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ON THE TOPOLOGY OF POINTWISE CONVERGENCE ON THE BOUNDARIES OF L_1-PREDUALS

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Abstract. In this paper we prove a theorem more general than the following: "If $(X, \|\cdot\|)$ is an L_1-predual, B is any boundary of X and $\{x_n : n \in \mathbb{N}\}$ is any subset of X, then the closure of $\{x_n : n \in \mathbb{N}\}$ with respect to the topology of pointwise convergence on B is separable with respect to the topology generated by the norm, whenever $\mathrm{Ext}(B_{X^*})$ is weak* Lindelöf." Several applications of this result are also presented.

1. Introduction

We shall say that a Banach space $(X, \|\cdot\|)$ is an L_1-*predual* if X^* is isometric to $L_1(\mu)$ for some suitable measure μ. Some examples of L_1-preduals include $(C(K), \|\cdot\|_\infty)$, and more generally, the space of continuous affine functions on a Choquet simplex (see [10] for the definition) endowed with the supremum norm (see, [4, Proposition 3.23]). We shall also consider the notion of a boundary. Specifically, for a nontrivial Banach space X over $\mathbb{R}$ we say that a subset B of B_{X^*}, the closed unit ball of X^*, is a *boundary* if for each $x \in X$ there exists a $b^* \in B$ such that $b^*(x) = \|x\|$. The prototypical example of a boundary is $\mathrm{Ext}(B_{X^*})$, the set of all extreme points of B_{X^*}, but there are many other interesting examples given in [9].

In the recent paper [9] the authors investigate the topology on a Banach space X that is generated by $\mathrm{Ext}(B_{X^*})$ and, more generally, the topology on X generated by an arbitrary boundary of X. This paper continues this study.

To be more precise we must first introduce some notation. For a nonempty subset Y of the dual of a Banach space X we shall denote by $\sigma(X, Y)$ the weakest linear topology on X that makes all the functionals from Y continuous. In [9] the authors show (see, [9, Theorem 2.2]) using [3, Lemma 1] that for any compact Hausdorff space K, any countable subset $\{x_n : n \in \mathbb{N}\}$ of $C(K)$ and any boundary B of $(C(K), \|\cdot\|_\infty)$, the closure of $\{x_n : n \in \mathbb{N}\}$ with respect to the $\sigma(C(K), B)$ topology is separable with respect to the topology generated by the norm. In this paper we extend this result by showing that if $(X, \|\cdot\|)$ is an L_1-predual, B is any boundary of X and $\{x_n : n \in \mathbb{N}\}$ is any subset of X, then the closure of $\{x_n : n \in \mathbb{N}\}$

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in the $\sigma(X,B)$ topology is separable with respect to the topology generated by the norm whenever $\text{Ext}(B_{X^*})$ is weak* Lindelöf.

We conclude this paper with some applications that indicate the utility of our results.

2. Preliminary results

Let X be a topological space and let $\mathcal{F}$ be a family of nonempty, closed and separable subsets of X. Then $\mathcal{F}$ is *rich* if the following two conditions are fulfilled:

(i) for every separable subspace Y of X, there exists a $Z \in \mathcal{F}$ such that $Y \subseteq Z$;
(ii) for every increasing sequence $(Z_n : n \in \mathbb{N})$ in $\mathcal{F}$, $\overline{\bigcup_{n\in\mathbb{N}} Z_n} \in \mathcal{F}$.

For any topological space X, the collection of all rich families of subsets forms a partially ordered set, under the binary relation of set inclusion. This partially ordered set has a greatest element, namely,

$$\mathcal{G}_X := \{S \subseteq X : S \text{ is a nonempty, closed and separable subset of } X\}.$$

On the other hand, if X is a separable space, then the partially ordered set has a least element, namely, $\mathcal{G}_\varnothing := \{X\}$.

The raison d'être for rich families is revealed next.

