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AFFINE INTERVAL EXCHANGE TRANSFORMATIONS WITH FLIPS AND WANDERING INTERVALS

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ABSTRACT. There exist uniquely ergodic affine interval exchange transformations of [0,1] with flips which have wandering intervals and are such that the support of the invariant measure is a Cantor set.

1. INTRODUCTION

Let N be a compact subinterval of either $\mathbb{R}$ or the circle S^1, and let $f : N \to N$ be piecewise continuous. We say that a subinterval $J \subset N$ is a *wandering interval* of the map f if the forward iterates $f^n(J)$, $n = 0, 1, 2, \ldots$, are pairwise disjoint intervals, each not reduced to a point, and the ω-limit set of J is an infinite set.

A great deal of information about the topological dynamics of a map $f : N \to N$ is revealed when one knows whether f has wandering intervals. This turns out to be a subtle question whose answer depends on both the topological and regularity properties of the map f.

The question of the existence of wandering intervals first arose in the case where f is a diffeomorphism of the circle S^1. The Denjoy counterexample shows that even a C^1 diffeomorphism $f : S^1 \to S^1$ may have wandering intervals. This behaviour is ruled out when f is smoother. More specifically, if f is a C^1 diffeomorphism of the circle such that the logarithm of its derivative has bounded variation, then f has no wandering intervals [6]. In this case the topological dynamics of f is simple: if f has no periodic points, then f is topologically conjugate to a rotation.

The first results ensuring the absence of wandering intervals on continuous maps satisfying some smoothness conditions were provided by Guckenheimer [8], Yoccoz [19], and Blokh and Lyubich [2]. Later on, de Melo et al. [13] generalised these results, proving that if N is compact and $f : N \to N$ is a C^2 map with non-flat critical points, then f has no wandering intervals. Concerning discontinuous maps, Berry and Mestel [1] found a condition which excludes wandering intervals in Lorenz maps — interval maps with a single discontinuity. Of course, conservative maps and, in particular, interval exchange transformations admit no wandering intervals. We consider the following generalisation of interval exchange transformations.

Let $0 \leq a < b$ and let $\{a, b\} \subset D \subset [a, b]$ be a discrete set consisting of $n + 1$ points. We say that an injective, continuously differentiable map $T : [a, b] \to [a, b]$

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defined on $\mathcal{D}(T) = [a, b] \setminus D$ is an *affine interval exchange transformation of n subintervals*, or *n-AIET* for short , if $|DT|$ is a positive, locally constant function such that $T([a, b] \setminus D)$ is all of $[a, b]$ except for finitely many points. We also assume that the points in $D \setminus \{a, b\}$ are non-removable discontinuities of T. We say that an AIET is *oriented* if $DT > 0$; otherwise we say that T has *flips*. An *isometric IET of n subintervals*, or n-IET for short, is an n-AIET satisfying $|DT| = 1$ everywhere.

Levitt [11] found an example of a non-uniquely ergodic oriented AIET with wandering intervals. Gutierrez and Camelier [4] constructed an AIET with wandering intervals that is semiconjugate to a self-similar IET. The regularity of conjugacies between AIETs and self-similar IETs was examined by Cobo [5] and by Liousse and Marzougui [12]. Recently, Bressaud, Hubert and Maass [3] provided sufficient conditions for a self-similar IET to have an AIET with a wandering interval semiconjugate to it.

In this paper we present an example of a self-similar IET with flips having the particular property that we can apply the main result of the work [3] to obtain a 5-AIET with flips that is semiconjugate to the IET and has densely distributed wandering intervals. The AIET so obtained is uniquely ergodic [17] (see [14, 18]), and the support of the invariant measure is a Cantor set.

A few remarks are due in order to place this example in context. The existence of minimal non-uniquely ergodic AIETs with flips and wandering intervals would follow by the argument of Levitt [11], provided we know a minimal non-uniquely ergodic IET with flips. However, no example of a minimal non-uniquely ergodic IET with flips is known, although it is possible to insert flips in the example of Keane [10] (for oriented IETs) to get a transitive non-uniquely IET with flips having saddle-connections. Computational evaluations indicate that it is impossible to obtain, via Rauzy induction, examples of self-similar 4-IETs with flips that meet the hypotheses of [3], despite this being possible in the case of oriented 4-IETs (see [4, 5]). Thus, the example we present here is the simplest possible in the sense that wandering intervals do not occur for AIETs with flips that are semiconjugate to a self-similar IET, obtained via Rauzy induction, defined on a smaller number of intervals.

