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ON THE LOCALIZATION PRINCIPLE FOR THE AUTOMORPHISMS OF PSEUDOELLIPSOIDS

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Abstract. We show that Alexander's extendibility theorem for a local automorphism of the unit ball is valid also for a local automorphism f of a pseudoellipsoid $\mathcal{E}^n_{(p_1,\dots,p_k)} \overset{\text{def}}{=} \{z \in \mathbb{C}^n : \sum_{j=1}^{n-k} |z_j|^2 + |z_{n-k+1}|^{2p_1} + \cdots + |z_n|^{2p_k} < 1\}$, provided that f is defined on a region $\mathcal{U} \subset \mathcal{E}^n_{(p)}$ such that: i) $\partial\mathcal{U} \cap \partial\mathcal{E}^n_{(p)}$ contains an open set of strongly pseudoconvex points; ii) $\mathcal{U} \cap \{ z_i = 0 \} \neq \emptyset$ for any $n-k+1 \leq i \leq n$. By the counterexamples we exhibit, such hypotheses can be considered as optimal.

1. Introduction

For a given k-tuple of integers $p = (p_1, \dots, p_k)$, with each $p_\ell \geq 2$, let us denote by $\mathcal{E}^n_{(p_1,\dots,p_k)}$ (or, more simply, $\mathcal{E}^n_{(p)}$) the pseudoellipsoid in $\mathbb{C}^n$ defined by

$$\mathcal{E}^n_{(p_1,\dots,p_k)} \overset{\text{def}}{=} \left\{ z \in \mathbb{C}^n \ : \ \sum_{j=1}^{n-k} |z_j|^2 + |z_{n-k+1}|^{2p_1} + \cdots + |z_n|^{2p_k} < 1 \right\} .$$

When $k = 0$, we assume $\mathcal{E}^n_{(p)}$ to be the unit ball $B^n = \{ z \in \mathbb{C}^n \ : \ |z| < 1 \}$. Now, let us consider the following definition.

Definition 1.1. We define a *local automorphism of* $\mathcal{E}^n_{(p)}$ to be any biholomorphic map $f : \mathcal{U}_1 \subset \mathcal{E}^n_{(p)} \to \mathcal{U}_2 \subset \mathcal{E}^n_{(p)}$ between two connected open subsets of $\mathcal{E}^n_{(p)}$ such that:

a) each of the intersections $\partial\mathcal{U}_i \cap \partial\mathcal{E}^n_{(p)}$, $i = 1, 2$, contains a boundary open set $\Gamma_i \subset \partial\mathcal{E}^n_{(p)}$;
b) there exists at least one sequence $\{ x_k \} \subset \mathcal{U}_1$ which converges to a point $x_o \in \Gamma_1$, which is not a limit point of $\partial\mathcal{U}_1 \cap \mathcal{E}^n_{(p)}$, and so that $\{ f(x_k) \}$ converges to a point $\hat{x}_o \in \Gamma_2$, which is not a limit point of $\partial\mathcal{U}_2 \cap \mathcal{E}^n_{(p)}$.

We say that *a local automorphism* $f : \mathcal{U}_1 \subset \mathcal{E}^n_{(p)} \to \mathcal{U}_2 \subset \mathcal{E}^n_{(p)}$ *extends to a global automorphism of* $\mathcal{E}^n_{(p)}$ if there exists some $F \in \mathrm{Aut}(\mathcal{E}^n_{(p)})$ such that $F|_{\mathcal{U}_1 \cap \mathcal{E}^n_{(p)}} = f|_{\mathcal{U}_1 \cap \mathcal{E}^n_{(p)}}$.

By a celebrated theorem of Alexander and its generalization obtained by Rudin ([Al, Ru]), when $\mathcal{E}^n_{(p)} = B^n$, any local automorphism extends to a global one.

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This crucial extendibility result is often referred to as the *localization principle for the automorphisms of* B^n, and it has been extended or established under different but similar hypotheses for a wide class of domains besides the unit balls (see e.g. [DS, Pi, Pi1]). On the other hand, even if it is known that the pseudoellipsoids $\mathcal{E}^n_{(p)}$ share many useful properties with B^n for what concerns the global automorphisms and the proper holomorphic maps (see for instance [We, La, LS, DS]), some simple examples show that Alexander's theorem cannot be true in full generality for a pseudoellipsoid $\mathcal{E}^n_{(p)}$ different from B^n (see e.g. Example 3.4 below).

