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THE BOUNDING GENERA AND w-INVARIANTS

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Abstract. In this paper, we give an estimate from below of the bounding genera for homology 3-spheres defined by Y. Matsumoto in terms of w-invariants. In particular, combining with Matsumoto's estimates we determine the values of the bounding genera for several infinite families of Brieskorn homology 3-spheres.

1. Introduction

In this paper, we give an estimate from below of the bounding genera for homology 3-spheres defined by Y. Matsumoto in terms of w-invariants. In particular, combining with Matsumoto's estimates we determine the values of the bounding genera for several infinite families of Brieskorn homology 3-spheres.

In 1982, Y. Matsumoto introduced the notion of a bounding genus for integral homology 3-spheres to study the kernel of the Rohlin invariant. Let Γ be a non-singular symmetric bilinear form over $\mathbb{Z}$. A homology 3-sphere Σ is said to bound the form Γ if and only if Σ bounds a compact, oriented, homologically 1-connected smooth 4-manifold W whose intersection form defined on $H_2(W)$ is isomorphic to Γ. Here a topological space X is said to be homologically 1-connected if it is connected and $H_1(X) = \{0\}$. Let H be the hyperbolic form, i.e., the intersection form of $S^2 \times S^2$. Then the bounding genus is defined as follows.

Definition 1.1 (Y. Matsumoto [11])**.** Let Σ be a homology 3-sphere. Then the bounding genus $|\Sigma|$ of Σ is defined to be

$$|\Sigma| := \begin{cases} \min\{n \,|\Sigma \text{ bounds } nH\}, & \mu(\Sigma) = 0, \\ +\infty, & \mu(\Sigma) = 1, \end{cases}$$

where $\mu(\Sigma)$ is the Rohlin invariant of Σ.

Remark 1.2. If the Rohlin invariant $\mu(\Sigma)$ of the homology 3-sphere Σ vanishes, then Σ bounds a smooth spin 4-manifold W with signature $\mathrm{Sign}(W)$ divisible by 16. By taking the connected sum with several copies of $K3$ surfaces or the $K3$ surface with reversed orientation, if necessary, we may assume that $\mathrm{Sign}(W) = 0$ and hence W is an indefinite spin 4-manifold. It is known that the intersection form of indefinite spin 4-manifolds is isomorphic to the direct sum of several copies of the hyperbolic form H.

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Remark 1.3. The bounding genus $|\Sigma|$ gives a homology cobordism invariant; i.e. it gives a map $|\ |: \Theta_3^H \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ from the homology cobordism group Θ_3^H of homology 3-spheres.

Remark 1.4. The bounding genus $|\Sigma|$ satisfies the triangle inequality $|\Sigma + \Sigma'| \leq |\Sigma| + |\Sigma'|$ and in fact gives a distance in Θ_3^H allowing the value to be infinity.

Remark 1.5. The notion of 1-connected bounding genus $\|\Sigma\|$ is also defined by replacing "homological 1-connectedness" by ordinary "1-connectedness" in the definition of the bounding genus $|\Sigma|$. Clearly the inequality $|\Sigma| \leq \|\Sigma\|$ holds.

Matsumoto gave upper estimates on the bounding genera for several families of homology 3-spheres using Dehn-Kirby calculus. For example, he gave the following estimates.

Proposition 1.6 (Y. Matsumoto [11, §4, Proposition 4.4])**.** $|\Sigma\,(2,7,14m-1)| \leq 3$ *for any positive odd integer* m*.*

For example, the bounding genus of the Brieskorn homology 3-sphere $\Sigma(2,7,13)$ satisfies $|\Sigma(2,7,13)| \leq 3$. Matsumoto called this estimate "hard-to-improve". In fact, R. Kirby proved that $\Sigma\,(2,7,13)$ bounds the plumbed 4-manifold $P\,(\Gamma_{16})$ associated to the intersection form Γ_{16}. Hence if $\Sigma\,(2,7,13)$ bounds $2 \cdot H$, then the closed 4-manifold $M = -P\,(\Gamma_{16}) \cup |2 \cdot H|$ obtained by gluing $-P\,(\Gamma_{16})$ and $|2 \cdot H|$ along the boundary $\Sigma\,(2,7,13)$ leads to the inequality

$$\frac{11}{8}\,|\mathrm{Sign}\,(M)| = \frac{11}{8} \cdot |-16| > 16 + 4 = b_2\,(M)\,,$$

which violates the following 11/8-conjecture proposed by Y. Matsumoto [11].

