d_1^{(i)} + m_2 d_2^{(i)} + \cdots + m_s d_s^{(i)}$):

$$h^{(i)} = \begin{bmatrix} h_{11}^{(i)} & h_{12}^{(i)} & \cdots & h_{1s}^{(i)} \\ 0 & h_{22}^{(i)} & \cdots & h_{2s}^{(i)} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{ss}^{(i)} \end{bmatrix}.$$

Hence, for $k = 1, 2, \dots, s$, we have the canonical projection $H_X \rightarrow H_{\tilde{X}_k}$ by $h = (h^{(i)})_{i \in \Gamma_0} \mapsto (h_{kk}^{(i)})_{i \in \Gamma_0}$ (here we will consider $h_{kk}^{(i)}$ to be trivial if $d_k^{(i)} = 0$). It follows from Lemma 2.5 that there exists $A_k \in GL(m_k)$ satisfying $h_{kk}^{(i)} = A_k \otimes I_{d_k^{(i)}}$ for any $i \in \Gamma_0$; hence we can define the character $\lambda_k(h) = \det A_k$ for $h \in H_X$. Then we see that each rational character group $\mathcal{X}(H_{\tilde{X}_k})$ is generated by λ_k and that $\lambda_1, \lambda_2, \dots, \lambda_s$ constitute a basis of the rational character group $\mathcal{X}(H_X)$; that is, it is a free abelian group of rank s . \square

For each $i \in \Gamma_0$, we define the character $\chi_i : G_{\mathbf{d}} \rightarrow \mathbb{K}^\times$ by $\chi_i(g) = \det g^{(i)}$ for $g = (g^{(i)})_{i \in \Gamma_0} \in G_{\mathbf{d}}$.

Corollary 2.7. *In the same notation as in Proposition 2.6, the representation matrix of the restriction map $\varphi_X : \mathcal{X}(G_{\mathbf{d}}) \rightarrow \mathcal{X}(H_X)$, with respect to bases $\chi_1, \chi_2, \dots, \chi_r$ and $\lambda_1, \lambda_2, \dots, \lambda_s$, is given by $[{}^t \mathbf{d}_1 | {}^t \mathbf{d}_2 | \cdots | {}^t \mathbf{d}_s]$, where \mathbf{d}_k is the positive root corresponding to X_k .*

Proof. Take an element $h = (h^{(i)})_{i \in \Gamma_0} \in H_X$. Then we have

$$\begin{aligned} \chi_i(h) &= \det(h_{11}^{(i)} h_{22}^{(i)} \cdots h_{ss}^{(i)}) = (\det h_{11}^{(i)}) (\det h_{22}^{(i)}) \cdots (\det h_{ss}^{(i)}) \\ &= (\lambda_1(h))^{d_1^{(i)}} (\lambda_2(h))^{d_2^{(i)}} \cdots (\lambda_s(h))^{d_s^{(i)}} \\ &= (\lambda_1^{d_1^{(i)}} \lambda_2^{d_2^{(i)}} \cdots \lambda_s^{d_s^{(i)}})(h) \end{aligned}$$

for each $i \in \Gamma_0$. Hence the representation matrix of φ_X with respect to such bases is given by

$$\begin{bmatrix} d_1^{(1)} & d_2^{(1)} & \cdots & d_s^{(1)} \\ d_1^{(2)} & d_2^{(2)} & \cdots & d_s^{(2)} \\ \vdots & \vdots & & \vdots \\ d_1^{(r)} & d_2^{(r)} & \cdots & d_s^{(r)} \end{bmatrix} = [{}^t \mathbf{d}_1 | {}^t \mathbf{d}_2 | \cdots | {}^t \mathbf{d}_s];$$

that is, for each k , the k -th column is nothing but the transpose of the positive root \mathbf{d}_k . □

3. CHARACTERIZATION OF SEMISIMPLE FPs

Now we are standing at the position required to prove our main theorem:

Theorem 3.1. *Let Γ be a finite-type quiver with r vertices. For a dimension vector \mathbf{d} , the following conditions are equivalent:*