Nonetheless, for each $\mathcal{E}^n_{(p)}$, it is possible to determine, precisely and in an efficient way, the class of local automorphisms that can be extended to global ones. In this short note we give a characterization of such local automorphisms by means of the following generalization of Alexander's theorem.

Theorem 1.2. *Let* $f : \mathcal{U}_1 \subset \mathcal{E}^n_{(p)} \to \mathcal{U}_2 \subset \mathcal{E}^n_{(p)}$ *be a local automorphism of a pseudoellipsoid* $\mathcal{E}^n_{(p)}$*, with* $p = (p_1, \dots, p_k)$*, and satisfying the following two conditions:*

i) *there exists a sequence* $\{x_i\}$ *as in (b) of Definition* 1.1*, whose limit point* $x_o \in \partial\mathcal{E}_{(p)}$ *is Levi non-degenerate;*

ii) *for any* $n-k+1 \leq i \leq n$*, the intersection* $\mathcal{U}_1 \cap \{\ z_i = 0\ \}$ *is not empty.*

Then f *extends to a global automorphism* $f \in \mathrm{Aut}(\mathcal{E}^n_{(p)})$.

We point out that the set $\partial\mathcal{E}^n_{(p)} \cap \bigcup_{i=n-k+1}^n\{\ z_i = 0\ \}$ coincides with the set of points of Levi degeneracy of $\partial\mathcal{E}^n_{(p)}$. So, Theorem 1.2 can be roughly stated by saying that f is globally extendible as soon as it admits a holomorphic extension to some open subset $\mathcal{U} \subset \mathcal{E}^n_{(p)}$, which intersects each of the hyperplanes containing the Levi degeneracy set of $\partial\mathcal{E}^n_{(p)}$ and, at the same time, the boundary $\partial\mathcal{U}$ contains an open set of strongly pseudoconvex points of $\partial\mathcal{E}^n_{(p)}$.

From Example 3.4, it will be clear that such hypotheses can be considered as optimal.

The properties of the pseudoellipsoid used in the proof are basically just two: (1) It admits a finite ramified covering over the unit ball; (2) Its automorphisms are "lifts" of the automorphisms of the unit ball that preserve the singular values of the covering. Since (2) is a consequence of (1), it is reasonable to expect that a similar result should be true for any arbitrary ramified covering of the unit ball.

About this more general problem, we refer to [KLS, KS] for what concerns the classification of the domains in $\mathbb{C}^2$ that admit a ramified holomorphic covering over B^2.

2. On the automorphisms of the unit ball

First of all, we need to recall some basic facts on the automorphisms of the unit ball. Let us denote by $\hat{\imath} : \mathbb{C}^n \to \mathbb{C}P^n$ the canonical embedding

$$\hat{\imath} : \mathbb{C}^n \to \mathbb{C}P^n\ , \qquad \hat{\imath}(z) = \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ 1 \end{bmatrix}$$

and let $\hat{\mathbb{C}}^n = \hat{\imath}(\mathbb{C}^n) = \mathbb{C}P^n \setminus \{[w]\ : w_{n+1} = 0\ \}$. We recall that, via the embedding, B^n corresponds to the projective open set $\hat{B}^n = \{\ [w] \in \mathbb{C}P^n\ :\ \langle w, w\rangle < 0\ \}$, where we denote by $\langle , \rangle$ the pseudo-Hermitian inner product on $\mathbb{C}^{n+1}$ defined by

$$\langle w, z\rangle = \bar{w}^t \cdot I_{n,1} \cdot z \ , \qquad \text{where } \ I_{n,1} \stackrel{\text{def}}{=} \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix} . \tag{2.1}$$

It is also known that a holomorphic map $F : B^n \to B^n$ is an automorphism of B^n if and only if the corresponding map $\hat{F} = \hat{\imath} \circ F \circ \hat{\imath}^{-1} : \hat{B}^n \to \hat{B}^n$ is a projective linear transformation which preserves the quadric $\partial \hat{B}^n = \{ \ [w] \ : \ \langle w, w\rangle = 0 \ \}$ (see e.g. [Ve]). This means that $\hat{F}$ is of the form

$$\hat{F}([z]) = [\mathbb{A} \cdot z], \tag{2.2}$$

where $\mathbb{A}$ is a matrix in $\mathrm{SU}_{n,1}$, i.e. such that $\overline{\mathbb{A}}^t I_{n,1} \mathbb{A} = I_{n,1}$ and with $\det \mathbb{A} = 1$.