2. Self-similar interval exchange transformations

Let $T : [a, b] \to [a, b]$ be an n-AIET defined on $[a, b] \setminus D$, where $D = \{x_0, \ldots, x_n\}$ and $a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$. Let $\beta_i \neq 0$ be the derivative of T on (x_{i-1}, x_i), $i = 1, 2 \ldots, n$. We shall refer to

$$x = (x_0, x_1, \ldots, x_n)$$

as the *D-vector* of T (i.e. the domain-of-definition-vector of T). The vectors

$$\gamma = (\log|\beta_1|, \log|\beta_2|, \ldots, \log|\beta_n|) \quad \text{and} \quad \tau = \left(\frac{\beta_1}{|\beta_1|}, \frac{\beta_2}{|\beta_2|}, \ldots, \frac{\beta_n}{|\beta_n|}\right)$$

will be called the *log-slope-vector* and the *flips-vector of T*, respectively. Notice that T has flips if and only if some coordinate of τ is equal to -1. Let

$$\{z_1, \ldots, z_n\} = \left\{T\left(\frac{x_0 + x_1}{2}\right), T\left(\frac{x_1 + x_2}{2}\right), \ldots, T\left(\frac{x_{n-1} + x_n}{2}\right)\right\}$$

be such that $0 < z_1 < z_2 < \ldots < z_n < 1$; we define the *permutation* π *associated to* T as the one that takes $i \in \{1, 2, \ldots, n\}$ to $\pi(i) = j$ if and only if $z_j = T((x_{i-1} + x_i)/2)$.

It should be remarked that an AIET $T : [a, b] \to [a, b]$ with flips-vector $\tau \in \{-1, 1\}^n$ which has the zero vector as the log-slope-vector is an IET (with flips-vector τ), and conversely. Let $J = [c, d]$ be a proper subinterval of $[a, b]$. We say that the IET E is *self-similar* (on J) if there exists an orientation preserving affine map $L : \mathbb{R} \to \mathbb{R}$ such that $L(J) = [a, b]$ and $L \circ \widetilde{E} = E \circ L$, where $\widetilde{E} : J \to J$ denotes the IET induced by E and $L(\mathcal{D}(\widetilde{E})) \subset \mathcal{D}(E)$. A self-similar IET $E : [a, b] \to [a, b]$ on a proper subinterval $J \subset [a, b]$ will be denoted by (E, J).

Given an AIET $T : [a, b] \to [a, b]$, the *orbit* of $p \in [a, b]$ is the set

$$O(p) = \{T^n(p) \mid n \in \mathbb{Z} \text{ and } p \in \mathcal{D}(T)\}.$$

The AIET T is called *transitive* if there exists an orbit of T that is dense in $[a, b]$. We say that the orbit of $p \in [a, b]$ is *finite* if $\#(O(p)) < \infty$. In this way, a point $p \in [a, b] - (\mathcal{D}(T) \cup \mathcal{D}(T^{-1}))$ has a finite orbit $O(p) = \{p\}$. A transitive AIET is *minimal* if it has no finite orbits.

Let $E : [a, b] \to [a, b]$ be an IET with D-vector $(x_0, x_1, \cdots, x_n)$. Denote by $J = [c, d]$ a proper subinterval of $[a, b]$. Suppose that E is self-similar (on J); then there exists an IET $\widetilde{E} : J \to J$ such that $L(J) = [a, b]$ and $L \circ \widetilde{E} = E \circ L$. For $i = 0, 1, \cdots, n$, let $y_i = L^{-1}(x_i)$. Thus, the sequence of discontinuities of $\widetilde{E}$ is $\{y_1, \cdots, y_{n-1}\}$.

We say that a non-negative matrix is *eventually positive* if some power of it is a positive matrix. A non-negative matrix is eventually positive if and only if it is both irreducible and aperiodic. Let A be an $n \times n$ non-negative matrix whose entries are:

$$A_{ji} = \#\{0 \le k \le N_i : E^k((y_{i-1}, y_i)) \subset (x_{j-1}, x_j)\},$$

where N_i is the smallest non-negative integer such that for some $y \in (y_{i-1}, y_i)$ (and therefore for all $y \in (y_{i-1}, y_i)$), $E^{N_i+1}(y) \in J$. We shall refer to A as the *matrix associated to* (E, J). Being self-similar, E is also transitive, which implies that A is eventually positive. Hence, by the Perron-Frobenius Theorem [7], A possesses exactly one probability right eigenvector $\alpha \in \Lambda_n$, where

$$\Lambda_n = \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_i > 0 \,\forall i\}.$$