- (1) *The scalar-removed representation $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is not an FP.*
- (2) *There exist some positive roots $\mathbf{d}_{i_1}, \mathbf{d}_{i_2}, \dots, \mathbf{d}_{i_p}$ of Γ such that $\mathbf{d} \in \langle \mathbf{d}_{i_1}, \mathbf{d}_{i_2}, \dots, \mathbf{d}_{i_p} \rangle_{\mathbb{Z}_{\geq 0}}$ and $\text{rank} [{}^t \mathbf{d}_{i_1} | {}^t \mathbf{d}_{i_2} | \cdots | {}^t \mathbf{d}_{i_p}] < r$.*

Proof. Assume that $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is not an FP. Then there exists a point $X \in R_{\mathbf{d}}(\Gamma)$ such that its $G_{\mathbf{d}}$ -orbit is decomposed into infinitely many $S_{\mathbf{d}}$ -orbits. By Lemma 2.3, this is equivalent to the condition that $\text{rank Im } \varphi_X < r$. Now we can choose some positive integers m_1, m_2, \dots, m_p and indecomposable representations $X_{i_1}, X_{i_2}, \dots, X_{i_p}$ such that $X \cong m_1 X_{i_1} \oplus m_2 X_{i_2} \oplus \cdots \oplus m_p X_{i_p}$. Here, as mentioned in Remark 2.2, we may assume that the X_k 's are numbered to satisfy the condition (2.1). Then it follows from Corollary 2.7 that $\text{rank Im } \varphi_X = \text{rank} [{}^t \mathbf{d}_{i_1} | {}^t \mathbf{d}_{i_2} | \cdots | {}^t \mathbf{d}_{i_p}]$, and we have $\mathbf{d} = \dim X = m_1 \mathbf{d}_{i_1} + m_2 \mathbf{d}_{i_2} + \cdots + m_p \mathbf{d}_{i_p}$; therefore we obtain (2).

Conversely, the condition $\mathbf{d} \in \langle \mathbf{d}_{i_1}, \mathbf{d}_{i_2}, \dots, \mathbf{d}_{i_p} \rangle_{\mathbb{Z}_{\geq 0}}$ implies that we can construct the representation $X = m_1 X_{i_1} \oplus m_2 X_{i_2} \oplus \cdots \oplus m_p X_{i_p} \in R_{\mathbf{d}}(\Gamma)$. Then the second condition means that the $G_{\mathbf{d}}$ -orbit of X is decomposed into infinitely many $S_{\mathbf{d}}$ -orbits; i.e., the representation $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is not an FP. □

In fact, condition (2) of Theorem 3.1 can be improved a little. Here we will review a few properties of positive roots.

Let E be a Euclidean space (over \mathbb{R}) endowed with an appropriate inner product. Fix a basis of E and define the lexicographical order with respect to the basis. Let Φ^+ be the set of all positive roots contained in a root system of E .

For a finite subset $M \subseteq \Phi^+$ we put $\Psi = \langle M \rangle_{\mathbb{R}} \cap \Phi^+$ and $\dim_{\mathbb{R}} \langle M \rangle_{\mathbb{R}} = p$; i.e., the subspace generated by M is of dimension p . Now we choose p positive roots $\alpha_1, \alpha_2, \dots, \alpha_p$ as follows:

$$\alpha_1 := \min \Psi, \text{ and } \alpha_k := \min (\Psi \setminus \langle \alpha_1, \dots, \alpha_{k-1} \rangle_{\mathbb{R}}) \text{ for } k = 2, 3, \dots, p.$$

Then we have the following lemma, which can be proved by induction on the dimension of $\langle M \rangle_{\mathbb{R}}$.

Lemma 3.2. *For any element $\alpha \in \Psi$, there exist non-negative integers k_1, k_2, \dots, k_p such that $\alpha = k_1\alpha_1 + k_2\alpha_2 + \dots + k_p\alpha_p$.*

Proposition 3.3. *Let $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_s$ be positive roots contained in a root system of a Euclidean space. If $\text{rank}_{\mathbb{Z}}\langle \mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_s \rangle_{\mathbb{Z}} = p$, then there exist p positive roots $\alpha_1, \alpha_2, \dots, \alpha_p$ such that $\langle \mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_s \rangle_{\mathbb{Z}} = \langle \alpha_1, \alpha_2, \dots, \alpha_p \rangle_{\mathbb{Z}}$. In particular, we have $\langle \mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_s \rangle_{\mathbb{Z}_{\geq 0}} \subseteq \langle \alpha_1, \alpha_2, \dots, \alpha_p \rangle_{\mathbb{Z}_{\geq 0}}$; that is, the lattice with coefficients of non-negative integers spanned by $\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_s$ is contained in one spanned by $\alpha_1, \alpha_2, \dots, \alpha_p$.*