The correspondence $F \mapsto \hat{F} = \hat{\imath} \circ F \circ \hat{\imath}^{-1}$ gives an isomorphism between $\mathrm{Aut}(B_n)$ and $\mathrm{SU}_{n,1} / K$, where $K = \left\{ \ e^{i \frac{2\pi k}{n+1}} I_{n+1} \ , \ 0 \leq k \leq n \ \right\}$.

The identification of the elements of $\mathrm{Aut}(B^n)$ with the corresponding projective linear transformations is often quite useful, for instance in order to establish the following fact (see also [We], §6).

Lemma 2.1. *Let $F = (F_1, \ldots, F_n) \in \mathrm{Aut}(B^n)$ be an automorphism such that*

$$F\left(B^n \cap \{ \ z_i = 0 \ \}\right) \subset \{ \ z_i = 0 \ \} \tag{2.3}$$

for all $n-k+1 \leq i \leq n$. Then the components F_i are of the following form:

$$F_j(z) = \frac{\sum_{\ell=1}^{n-k} A^\ell_j z_\ell + b_j}{\sum_{\ell=1}^{n-k} c^\ell z_\ell + d} \ , \qquad \text{for } 1 \leq j \leq n-k, \tag{2.4}$$

$$F_j(z) = e^{i\theta_j} z_j \frac{1}{\sum_{\ell=1}^{n-k} c^\ell z_\ell + d} \ , \qquad \text{for } n-k+1 \leq j \leq n, \tag{2.5}$$

for some $\theta_j \in \mathbb{R}$ and where $A = (A^i_j)$, $b = (b_j)$, $c = (c^\ell)$ and d are such that $\begin{pmatrix} A & b \\ c & d \end{pmatrix} \in \mathrm{SU}_{n-k,1}$. In particular, the maps F_j, $1 \leq j \leq n-k$, coincide with the components of an element of $\mathrm{Aut}(B^{n-k})$*, while $\sum_{j=1}^{n-k} c^j z_j + d \neq 0$ for any $z \in B^n$.*

Proof. By hypothesis, the corresponding automorphism $\hat{F} = \hat{\imath} \circ F \circ \hat{\imath}^{-1} \in \mathrm{Aut}(\hat{B}^n)$ maps all hyperplanes $H_i = \{ \ [w] \in \mathbb{C}P^n \ : \ w_i = 0 \ \}$ into themselves and hence fixes their poles relative to the quadric $\partial \hat{B}^n$, i.e. fixes all the points

$$[e_i] = [0 : \ldots : 0 : \underset{i-th\ place}{1} : 0 : \ldots : 0] \ , \qquad n-k+1 \leq i \leq n \ .$$

This implies that the matrix $\mathbb{A}$ which determines the projective transformation $\hat{F}$ is of the form

$$\mathbb{A} = \begin{pmatrix} A & 0 & \ldots & 0 & b \\ 0 & e^{i\theta_{n-k+1}} & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & e^{i\theta_n} & 0 \\ c & 0 & \ldots & 0 & d \end{pmatrix} ,$$

where A, b, c and d are such that $\mathbb{A}' \stackrel{\text{def}}{=} \begin{pmatrix} A & b \\ c & d \end{pmatrix}$ belongs to $\mathrm{SU}_{n-k,1}$. From this, (2.4) and (2.5) follow immediately. The last claim follows from the fact that the value $\sum_{\ell=1}^{n-k} c^\ell z_\ell + d$ is the last homogeneous coordinate of the element

$[\mathbb{A}' \cdot (z_1 : \ldots : z_{n-k} : 1)] \in \mathbb{C}P^{n-k}$ and it is clearly different from 0, since the map $[w] \mapsto [\mathbb{A}' \cdot w]$ is an automorphism of $\hat{B}^{n-k} \subset \mathbb{C}P^{n-k} \setminus \{ w_{n-k+1} \neq 0 \}$. □