Proposition 1. *Suppose that X is a topological space. If $\{\mathcal{F}_n : n \in \mathbb{N}\}$ are rich families of X, then so is $\bigcap_{n\in\mathbb{N}} \mathcal{F}_n$.*

For a proof of this proposition see [2, Proposition 1.1].

Throughout this paper we will be primarily working with Banach spaces, so a natural class of rich families, given a Banach space X, is the family of all closed separable linear subspaces of X, which we denote by $\mathcal{S}_X$. There are however many other interesting examples of rich families that can be found in [2] and [7].

For our first result we will provide another nontrivial example of a rich family, but to achieve this we first need a preliminary result that characterises when a given Banach space is an L_1-predual.

Lemma 1 ([6, §21, Theorem 7])**.** *For a Banach space X the following are equivalent:*

(i) *X is an L_1-predual;*
(ii) *for each weak* continuous convex function f on B_{X^*},*

$$f^*(0) = \frac{1}{2}\max\{f(x^*) + f(-x^*) : x^* \in B_{X^*}\},$$

where $f^* = \inf\{h : h \geq f$ and h is weak continuous and affine on $B_{X^*}\}$.*

Before proceeding further we shall introduce the following notation. If X is a normed linear space, then each $x \in X$ defines a weak* continuous affine function $\widehat{x}$ on B_{X^*} via the canonical embbeding, that is, $\widehat{x}(x^*) := x^*(x)$ for all $x^* \in B_{X^*}$.

Theorem 1. *Let X be an L_1-predual. Then the set of all closed separable linear subspaces of X that are themselves L_1-preduals forms a rich family.*

Proof. Let $\mathscr{L} := \{Z \in \mathcal{S}_X : Z \text{ is an } L_1\text{-predual}\}$. We shall verify that $\mathscr{L}$ is a rich family. So first let us consider an arbitrary separable closed linear subspace Y of X. Then by [6, §23, Lemma 1] there exists a closed separable subspace $Z \in \mathscr{L}$ such that $Y \subseteq Z$. Next, let us consider an increasing sequence $(Z_n : n \in \mathbb{N})$ in $\mathscr{L}$

and let $Z := \overline{\bigcup_{n\in\mathbb{N}} Z_n}$. To show that $Z \in \mathscr{L}$ we shall appeal to Lemma 1. Let f be a weak* continuous convex function on B_{Z^*}. Since

$$\frac{1}{2}\max\{f(x^*)+f(-x^*) : x^* \in B_{Z^*}\} \le f^*(0),$$

it is enough to verify that for each $\varepsilon > 0$, $f^*(0) \le \frac{1}{2}\max\{f(x^*)+f(-x^*) : x^* \in B_{Z^*}\}+\varepsilon$. To this end, suppose that $\varepsilon > 0$. Since f is weak* continuous and convex and B_{Z^*} is weak* compact, by [1, Corollary I.1.3] there exist $z_i \in Z$ and $c_i \in \mathbb{R}$, $i = 1, \dots, n$, such that the weak* convex continuous $g : B_{Z^*} \to \mathbb{R}$ defined by

$$g := \max\{\widehat{z_1}+c_1, \widehat{z_2}+c_2, \dots, \widehat{z_n}+c_n\}$$

satisfies

$$f(z^*)-\varepsilon < g(z^*) < f(z^*), \quad z^* \in B_{Z^*}.$$

Since $\bigcup_{n\in\mathbb{N}} Z_n$ is dense in Z we may further assume that all the elements z_i are contained in some fixed Z_j, $j \in \mathbb{N}$.