Proof. Put $M := \{\mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_s\}$. Since $\langle M \rangle_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} \langle \mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_s \rangle_{\mathbb{Z}}$, we have $\dim_{\mathbb{R}}\langle M \rangle_{\mathbb{R}} = \text{rank}_{\mathbb{Z}}\langle \mathbf{d}_1, \mathbf{d}_2, \dots, \mathbf{d}_s \rangle_{\mathbb{Z}} = p$. By Lemma 3.2, each \mathbf{d}_k can be expressed as a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_p$ with coefficients of non-negative integers. Thus we obtain our assertion. \square

Therefore we have gained a more sophisticated characterization of scalar-removed FPs associated with finite-type quivers.

Theorem 3.4. *Let Γ be a finite-type quiver with r vertices. For a dimension vector \mathbf{d} , the following conditions are equivalent:*

- (1) *The scalar-removed representation $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is not an FP.*
- (2) *There exist $r - 1$ positive roots $\mathbf{d}_{i_1}, \mathbf{d}_{i_2}, \dots, \mathbf{d}_{i_{r-1}}$ of Γ satisfying $\mathbf{d} \in \langle \mathbf{d}_{i_1}, \mathbf{d}_{i_2}, \dots, \mathbf{d}_{i_{r-1}} \rangle_{\mathbb{Z}_{\geq 0}}$; that is, \mathbf{d} can be expressed as a linear combination of $r - 1$ positive roots with coefficients of non-negative integers.*

In particular, the condition whether $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is an FP or not does not depend on the choice of an orientation of Γ .

4. EXAMPLES OF D_4 -TYPE

In this section, we give some examples of D_4 -type. Let Γ be the D_4 -type quiver mentioned in Example 2.1. We are interested in lattices of small rank because we will determine dimension \mathbf{d} such that $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is not an FP.

First we note that there exist twenty distinct lattices of rank three (with each component of the sum of generators being positive) spanned by positive roots of D_4 -type.

- | | | |
|---------------------------------------|---|--|
| (1) $L_1 = \langle 2, 11, 12 \rangle$ | (8) $L_8 = \langle 1, 7, 8 \rangle$ | (15) $L_{15} = \langle 1, 6, 11 \rangle$ |
| (2) $L_2 = \langle 1, 8, 12 \rangle$ | (9) $L_9 = \langle 2, 7, 11 \rangle$ | (16) $L_{16} = \langle 2, 8, 9 \rangle$ |
| (3) $L_3 = \langle 1, 7, 11 \rangle$ | (10) $L_{10} = \langle 4, 5, 10 \rangle$ | (17) $L_{17} = \langle 7, 8, 9 \rangle$ |
| (4) $L_4 = \langle 3, 5, 12 \rangle$ | (11) $L_{11} = \langle 3, 4, 5 \rangle$ | (18) $L_{18} = \langle 2, 7, 9 \rangle$ |
| (5) $L_5 = \langle 1, 5, 10 \rangle$ | (12) $L_{12} = \langle 3, 5, 10 \rangle$ | (19) $L_{19} = \langle 2, 7, 8 \rangle$ |
| (6) $L_6 = \langle 4, 5, 11 \rangle$ | (13) $L_{13} = \langle 6, 11, 12 \rangle$ | (20) $L_{20} = \langle 3, 4, 10 \rangle$ |
| (7) $L_7 = \langle 2, 8, 12 \rangle$ | (14) $L_{14} = \langle 1, 6, 12 \rangle$ | |

In the above list, for example, $L_1 = \langle 2, 11, 12 \rangle$ means that the lattice (free \mathbb{Z} -module) L_1 is spanned by three roots $\mathbf{d}_2, \mathbf{d}_{11}, \mathbf{d}_{12}$ (we recall that the roots of D_4 -type Γ have been numbered in Example 2.1). Therefore a dimension vector (i.e., a four-tuple of positive integers) $\mathbf{d} = (d^{(1)}, d^{(2)}, d^{(3)}; d^{(4)})$ is contained in L_1 if and only if $d^{(1)} = d^{(2)}$. Thus we obtain the following theorem for D_4 -type:

Theorem 4.1. *Let Γ be a D_4 -type quiver. Then, for a given dimension vector $\mathbf{d} = (d^{(1)}, d^{(2)}, d^{(3)}; d^{(4)})$, the scalar-removed representation $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is not an FP if and only if at least one of the following twenty conditions is satisfied:*

- | | |
|--|---|
| (1) $d^{(1)} = d^{(2)}$ | (14) $2d^{(3)} = d^{(2)}$ |
| (2) $d^{(3)} = d^{(2)}$ | and $d^{(3)} < \min\{d^{(4)}, d^{(1)}\}$ |
| (3) $d^{(4)} = d^{(2)}$ | (15) $2d^{(4)} = d^{(2)}$ |
| (4) $d^{(1)} = d^{(3)} < d^{(2)}$ | and $d^{(4)} < \min\{d^{(1)}, d^{(3)}\}$ |
| (5) $d^{(3)} = d^{(4)} < d^{(2)}$ | (16) $d^{(1)} + d^{(3)} = d^{(4)} + d^{(2)}$ |
| (6) $d^{(4)} = d^{(1)} < d^{(2)}$ | and $\max\{d^{(1)}, d^{(3)}\} < d^{(2)}$ |
| (7) $d^{(1)} + d^{(3)} = d^{(2)}$ | (17) $d^{(3)} + d^{(4)} = d^{(1)} + d^{(2)}$ |
| (8) $d^{(3)} + d^{(4)} = d^{(2)}$ | and $\max\{d^{(3)}, d^{(4)}\} < d^{(2)}$ |
| (9) $d^{(4)} + d^{(1)} = d^{(2)}$ | (18) $d^{(4)} + d^{(1)} = d^{(3)} + d^{(2)}$ |
| (10) $d^{(1)} + d^{(3)} = d^{(4)} < d^{(2)}$ | and $\max\{d^{(4)}, d^{(1)}\} < d^{(2)}$ |
| (11) $d^{(3)} + d^{(4)} = d^{(1)} < d^{(2)}$ | (19) $d^{(1)} + d^{(3)} + d^{(4)} = d^{(2)}$ |
| (12) $d^{(4)} + d^{(1)} = d^{(3)} < d^{(2)}$ | (20) $d^{(1)} + d^{(3)} + d^{(4)} = 2d^{(2)}$ |
| (13) $2d^{(1)} = d^{(2)}$ | and $\max\{d^{(1)}, d^{(3)}, d^{(4)}\} < d^{(2)}$ |
| and $d^{(1)} < \min\{d^{(3)}, d^{(4)}\}$ | |

Note that Theorem 4.1 was independently obtained by Dr. Tomohiro Kamiyoshi, a researcher (non-full-time) at the University of Tsukuba. He has investigated representations associated with D_4 -type quivers *under various scalar restrictions* (see [4]).

Among D_4 -type FPs $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$, we are interested in representations of dimension $\mathbf{d} = (d^{(1)}, d^{(2)}, d^{(3)}; d^{(4)})$ satisfying $d^{(2)} > \max\{d^{(1)}, d^{(3)}, d^{(4)}\}$, because if an A_3 -type representation of dimension $(d^{(1)}, d^{(2)}, d^{(3)})$ is an FP, then so is any D_4 -type with dimensional condition $d^{(2)} < d^{(4)}$. (Recall the elementary transformations of matrices. A precise statement is mentioned in, for example, [5, Proposition 1.3].)

Example 4.2. For $\mathbf{d} = (2, 8, 3; 4)$, we have $\mathbf{d} = -2\mathbf{d}_1 + 4\mathbf{d}_6 - \mathbf{d}_{11}$ and hence $\mathbf{d} \in L_{15}$. However, we can conclude that $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is an FP, because \mathbf{d} cannot be expressed as a linear combination of positive roots with coefficients of *non-negative* integers (i.e., the dimension \mathbf{d} does not satisfy any of the twenty conditions listed in Theorem 4.1). In fact, $R_{\mathbf{d}}(\Gamma)$ is decomposed into 439 $S_{\mathbf{d}}$ -orbits.

Thus we realize that the conditions on \mathbf{d} whether $(S_{\mathbf{d}}, R_{\mathbf{d}}(\Gamma))$ is an FP or not can be obtained in this way. To know such conditions, it is sufficient to list lattices of small rank. For example, there exist 26 (resp. 76, 633) lattices of A_5 -type (resp. D_5 , E_6 -type) of small rank with each component of the sum of generators being positive.

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