3. Proof of Theorem 1.2

First of all, we need to introduce the following notation. For any $p = (p_1, \ldots, p_k)$, we will use the symbol $\pi^{(p)}$ to denote the map

$$\pi^{(p)} : \mathbb{C}^n \to \mathbb{C}^n \ , \qquad \pi^{(p)}(z) = (z_1, \ldots, z_{n-k}, z_{n-k+1}^{p_1}, \ldots, z_n^{p_k}) \ .$$

We recall that the restriction $\pi^{(p)}\big|_{\mathcal{E}^n_{(p)}}$ gives a proper holomorphic map $\pi^{(p)} : \mathcal{E}^n_{(p)} \longrightarrow B^n$.

Secondly, we need to recall a useful theorem by Forstneric and Rosay ([FR]). Given a domain $D \subset \mathbb{C}^n$, we say that a boundary point $z_o \in \partial D$ *satisfies the condition* (P) if:

- ∂D is of class $\mathcal{C}^{1+\varepsilon}$ near z_o for some $\varepsilon > 0$;
- there exist a continuous negative plurisubharmonic function ρ on D and a neighborhood $\mathcal{U}$ of z_o so that $\rho(z) \geq -c\, d(z, \partial D)$ at all points of $\mathcal{U} \cap D$ for some constant $c > 0$.

Theorem 1.1 and some related remarks of [FR] can be summarized as follows.

Theorem 3.1. *Let $h : D \to D'$ be a proper holomorphic map between two domains of $\mathbb{C}^n$ and let $z_o \in \partial D$ be a point that satisfies the condition (P).*

If there exists a sequence $\{z_j\} \subset D$ so that $\lim_{j\to\infty} z_j = z_o$ and $\lim_{j\to\infty} h(z_j) = \hat{z}_o$ for some $\hat{z}_o \in \partial D'$ at which $\partial D'$ is $\mathcal{C}^2$ and strictly pseudoconvex, then h extends continuously to all points of a neighborhood $\mathcal{V}$ of z_o in $\overline{D}$.

We may now prove the following lemma.

Lemma 3.2. *Let $f : \mathcal{U}_1 \subset \mathcal{E}^n_{(p)} \to \mathcal{U}_2 \subset \mathcal{E}^n_{(p)}$ be a local automorphism of a pseudoellipsoid $\mathcal{E}^n_{(p)}$ with $p = (p_1, \ldots, p_k)$ and assume that*

i) *there exists a sequence $\{x_i\}$ as in (b) of Definition* 1.1*, whose limit point $x_o \in \partial\mathcal{E}_{(p)}$ is Levi non-degenerate;*

ii) *for any $n-k+1 \leq i \leq n$, the intersection $\mathcal{U}_1 \cap \{ z_i = 0 \}$ is not empty.*

Then, up to composition with a coordinate permutation,

$$(z_1, \ldots, z_n) \mapsto (z_{\sigma(1)}, \ldots, z_{\sigma(n)}), \tag{3.1}$$

the map f sends the points of the hyperplane $\{ z_i = 0 \}$ into the same hyperplane for any $n-k+1 \leq i \leq n$.

Proof. In all the following we will use the symbols Γ_i, x_o and $\hat{x}_o$ with the same meaning as in Definition 1.1.

First of all, notice that $\hat{x}_o \in \Gamma_2 \subset \partial\mathcal{U}_2$ satisfies the condition (P) and hence, by Theorem 3.1, for any sufficiently small ball $B_\varepsilon(\hat{x}_o)$, centered at $\hat{x}_o$ and of radius ε, the holomorphic map $f^{-1} : \mathcal{U}_2 \to \mathcal{U}_1$ extends continuously to all points of $\overline{B_\varepsilon(\hat{x}_o)} \cap \Gamma_2$. In particular, we may assume that $f^{-1}(\overline{B_\varepsilon(\hat{x}_o) \cap \Gamma_2})$ is contained in a neighborhood of $x_o = f^{-1}(\hat{x}_o)$ in Γ_1 in which there are no Levi degenerate points.