Next, let $r : B_{Z^*} \to B_{Z_j^*}$ be the restriction mapping (i.e., $r(z^*) = z^*|_{Z_j}$ for all $z^* \in B_{Z^*}$) and let $h : B_{Z_j^*} \to \mathbb{R}$ be defined by $h := \max\{\widehat{z_1}+c_1, \widehat{z_2}+c_2, \dots, \widehat{z_n}+c_n\}$. Then h is weak* continuous and convex on $B_{Z_j^*}$ and $g = h \circ r$. Moreover, by the definition of g (and the fact that r is weak*-to-weak* continuous and linear) we have that $g^*(z^*) \le h^*(r(z^*))$ for all $z^* \in B_{Z^*}$. Now, by the assumption that Z_j is an L_1-predual (and Lemma 1) there exists a $y^* \in B_{Z_j^*}$ such that

$$h^*(0) = \frac{1}{2}[h(y^*)+h(-y^*)].$$

Choose $z^* \in r^{-1}(y^*)$, which is nonempty by the Hahn-Banach extension theorem. Then,

$$g^*(0) \le h^*(0) = \frac{1}{2}[h(y^*)+h(-y^*)] = \frac{1}{2}[g(z^*)+g(-z^*)] \le g^*(0).$$

Therefore,

$$\begin{aligned} f^*(0)-\varepsilon = (f-\varepsilon)^*(0) &\le g^*(0) = \frac{1}{2}[g(z^*)+g(-z^*)] \\ &\le \frac{1}{2}[f(z^*)+f(-z^*)] \\ &\le \frac{1}{2}\max\{f(x^*)+f(-x^*) : x^* \in B_{Z^*}\}. \end{aligned}$$

That is, $f^*(0) \le \frac{1}{2}\max\{f(x^*)+f(-x^*) : x^* \in B_{Z^*}\}+\varepsilon$, which completes the proof. □

Before we can introduce another class of rich families, we require the following lemma, which is a Banach space version of [11, Theorem 2.10].

Lemma 2. *Let Y be a closed separable linear subspace of a Banach space X and suppose that $L \subseteq \mathrm{Ext}(B_{X^*})$ is weak* Lindelöf. Then there exists a closed separable linear subspace Z of X, containing Y, such that for any $l^* \in L$ and any x^*, $y^* \in B_{Z^*}$, if $l^*|_Z = \frac{1}{2}(x^*+y^*)$, then $x^*|_Y = y^*|_Y$.*

Proof. Let $\mathscr{B}$ be a countable base for the topology on $(B_{Y^*}, \mathrm{weak}^*)$ consisting of closed convex sets. Recall that such a base exists because $(B_{Y^*}, \mathrm{weak}^*)$ is compact, by the Banach-Alaoglu Theorem, and $(B_{Y^*}, \mathrm{weak}^*)$ is metrizable, since Y is separable. Let:

(i) $\mathscr{F} := \{r^{-1}(B) : B \in \mathscr{B}\}$, where $r : B_{X^*} \to B_{Y^*}$ is the restriction mapping;
(ii) $\mathscr{R} := \{\frac{1}{2}(F_1 + F_2) : F_1, F_2 \in \mathscr{F}$ and $F_1 \cap F_2 = \varnothing\}$.

By construction $\bigcup \mathscr{R} \subseteq B_{X^*} \setminus \mathrm{Ext}(B_{X^*})$ and so $L \cap \bigcup \mathscr{R} = \varnothing$. Furthermore, for each $l^* \in L$ and $F \in \mathscr{R}$ there exists a $y \in X$ such that

$$\sup\{\widehat{y}(f^*) : f^* \in F\} < \widehat{y}(l^*).$$

Therefore, since L is weak* Lindelöf for each $F \in \mathscr{R}$ there exists a countable subset C_F in X such that, for each $l^* \in L$ there exists a $y \in C_F$ such that $\sup\{\widehat{y}(f^*) : f^* \in F\} < \widehat{y}(l^*)$. If we set $C := \bigcup\{C_F : F \in \mathscr{R}\}$ and $Z := \overline{\mathrm{span}}(C \cup X)$, then $X \subseteq Z$ and Z is a closed separable linear subspace of X.