Moreover, the eigenvalue μ corresponding to α is simple, real and greater than 1; also, all other eigenvalues of A have absolute value less than μ. It was proved by Veech [17] (see also [14, 18]) that every self-similar IET is minimal and uniquely ergodic. Furthermore, following Rauzy [16], we conclude that

$$\alpha = (x_1 - x_0, x_2 - x_1, \cdots, x_n - x_{n-1}).$$

3. The theorem of Bressaud, Hubert and Maass

Let $A \in SL_n(\mathbb{Z})$ and let $\mathbb{Q}[t]$ be the ring of polynomials with rational coefficients in one variable. We say that two real eigenvalues θ_1 and θ_2 of A are *conjugate* if there exists an irreducible polynomial $f \in \mathbb{Q}[t]$ such that $f(\theta_1) = f(\theta_2) = 0$. We say that an AIET T of $[0, 1]$ is *semiconjugate* (resp. *conjugate*) to an IET E of $[0, 1]$ if there exists a non-decreasing (resp. bijective) continuous map $h : [0, 1] \to [0, 1]$ such that $h(\mathcal{D}(T)) \subset \mathcal{D}(E)$ and $E \circ h = h \circ T$.

Theorem 1 (Bressaud, Hubert and Maass, 2007)**.** *Let J be a proper subinterval of $[0,1]$, let $E : [0,1] \to [0,1]$ be an interval exchange transformation which is self-similar on J, and let A be the matrix associated to (E,J). Let θ_1 be the Perron-Frobenius eigenvalue of A. Assume that A has a real eigenvalue θ_2 such that*

(1) $1 < \theta_2\ (< \theta_1)$;
(2) θ_1 *and* θ_2 *are conjugate.*

Then there exists an affine interval exchange transformation T of $[0,1]$ with wandering intervals that is semiconjugate to E.

Proof. This theorem was proved in [3] for oriented IETs. The same proof holds word for word for IETs with flips. In this case, the AIET T inherits its flips from the IET E through the previously constructed semiconjugacy. □

4. The interval exchange transformation E

In this section we shall present the IET we shall use to construct the AIET with flips and wandering intervals. We shall need the Rauzy induction [16, 15, 9] to obtain a minimal, self-similar IET whose associated matrix satisfies all the hypotheses of Theorem 1.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \Lambda_5$ be the probability Perron-Frobenius right eigenvector of the matrix

$$A = \begin{pmatrix} 2 & 4 & 6 & 5 & 2 \\ 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 & 0 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 3 & 5 & 4 & 2 \end{pmatrix}.$$

The eigenvalues $\theta_1, \theta_2, \rho_1, \rho_2, \rho_3$ of A are real and have approximate values

$$\theta_1 = 7.829,\ \theta_2 = 1.588,\ \rho_1 = 1,\ \rho_2 = 0.358,\ \rho_3 = 0.225,$$

and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, the probability right eigenvector associated to θ_1, has approximate value

$$\alpha = (0.380, 0.091, 0.070, 0.170, 0.289).$$

In what follows we represent a permutation π of the set $\{1, 2, \ldots, n\}$ by the n-tuple $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$.

We consider the IET $E : [0,1] \to [0,1]$, which is determined by the following conditions:

(1) E has the D-vector $x = (x_0, x_1, x_2, x_3, x_4, x_5)$, where

$$x_0 = 0, \quad x_i = \sum_{k=1}^{i} \alpha_k \text{ for } i = 1, \ldots, 5;$$

(2) E has associated permutation $(5, 3, 2, 1, 4)$;
(3) E has flips-vector $(-1, -1, 1, 1, -1)$.

Lemma 2. *The map E is self-similar on the interval $J = [0, 1/\theta_1]$, and A is precisely the matrix associated to (E, J).*

Proof. We apply the Rauzy algorithm to the IET E. We represent $E : I \to I$ by the pair $E^{(0)} = (\alpha^{(0)}, p^{(0)})$, where $\alpha^{(0)} = \alpha$ is its length vector and $p^{(0)} = (-5, -3, 2, 1, -4)$ is its signed permutation obtained by elementwise multiplication of the permutation $(5, 3, 2, 1, 4)$ with the flips-vector $(-1, -1, 1, 1, -1)$. We shall apply the Rauzy procedure fourteen times, obtaining IETs $E^{(k)} = (\alpha^{(k)}, p^{(k)})$, $k = 0, \ldots, 14$, with D-vector $x^{(k)}$ given by $x_0^{(k)} = 0$ and $x_i^{(k)} = \sum_{j=1}^{i} \alpha_j^{(k)}$ for $i = 1, 2, \ldots, 5$.

k	$p^{(k)}$	$t^{(k)}$
0	-5 -3 2 1 -4	1
1	4 -5 -3 2 1	0
2	5 -2 -4 3 1	1
3	5 1 -2 -4 3	1
4	5 3 1 -2 -4	1
5	5 -4 3 1 -2	0
6	-2 -5 4 1 -3	1
7	-2 3 -5 4 1	0
8	-3 4 -2 5 1	1
9	-3 4 -2 5 1	1
10	-3 4 -2 5 1	0
11	-4 5 -3 2 1	1
12	-4 5 1 -3 2	1
13	-4 5 2 1 -3	0
14	-5 -3 2 1 -4	1

TABLE 1. Rauzy cycle with associated matrix A.