Pick a Levi non-degenerate point $\hat{x}'_o \in \overline{B_\varepsilon(\hat{x}_o)} \cap \Gamma_2$ and consider a sequence $\{\hat{x}'_k\} \subset \overline{B_\varepsilon(\hat{x}_o)} \cap \mathcal{U}_2$ which converges to $\hat{x}'_o$. By construction, the sequence $\{x'_k = f^{-1}(\hat{x}'_k)\} \subset \mathcal{U}_1$ converges to the Levi non-degenerate point $x'_o = f^{-1}(\hat{x}'_o) \in \Gamma_1$. It follows that, replacing x_o by x'_o and $\hat{x}_o$ by $\hat{x}'_o$ and by Theorem 3.1 applied to

f and f^{-1}, there is no loss of generality if we assume that x_o and $\hat{x}_o$ are both Levi non-degenerate and that, for any sufficiently small $\varepsilon_1 > 0$, the map f extends continuously to a map

$$f : \mathcal{U}_1 \cup \left(\overline{B_{\varepsilon_1}(x_o)} \cap \Gamma_1 \right) \to \mathcal{U}_2 \cup \left(B_{\varepsilon}(\hat{x}_o) \cap \Gamma_2 \right),$$

which is a homeomorphism onto its image.

Since the complex Jacobian matrices $J\pi^{(p)}\big|_{x_o}$ and $J\pi^{(p)}\big|_{\hat{x}_o}$ are of maximal rank (recall that x_o and $\hat{x}_o \in \partial\mathcal{E}^n_{(p)}$ are both Levi non-degenerate), from the fact that x_o is not a limit point of $\partial\mathcal{U}_1 \cap \mathcal{E}^n_{(p)}$ and by the continuity of f and f^{-1} around x_o and $\hat{x}_o$, respectively, we may choose ε_1 and ε_2 so that:

a) $\pi^{(p)}|_{B_{\varepsilon_1}(x_o)}$ and $\pi^{(p)}|_{B_{\varepsilon_2}(\hat{x}_o)}$ are both biholomorphisms onto their images;

b) $\overline{f(B_{\varepsilon_1}(x_o) \cap \mathcal{U}_1)} \subset B_{\varepsilon_2}(\hat{x}_o)$ and $f|_{B_{\varepsilon_1}(x_o)\cap\mathcal{U}_1}$ extends to a homeomorphism between $\overline{B_{\varepsilon_1}(x_o) \cap \mathcal{U}_1}$ and $\overline{f(B_{\varepsilon_1}(x_o) \cap \mathcal{U}_1)}$ which induces a homeomorphism between $B_{\varepsilon_1}(x_o) \cap \Gamma_1$ and $f(B_{\varepsilon_1}(x_o) \cap \Gamma_1) \subset \Gamma_2$.

Notice that, by definition, x_o is not a limit point of $\partial\left(B_{\varepsilon_1}(x_o) \cap \mathcal{U}_1\right) \cap \mathcal{E}^n_{(p)}$ and, by (b), $\hat{x}_o$ is not a limit point of $\partial f\left(B_{\varepsilon_1}(x_o) \cap \mathcal{U}_1\right) \cap \mathcal{E}^n_{(p)}$. So, if we set

$$\mathcal{U}'_1 \overset{\text{def}}{=} B_{\varepsilon_1}(x_o) \cap \mathcal{U}_1 , \quad \mathcal{U}'_2 \overset{\text{def}}{=} f(\mathcal{U}'_1) \subset B_{\varepsilon_2}(\hat{x}_o) , \quad \mathcal{V}_i \overset{\text{def}}{=} \pi^{(p)}(\mathcal{U}'_i) \quad i = 1, 2 ,$$

then the maps

$$f|_{\mathcal{U}'_1} : \mathcal{U}'_1 \to \mathcal{U}'_2$$

and

$$\tilde{f} = \pi^{(p)} \circ f \circ \pi^{(p)-1}\Big|_{\mathcal{V}_1} : \mathcal{V}_1 \subset B^n \longrightarrow \mathcal{V}_2 \subset B^n$$

are local automorphisms of $\mathcal{E}^n_{(p)}$ and of the unit ball, respectively.