It now only remains to verify that if $l^* \in L$, x^*, $y^* \in B_{Z^*}$ and $l^*|_Z = \frac{1}{2}(x^* + y^*)$, then $x^*|_Y = y^*|_Y$. So, in order to obtain a contradiction, suppose that for some $l^* \in L$ and x^*, $y^* \in B_{Z^*}$, $l^*|_Z = \frac{1}{2}(x^* + y^*)$ but $x^*|_Y \neq y^*|_Y$. Then there exists $B_1, B_2 \in \mathscr{B}$ such that $x^*|_Y \in B_1$ and $y^*|_Y \in B_2$ and $B_1 \cap B_2 = \varnothing$. Set $F_1 := r^{-1}(B_1)$ and $F_2 := r^{-1}(B_2)$. Then $F_1, F_2 \in \mathscr{F}$ and $F_1 \cap F_2 = \varnothing$. Now, by the Hahn-Banach Extension Theorem there exist $x_1^* \in B_{X^*}$ and $y_1^* \in B_{X^*}$ such that $x_1^*|_Z = x^*$ and $y_1^*|_Z = y^*$. Moreover,

$$x_1^*|_Y = (x_1^*|_Z)|_Y = x^*|_Y \in B_1 \quad \text{ and } \quad y_1^*|_Y = (y_1^*|_Z)|_Y = y^*|_Y \in B_2.$$

That is, $x_1^* \in F_1$ and $y_1^* \in F_2$. Therefore, $\frac{1}{2}(x_1^* + y_1^*) \in \frac{1}{2}(F_1 + F_2) =: F$. Since $F \in \mathscr{R}$, by the construction there exists a $y \in C_F \subseteq C \subseteq Z$ such that $\sup\{\widehat{y}(f^*) : f^* \in F\} < \widehat{y}(l^*)$. In particular,

$$\frac{1}{2}(x^* + y^*)(y) = \widehat{y}(\frac{1}{2}(x_1^* + y_1^*)) < l^*(y) = (l^*|_Z)(y).$$

However, this contradicts the fact that $\frac{1}{2}(x^* + y^*) = l^*|_Z$. $\square$

Theorem 2. *Let X be a Banach space and let $L \subseteq \mathrm{Ext}(B_{X^*})$ be a weak* Lindelöf subset. Then the set of all Z in $\mathcal{S}_X$ such that $\{l^*|_Z : l^* \in L\} \subseteq \mathrm{Ext}(B_{Z^*})$ forms a rich family.*

Proof. Let $\mathscr{L}$ denote the family of all closed separable linear subspaces Z of X such that $\{l^*|_Z : l^* \in L\} \subseteq \mathrm{Ext}(B_{Z^*})$. We shall verify that $\mathscr{L}$ is a rich family of closed separable linear subspaces of X. So first let us consider an arbitrary closed separable linear subspace Y of X, with the aim of showing that there exists a subspace $Z \in \mathscr{L}$ such that $Y \subseteq Z$. We begin by inductively applying Lemma 2 to obtain an increasing sequence $(Z_n : n \in \mathbb{N})$ of closed separable linear subspaces of X such that: $Y \subseteq Z_1$ and for any $l^* \in L$ and any $x^*, y^* \in B_{Z_{n+1}^*}$, if $l^*|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*)$, then $x^*|_{Z_n} = y^*|_{Z_n}$.

We now claim that if $Z := \overline{\bigcup_{n\in\mathbb{N}} Z_n}$, then $l^*|_Z \in \mathrm{Ext}(B_{Z^*})$ for each $l^* \in L$. To this end, suppose that $l^* \in L$ and $l^*|_Z = \frac{1}{2}(x^* + y^*)$ for some $x^*, y^* \in B_{Z^*}$. Then for each $n \in \mathbb{N}$,

$$l^*|_{Z_{n+1}} = (l^*|_Z)|_{Z_{n+1}} = \frac{1}{2}(x^* + y^*)|_{Z_{n+1}} = \frac{1}{2}(x^*|_{Z_{n+1}} + y^*|_{Z_{n+1}})$$