Given an IET $E^{(k)}$, defined on an interval $[0, L^{(k)}]$ and represented by the pair $(\alpha^{(k)}, p^{(k)})$, the IET $E^{(k+1)}$ is defined to be the map induced on the interval $[0, L^{(k+1)}]$ by $E^{(k)}$, where $L^{(k+1)} = L^{(k)} - \min\{\alpha_5^{(k)}, \alpha_s^{(k)}\}$ and s is such that $|p_n^{(k)}(s)| = 5$. We say that the type $t^{(k)}$ of $E^{(k)}$ is 0 if $\alpha_5^{(k)} > \alpha_s^{(k)}$ and 1 if $\alpha_5^{(k)} < \alpha_s^{(k)}$. Notice that $\sum_{i=1}^{5} \alpha_i^{(k)} = L^{(k)}$.

The new signed permutations $p^{(k)}$ obtained by this procedure are given in Table 1, along with the type $t^{(k)}$ of $E^{(k)}$. The length vector $\alpha^{(k+1)}$ is obtained from $\alpha^{(k)}$ by the equation $\alpha^{(k)} = M(p^{(k)}, t^{(k)}) \cdot \alpha^{(k+1)}$, where $M(p^{(k)}, t^{(k)}) \in SL_n(\mathbb{Z})$ is a certain elementary matrix (see [9]). Moreover, we have that

$$M(p^{(0)}, t^{(0)}) \cdot \cdots \cdot M(p^{(13)}, t^{(13)}) = A.$$

Thus $\alpha^{(14)} = A^{-1} \cdot \alpha^{(0)} = \alpha^{(0)}/\theta_1$, and $J = [0, L^{(14)}]$. Notice that $p^{(14)} = p^{(0)}$, so we have a Rauzy cycle: $E^{(14)}$ and $E^{(0)}$ have the same flips-vector and permutation. Hence $\widetilde{E} = E^{(14)}$ is a $1/\theta_1$-scaled copy of $E = E^{(0)}$, and so E is self-similar on the interval J.

As remarked earlier, since E self-similar, we have that the matrix associated to (E, J) is eventually positive. In fact, we have that A is the matrix associated to (E, J). To see this, for $i \in \{0, \ldots, 5\}$ let $y_i = x_i/\theta_1$ be the points of discontinuity for $\widetilde{E}$. Table 2 shows the itinerary $I(i) = \{I(i)_k\}_{k=1}^{N_i}$ of each interval

(y_{i-1}, y_i), where $N_i = \min\{n > 1 : E^{n+1}((y_{i-1}, y_i)) \subset J\}$ and $I(i)_k = r$ if and only if $E^k((y_{i-1}, y_i)) \subset (x_{r-1}, x_r)$.

i	N_i	$I(i)$
1	4	1 5 1 4
2	11	1 5 2 1 4 1 5 2 1 5 4
3	17	1 5 2 1 4 1 5 3 1 5 3 1 5 3 1 5 4
4	14	1 5 2 1 4 1 5 3 1 5 3 1 5 4
5	6	1 5 2 1 5 4

TABLE 2. Itineraries $I(i)$, $i \in \{1, \ldots, 5\}$.

The number of times that j occurs in $I(i)$, for $i, j \in \{1, \ldots, 5\}$, is precisely A_{ji}; thus A is the matrix associated to the pair (E, J), as required. □

Theorem A. *There exists a uniquely ergodic affine interval exchange transformation of* $[0, 1]$ *with flips that has wandering intervals and is such that the support of the invariant measure is a Cantor set.*

Proof. By construction, the matrix A associated to (E, J) satisfies hypothesis (1) of Theorem 1. The characteristic polynomial $p(t)$ of A can be written as the product of two irreducible polynomials over $\mathbb{Q}[t]$:

$$p(t) = (1 - t)(1 - 8t + 18t^2 - 10t^3 + t^4).$$

Thus the eigenvalues θ_1 and θ_2 are zeros of the same irreducible polynomial of degree four and so are conjugate. Hence, A also satisfies hypothesis (2) of Theorem 1, which finishes the proof. □

Note that for an AIET T, the forward and backward iterates of a wandering interval J form a pairwise disjoint collection of intervals. Moreover, when T is semiconjugate to a transitive IET, as is the case in Theorem A, the α-limit set and ω-limit set of J coincide.

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