By Rudin's generalization of Alexander's theorem ([Ru]), this implies that $\tilde{f}$ extends to a global automorphism of B^n, which we denote by $\tilde{f}$ as well. By construction, for any $z \in \mathcal{U}'_1 = \pi^{(p)-1}(\mathcal{V}_1)$, we have

$$\tilde{f} \circ \pi^{(p)}(z) = \pi^{(p)} \circ f(z), \tag{3.2}$$

but since both sides have a holomorphic extension on $\mathcal{U}_1$, we get that (3.2) must be true also for any z in such a larger set.

In particular,

$$J(\tilde{f})|_{\pi^{(p)}(z)} \cdot J(\pi^{(p)})|_z = J(\pi^{(p)})|_{f(z)} \cdot J(f)|_z , \qquad \text{for any } z \in \mathcal{U}_1. \tag{3.3}$$

Since for any $z \in \mathcal{U}_1$, $\det J(f)|_z \neq 0$ and

$$\{ \; J(\pi^{(p)})|_z = 0 \; \} = \bigcup_{i=n-k+1}^{n} \{ \; z_i = 0 \; \}, \tag{3.4}$$

equality (3.3) implies that, for any $n-k+1 \leq i \leq n$ and $z \in \mathcal{U}_1 \cap \{ \; z_i = 0 \; \}$, the value of $J(\pi^{(p)})|_{f(z)}$ is 0. By (3.4), this means that $f\left(\mathcal{U}_1 \cap \{ \; z_i = 0 \; \}\right)$ is contained in the union $\bigcup_{j=n-k+1}^{n} \{ \; z_j = 0 \; \}$. Indeed, it is contained in exactly one of the hyperplanes $\{z_j = 0\}$, because f is a biholomorphism and consequently $f\left(\mathcal{U}_1 \cap \{ \; z_i = 0 \; \}\right)$ is an irreducible analytic variety. From this the conclusion follows. □

We proceed by defining a rule that associates an automorphism of B^n with any local automorphism of a pseudoellipsoid (see also [We], §6). Given a local automorphism $f : \mathcal{U} \to \mathbb{C}^n$ of $\mathcal{E}^n_{(p)}$, pick a point $x_o \in \mathcal{U} \cap \partial\mathcal{E}^n_{(p)}$ for which (b) of

Definition 1.1 holds and determine a small ball $B_\varepsilon(x_o)$ centered in x_o as in the proof of the previous lemma. Then, we denote by $\tilde f \in \mathrm{Aut}(B^n)$ the global automorphism of the unit ball that extends $\tilde f \overset{\mathrm{def}}{=} \pi^{(p)} \circ f \circ \pi^{(p)-1}|_{\pi^{(p)}(\mathcal{V})}$, with $\mathcal{V} \overset{\mathrm{def}}{=} B_\varepsilon(x_o) \cap \mathcal{E}^n_{(p)}$. By the identity principle of the holomorphic maps, such an automorphism $\tilde f$ depends only on f and will be called *the (global) automorphism of B^n associated with f.*

With the help of such a correspondence, we may state the following criterion for extendibility of local automorphisms.

Proposition 3.3. *A local automorphism $f : \mathcal{U}_1 \subset \mathcal{E}^n_{(p)} \to \mathcal{U}_2 \subset \mathcal{E}^n_{(p)}$ of a pseudoellipsoid $\mathcal{E}^n_{(p)}$, $p = (p_1, \ldots, p_k)$, extends to a global automorphism $f \in \mathrm{Aut}(\mathcal{E}^n_{(p)})$ if and only if its associated automorphism $\tilde f \in \mathrm{Aut}(B^n)$ satisfies* (2.3) *for any $n - k + 1 \le i \le n$, up to composition with a permutation of those coordinates z_{n-k+j}, for which the integers p_j are of the same value.*

Proof. Assume that the local automorphism $f : \mathcal{U} \to \mathbb{C}^n$ extends to a global automorphism $f \in \mathrm{Aut}(\mathcal{E}^n_{(p)})$ and recall that, by construction, the associated automorphism $\tilde f \in \mathrm{Aut}(B^n)$ satisfies (3.2) at all points where f is defined (in this case, at all points of $\mathcal{E}^n_{(p)}$). Then, by Lemma 3.2 and the fact that $\pi^{(p)}\left(\mathcal{E}^n_{(p)} \cap \{\ z_i = 0\ \}\right) = B^n \cap \{\ z_i = 0\ \}$, the equality (3.3) implies that, up to a suitable permutation of coordinates, $\tilde f$ satisfies (2.3) for any $n - k + 1 \le i \le n$.