and $x^*|_{Z_{n+1}}, y^*|_{Z_{n+1}} \in B_{Z_{n+1}^*}$ Therefore, by construction, $x^*|_{Z_n} = y^*|_{Z_n}$. Now since $\bigcup_{n\in\mathbb{N}} Z_n$ is dense in Z and both x^* and y^* are continuous, we may deduce that $x^* = y^*$, which in turn implies that $l^*|_Z \in \mathrm{Ext}(B_{Z^*})$. This shows that $Y \subseteq Z$ and $Z \in \mathscr{L}$.

To complete this proof we must verify that for each increasing sequence of closed separable subspaces $(Z_n : n \in \mathbb{N})$ in $\mathscr{L}$, $\overline{\bigcup_{n\in\mathbb{N}} Z_n} \in \mathscr{L}$. This, however, follows easily from the definition of the family $\mathscr{L}$. $\square$

Let X be a normed linear space. Then we say that an element $x^* \in B_{X^*}$ is *weak* exposed* if there exists an element $x \in X$ such that $y^*(x) < x^*(x)$ for all $y^* \in B_{X^*} \setminus \{x^*\}$. It is not difficult to show that if $\mathrm{Exp}(B_{X^*})$ denotes the set of all weak* exposed points of B_{X^*}, then $\mathrm{Exp}(B_{X^*}) \subseteq \mathrm{Ext}(B_{X^*})$. However, if X is a separable L_1-predual, then the relationship between $\mathrm{Exp}(B_{X^*})$ and $\mathrm{Ext}(B_{X^*})$ is much closer.

Lemma 3 ([13, Lemma 3.3(b)]). *If X is a separable L_1-predual, then* $\mathrm{Exp}(B_{X^*}) = \mathrm{Ext}(B_{X^*})$.

Let us also pause for a moment to recall that if B is any boundary of a Banach space X, then

$$\mathrm{Exp}(B_{X^*}) \subseteq B \cap \mathrm{Ext}(B_{X^*}) \subseteq \mathrm{Ext}(B_{X^*}) \subseteq \overline{B}^{\mathrm{weak}^*}.$$

The fact that $\mathrm{Ext}(B_{X^*}) \subseteq \overline{B}^{\mathrm{weak}^*}$ follows from Milman's theorem, [10, page 8] and the fact that $B_{X^*} = \overline{\mathrm{co}}^{\mathrm{weak}^*}(B)$, which in turn follows from a separation argument. Let us also take this opportunity to observe that if B_X denotes the closed unit ball in X, then B_X is closed in the $\sigma(X, B)$ topology for any boundary B of X. Finally, let us end this section with one more simple observation that will turn out to be useful in our later endeavours.

Proposition 2. *Suppose that Y is a linear subspace of a Banach space $(X, \|\cdot\|)$ and B is any boundary for X. Then for each $e^* \in \mathrm{Exp}(B_{Y^*})$ there exists $b^* \in B$ such that $e^* = b^*|_Y$.*

Proof. Suppose that $e^* \in \mathrm{Exp}(B_{Y^*})$. Then there exists an $x \in Y$ such that $y^*(x) < e^*(x)$ for each $y^* \in B_{Y^*} \setminus \{e^*\}$. By the fact that B is a boundary of $(X, \|\cdot\|)$ there exists a $b^* \in B$ such that $b^*(x) = \|x\| \neq 0$. Then for any $y^* \in B_{Y^*}$ we have

$$y^*(x) \leq |y^*(x)| \leq \|y^*\|\|x\| \leq \|x\| = b^*(x) = (b^*|_Y)(x).$$

In particular, $e^*(x) \leq b^*|_Y(x)$. Since $b^*|_Y \in B_{Y^*}$ and $y^*(x) < e^*(x)$ for all $y^* \in B_{Y^*} \setminus \{e^*\}$, it must be the case that $e^* = b^*|_Y$. $\square$

This ends our preliminary section.