Conversely, assume that $f = (f_1, \ldots, f_n) : \mathcal{U}_1 \subset \mathcal{E}^n_{(p)} \to \mathcal{U}_2 \subset \mathcal{E}^n_{(p)}$ is a local automorphism of $\mathcal{E}^n_{(p)}$ such that (up to a suitable permutation of coordinates) the associated automorphism $\tilde f = (\tilde f_1, \ldots, \tilde f_n) \in \mathrm{Aut}(B^n))$ satisfies (2.3) for any $n - k + 1 \le i \le n$. From (2.4), (2.5) and (3.2), it follows that the components f_j of f are of the form

$$f_j(z) = \frac{\sum_{\ell=1}^{n-k} A^\ell_j z_\ell + b_j}{\sum_{\ell=1}^{n-k} c^\ell z_\ell + d}\ , \qquad \text{for } 1 \le j \le n-k, \tag{3.5}$$

$$f_{n-k+j}(z) = e^{i\theta_j} z_j \frac{1}{\left(\sum_{\ell=1}^{n-k} c^\ell z_\ell + d\right)^{\frac{1}{p_j}}}\ , \qquad \text{for } 1 \le j \le k, \tag{3.6}$$

for some fixed definitions of the p_j-th roots $w \mapsto w^{\frac{1}{p_j}}$.

From (3.5) and (3.6) it follows immediately that f coincides with a globally defined automorphism of $\mathcal{E}^n_{(p)}$ (for the general expressions of the elements in $\mathrm{Aut}(\mathcal{E}^n_{(p)})$, see [We, La]). □

Now, Theorem 1.2 follows almost immediately. In fact, if $f : \mathcal{U}_1 \subset \mathcal{E}^n_{(p)} \to \mathcal{U}_2 \subset \mathcal{E}^n_{(p)}$ is a local automorphism satisfying the hypotheses of the theorem, by Lemma 3.2 and (3.2), the associated automorphism $\tilde f \in \mathrm{Aut}(B^n)$ satisfies the hypotheses of Proposition 3.3 and the claim follows.

We conclude with the following simple construction of non-extendible local automorphisms of pseudoellipsoids.

Example 3.4. Let $\tilde f \in \mathrm{Aut}(B^n)$ be an automorphism which does not satisfy (2.3) for some $n - k + 1 \le j \le n$. Pick a point $w_o \in \partial B \cap \{\ \prod_{j=n-k+1}^n z_j \ne 0\ \}$ so that also its image $\tilde f(w_o)$ is in $\partial B \cap \{\ \prod_{j=n-k+1}^n z_j \ne 0\ \}$. Then, let $z_o \in \partial \mathcal{E}^n_{(p)}$ so that $\pi^{(p)}(z_o) = w_o$ and consider a connected neighborhood $\mathcal{U}$ of z_o with the following two properties: a) $\pi^{(p)}|_{\mathcal{U}}$ is a biholomorphism between $\mathcal{U}$ and its image $\pi^{(p)}(\mathcal{U})$; b) $\tilde{f}(\pi^{(p)}(\mathcal{U}))$ does not intersect $\left\{ \prod_{j=n-k+1}^{n} z_j = 0 \right\}$ (a sufficiently small neighborhood $\mathcal{U}$ surely satisfies both requirements). Then, we may consider the map

$$f : \mathcal{U}_1 = \mathcal{U} \cap \mathcal{E}^n_{(p)} \to \mathcal{U}_2 = f(\mathcal{U}) \cap \mathcal{E}^n_{(p)} \ , \qquad f \overset{\text{def}}{=} \pi^{(p)-1} \circ \tilde{f} \circ \pi^{(p)} \ .$$

By construction, f is a local automorphism of $\mathcal{E}^n_{(p)}$ and its associated automorphism of $\mathrm{Aut}(B^n)$ is $\tilde{f}$. By the hypotheses on $\tilde{f}$ and by Proposition 3.3, f cannot extend to a global automorphism of $\mathcal{E}^n_{(p)}$.

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