3. The main results

Theorem 3. *Let B be any boundary for a Banach space X that is an L_1-predual and suppose that $\{x_n : n \in \mathbb{N}\} \subseteq X$. Then*

$$\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \subseteq \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}.$$

Proof. In order to obtain a contradiction let us suppose that

$$\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \not\subseteq \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}.$$

Choose $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \setminus \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$. Then there exists a finite set $\{e_1^*, e_2^*, \ldots, e_m^*\} \subseteq \mathrm{Ext}(B_{X^*})$ and $\varepsilon > 0$ so that

$$\{y \in X : |e_k^*(x) - e_k^*(y)| < \varepsilon \text{ for all } 1 \leq k \leq m\} \cap \{x_n : n \in \mathbb{N}\} = \varnothing.$$

Let $Y := \overline{\mathrm{span}}(\{x_n : n \in \mathbb{N}\} \cup \{x\})$, let $\mathcal{F}_1$ be any rich family of L_1-preduals whose existence is guaranteed by Theorem 1, and let $\mathcal{F}_2$ be any rich family such that for every $Z \in \mathcal{F}_2$ and every $1 \leq k \leq m$, $e_k^*|_Z \in \mathrm{Ext}(B_{Z^*})$, whose existence is guaranteed by Theorem 2. Next, let us choose $Z \in \mathcal{F}_1 \cap \mathcal{F}_2$ so that $Y \subseteq Z$. Recall that this is possible because, by Proposition 1, $\mathcal{F}_1 \cap \mathcal{F}_2$ is a rich family. Since Z is a separable L_1-predual we have by Lemma 3 that $e_k^*|_Z \in \mathrm{Exp}(B_{Z^*})$ for each $1 \leq k \leq m$. Now, by Proposition 2, for each $1 \leq k \leq m$ there exists a $b_k^* \in B$ such that $e_k^*|_Z = b_k^*|_Z$. Therefore,

$$\begin{aligned} |b_k^*(x) - b_k^*(x_j)| &= |(b_k^*|_Z)(x) - (b_k^*|_Z)(x_j)| \\ &= |(e_k^*|_Z)(x) - (e_k^*|_Z)(x_j)| = |e_k^*(x) - e_k^*(x_j)| \end{aligned}$$

for all $j \in \mathbb{N}$ and all $1 \leq k \leq m$. Thus,

$$\{y \in X : |b_k^*(x) - b_k^*(y)| < \varepsilon \text{ for all } 1 \leq k \leq m\} \cap \{x_n : n \in \mathbb{N}\} = \varnothing.$$

This contradicts the fact that $x \in \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$, which completes the proof. □

Corollary 1 ([13, Theorem 1.1(a)]). *Let B be any boundary for a Banach space X that is an L_1-predual. Then every relatively countably $\sigma(X,B)$-compact subset is relatively countably $\sigma(X, \mathrm{Ext}(B_{X^*}))$-compact. In particular, every norm bounded, relatively countably $\sigma(X,B)$-compact subset is relatively weakly compact.*

Proof. Suppose that a nonempty set $C \subseteq X$ is relatively countably $\sigma(X,B)$-compact. Let $\{c_n : n \in \mathbb{N}\}$ be any sequence in C. Then by Theorem 3

$$\varnothing \neq \bigcap_{n \in \mathbb{N}} \overline{\{c_k : k \geq n\}}^{\sigma(X,B)} \subseteq \bigcap_{n \in \mathbb{N}} \overline{\{c_k : k \geq n\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}.$$

Hence C is relatively countably $\sigma(X, \mathrm{Ext}(B_{X^*}))$-compact. In the case when C is also norm bounded the result follows from either [5] or [8]. □

Recall that a *network* for a topological space X is a family $\mathscr{N}$ of subsets of X such that for any point $x \in X$ and any open neighbourhood U of x there is an $N \in \mathscr{N}$ such that $x \in N \subseteq U$, and a topological space X is said to be $\aleph_0$-*monolithic* if the closure of every countable set has a countable network.

The next corollary generalises [9, Theorem 2.2].

Corollary 2. *Let B be any boundary for a Banach space X that is an L_1-predual and suppose that $\{x_n : n \in \mathbb{N}\} \subseteq X$. Then $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$ is norm separable whenever X is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology. In particular, $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$ is norm separable whenever $\mathrm{Ext}(B_{X^*})$ is weak* Lindelöf.*

Proof. From Theorem 3, $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)} \subseteq \overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$. Since X is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,\mathrm{Ext}(B_{X^*}))}$ has a countable network with respect to the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology and hence so does $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,B)}$. Let $\mathcal{A}(B_{X^*})$ denote the set of all weak* continuous real-valued affine mappings on B_{X^*}. Then the mapping $T : (X, \|\cdot\|) \to (\mathcal{A}(B_{X^*}), \|\cdot\|_\infty)$ defined by $T(x)(x^*) := x^*(x)$ for all $x^* \in B_{X^*}$ is a homeomorphic embedding with respect to both (i) the norm topologies on X and $\mathcal{A}(B_{X^*})$ and (ii) the $\sigma(X, \mathrm{Ext}(X^*))$ and $\tau_p(\mathrm{Ext}(B_{X^*}))$ topologies on X and $\mathcal{A}(B_{X^*})$ respectively. The result then follows from [9, Theorem 2.6]. The last claim follows from [9, Theorem 2.14], where it is shown that if $\mathrm{Ext}(B_{X^*})$ is weak* Lindelöf, then X is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology. $\square$

In [9] many conditions are given under which $\sigma(X, \mathrm{Ext}(B_{X^*}))$ is $\aleph_0$-monolithic.

To demonstrate how this last theorem may be applied we shall present some sample applications.

4. Applications

Our first application is to metrizability of compact convex sets. If K is a compact convex set in a real locally convex space, let $\mathcal{A}(K)$ stand for the space of all affine continuous functions on K.

Proposition 3. *Let K be a Choquet simplex in a separated locally convex space (over $\mathbb{R}$) such that every regular Borel probability measure carried on $\mathrm{Ext}(K)$ is atomic. Then K is metrizable if, and only if, the space $(B_{\mathcal{A}(K)}, \sigma(\mathcal{A}(K), B))$ is separable, for some boundary B of $(\mathcal{A}(K), \|\cdot\|_\infty)$.*

Proof. This follows directly from Theorem 3 and [9, Theorem 2.19]. $\square$

We remark that there exists a nonmetrizable Choquet simplex K and a boundary B of $(\mathcal{A}(K), \|\cdot\|_\infty)$ such that $(B_{\mathcal{A}(K)}, \sigma(A(K), B))$ is separable. (It is shown in [13, Section 4] that the construction of [9, Example 2.10] yields the required example.)

Our final few results concern automatic continuity. In particular, the next result improves [12, Theorem 6].

Proposition 4. *Let B be any boundary for a Banach space X that is an L_1-predual and suppose that A is a a separable Baire space. If X is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology, then for each continuous mapping $f : A \to (X, \sigma(X, B))$ there exists a dense subset D of A such that f is continuous with respect to the norm topology on X at each point of D.*

Proof. Fix $\varepsilon > 0$ and consider the open set:

$$O_\varepsilon := \bigcup \{U \subseteq A : U \text{ is open and } \|\cdot\|\text{-diam}[f(U)] \leq 2\varepsilon\}.$$

We shall show that O_ε is dense in A. To this end, let W be a nonempty open subset of A and let $\{a_n : n \in \mathbb{N}\}$ be a countable dense subset of W. Then by continuity

$$f(W) \subseteq \overline{\{f(a_n) : n \in \mathbb{N}\}}^{\sigma(X,B)},$$

which is norm separable by Corollary 2. Therefore there exists a countable set $\{x_n : n \in \mathbb{N}\}$ in X such that $f(W) \subseteq \bigcup_{n \in \mathbb{N}} (x_n + \varepsilon B_X)$. For each $n \in \mathbb{N}$, let $C_n := f^{-1}(x_n + \varepsilon B_X)$. Since each $x_n + \varepsilon B_X$ is closed in the $\sigma(X, B)$ topology each set C_n is closed in A and, moreover, $W \subseteq \bigcup_{n \in \mathbb{N}} C_n$. Since W is of the second Baire category in A there exist a nonempty open set $U \subseteq W$ and a $k \in \mathbb{N}$ such that $U \subseteq C_k$. Then $U \subseteq O_\varepsilon \cap W$ and O_ε is indeed dense in A. Hence f is $\|\cdot\|$-continuous at each point of $\bigcap_{n \in \mathbb{N}} O_{1/n}$. $\square$

Theorem 4. *Suppose that A is a topological space with countable tightness that possesses a rich family $\mathcal{F}$ of Baire subspaces and suppose that X is an L_1-predual. Then for any boundary B of X and any continuous function $f : A \to (X, \sigma(X, B))$ there exists a dense subset D of A such that f is continuous with respect to the norm topology on X at each point of D provided X is $\aleph_0$-monolithic in the $\sigma(X, \mathrm{Ext}(B_{X^*}))$ topology.*

Proof. In order to obtain a contradiction let us suppose that f does not have a dense set of points of continuity with respect to the norm topology on X. Since A is a Baire space (by [7, Theorem 3.3]), this implies that for some $\varepsilon > 0$ the open set

$$O_\varepsilon := \bigcup\{U \subseteq A : \ U \text{ is open and } \|\cdot\|\text{-diam}[f(U)] \leq 2\varepsilon\}$$

is not dense in A. That is, there exists a nonempty open subset W of A such that $W \cap O_\varepsilon = \varnothing$. For each $x \in A$, let $F_x := \{y \in A : \|f(y) - f(x)\| > \varepsilon\}$. Then $x \in \overline{F_x}$ for each $x \in W$. Moreover, since A has countable tightness, for each $x \in W$, there exists a countable subset C_x of F_x such that $x \in \overline{C_x}$.

Next, we inductively define an increasing sequence of separable subspaces $(F_n : n \in \mathbb{N})$ of A and countable sets $(D_n : n \in \mathbb{N})$ in A such that:

(i) $W \cap F_1 \neq \varnothing$;
(ii) $\bigcup\{C_x : x \in D_n \cap W\} \cup F_n \subseteq F_{n+1} \in \mathcal{F}$ for all $n \in \mathbb{N}$, where D_n is any countable dense subset of F_n.

Note that since the family $\mathcal{F}$ is rich, this construction is possible.

Let $F := \overline{\bigcup_{n\in\mathbb{N}} F_n}$ and $D := \bigcup_{n\in\mathbb{N}} D_n$. Then $\overline{D} = F \in \mathcal{F}$ and $\|\cdot\|\text{-diam}[f(U)] \geq \varepsilon$ for every nonempty open subset U of $F \cap W$. Therefore, $f|_F$ has no points of continuity in $F \cap W$ with respect to the $\|\cdot\|$-topology. This, however, contradicts Proposition 4. □

Our final result improves [7, Theorem 4.7].

Corollary 3. *Suppose that A is a topological space with countable tightness that possesses a rich family of Baire subspaces and suppose that K is a compact Hausdorff space. Then for any boundary of $(C(K), \|\cdot\|_\infty)$ and any continuous function $f : A \to (C(K), \sigma(C(K), B))$ there exists a dense subset D of A such that f is continuous with respect to the $\|\cdot\|_\infty$-topology at each point of D.*

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