Conjecture 1.7 (Y. Matsumoto [11])**.** *Let* M *be a closed spin* 4*-manifold. Then the following inequality holds:*

$$\frac{11}{8}|\mathrm{Sign}(M)| \leq b_2\,(M)\,.$$

To determine the bounding genera we need an estimate from below. In fact, M. Furuta proved an inequality called the 10/8-inequality close to the 11/8-conjecture by using the finite-dimensional approximation of the Seiberg-Witten monopole equation on closed spin 4-manifolds.

Theorem 1.8 (M. Furuta [9])**.** *For any closed spin* 4*-manifold* M *with* $\mathrm{Sign}\,(M) \neq 0$*, the following inequality holds:*

$$\frac{10}{8}\,|\mathrm{Sign}\,(M)| + 2 \leq b_2\,(M)\,.$$

If we apply this inequality to $M = -P\,(\Gamma_{16}) \cup |2 \cdot H|$, then we have the inequality

$$\frac{10}{8}\,|-16| + 2 = 22 > 20 = 16 + 4$$

violating the 10/8-inequality above and hence $|\Sigma(2,7,13)| = 3$. For other Brieskorn homology 3-spheres Σ, we need to find "good" spin 4-manifolds such as $P\,(\Gamma_{16})$ which Σ bounds.

In a joint work with M. Furuta [7], we used a V-manifold version of the 10/8-inequality to define a homology cobordism invariant for a class of homology 3-spheres which we call the w-invariant. The notion of V-manifold is defined by I. Satake [15] as a generalization of manifolds which allows neighborhoods to be the quotients of Euclidean spaces divided by finite group actions. The w-invariant can be considered as the Seiberg-Witten theory counterpart of the invariant [3] of R. Fintushel and R. Stern defined by using the Donaldson theory. In fact, the w-invariant is defined for a triple (Σ, X, c) composed of a homology 3-sphere Σ, a compact smooth spin 4-V-manifold X with boundary Σ, and a V-spin^c structure c on X, and it takes values in the integers, $w(\Sigma, X, c) \in \mathbb{Z}$. If the V-spin^c structure c comes from the V-spin structure on X, then the value $w(\Sigma, X, c)$ modulo 2 is equal to Rohlin's μ-invariant.

By using this invariant $w(\Sigma, X, c)$, we give the following estimate on bounding genera from below whose proof will be given in Section 2.

Theorem 1.9. *Let Σ be an integral homology 3-sphere bounding a compact smooth spin 4-V-manifold X with V-spin^c structure c which comes from a V-spin structure on X. Then the following inequalities hold:*

(1) *If $w(\Sigma, X, c) > 0$, then $|\Sigma| \geq w(\Sigma, X, c) - b_2^+(X) + 1$.*
(2) *If $w(\Sigma, X, c) < 0$, then $|\Sigma| \geq -w(\Sigma, X, c) - b_2^-(X) + 1$.*

As in the case of smooth manifolds, we need to find a "good" spin 4-V-manifold X to give an efficient estimate. However, for Seifert homology 3-spheres $\Sigma = \Sigma(a_1, \ldots, a_n)$, we can take X to be the canonical D^2-V-bundle $X \to S^2$ over S^2 associated to the Seifert fibration $\Sigma \to S^2$. Then X is a 4-V-manifold with n-singular points which are cones over lens spaces and with $b_2^+(X) = 0$, $b_2^-(X) = 1$. If one of the a_i's is even, then X admits a unique V-spin structure c on X. For example, the value of the w-invariant of the Brieskorn homology 3-sphere $\Sigma(2,7,13)$ is $w(\Sigma(2,7,13), X, c) = 2 > 0$. Hence by Theorem 1.9, we see that $|\Sigma(2,7,13)| \geq 2 - 0 + 1 = 3$. Therefore the bounding genus of $\Sigma(2,7,13)$ is certainly $|\Sigma(2,7,13)| = 3$. In Section 3, we will prove the following:

Proposition 1.10. $|\Sigma(2,7,14m-1)| = 3$ *for any positive odd integer m.*

R. Fintushel and R. Stern defined the invariant $R(a_1, \ldots, a_n)$ for Seifert homology 3-spheres $\Sigma(a_1, \ldots, a_n)$ by using the Donaldson theory and proved that if $R(a_1, \ldots, a_n) > 0$, then $\Sigma(a_1, \ldots, a_n)$ cannot be the boundary of an acyclic 4-manifold [3]. Hence if $R(a_1, \ldots, a_n) > 0$, then we can show that $|\Sigma(a_1, \ldots, a_n)| \geq 1$. Matsumoto proved for example that $|\Sigma(2,3,12k-1)| \leq 1$, whereas the w-invariant of $\Sigma(2,3,11)$ is zero, and hence the above Theorem 1.9 cannot be applied. However we see that $R(2,3,11) > 0$ and therefore $|\Sigma(2,3,11)| = 1$. In Section 3, we will prove the following:

Proposition 1.11. $|\Sigma(2,3,12k-1)| = 1$ *for any non-negative integer k.*

2. Bounding genera and w-invariants

First we recall the V-manifold version of the 10/8-inequality.

Theorem 2.1 ([7])**.** *Let X be a closed smooth spin 4-V-manifold. Fix a Riemannian V-metric on X and let $D(X)$ be the positive chiral Dirac operator. Suppose the V-index of the Dirac operator is positive:* $\mathrm{ind}_V D(X) > 0$. *Then the following inequality holds:*

$$\mathrm{ind}_V D(X) + 1 \leq b_2^+(X).$$

The w-invariant is defined as follows.

Definition 2.2. Let (Σ, X, c) be a triple composed of a homology 3-sphere Σ, a compact smooth spin 4-V-manifold X with boundary $\partial X \cong \Sigma$, and a V-spin^c structure c on X. Then we define

$$w(\Sigma, X, c) := \mathrm{ind}_V D(X \cup_\Sigma W) + \frac{\mathrm{Sign}(W)}{8},$$

where W is a smooth spin 4-manifold with boundary $\partial W \cong -\Sigma$.

Remark 2.3. $w(\Sigma, X, c)$ does not depend on the choice of W and its spin structure by the excision properties of V-indices and the fact that the L-genus is (-8)-times the $\hat{A}$-genus. Moreover, if the V-spin^c structure c comes from a V-spin structure, then $w(\Sigma, X, c)$ is equal to the Rohlin invariant $\mu(\Sigma)$ modulo 2. $w(\Sigma, X, c)$ may depend on the choice of X and c, but Theorem 2.1 implies a homology cobordism invariance of $w(\Sigma, X, c)$ in a certain class of homology 3-spheres Σ including the set of all Seifert homology 3-spheres [7].

By using this Theorem 2.1, we give a proof of Theorem 1.9.

Proof of Theorem 1.9. Let $m = |\Sigma|$ be the bounding genus of Σ. Then Σ bounds a homologically 1-connected compact oriented smooth spin 4-manifold W_m with intersection form mH. This implies that $b_2^\pm(W_m) = m$ and Sign $W_m = 0$. On the other hand, let X be a closed spin 4-V-manifold X with V-spin structure c with boundary $\partial X \cong \Sigma$. Let Z be a closed spin 4-V-manifold obtained by gluing X and $-W_m$ along the boundary Σ. Then we have $b_2^\pm(Z) = b_2^\pm(X) + m$. Note that

$$w(\Sigma, X, c) = \mathrm{ind}_V D(Z) + \frac{\mathrm{Sign}\ W_m}{8} = \mathrm{ind}_V D(Z).$$

Suppose that $w(\Sigma, X, c) > 0$. Then by Theorem 2.1 we have

$$w(\Sigma, X, c) = \mathrm{ind}_V D(Z) \leq b_2^+(Z) - 1 = b_2^+(X) + m - 1.$$

Similarly, if $w(\Sigma, X, c) < 0$, then we apply Theorem 2.1 by replacing X with $-X$ and by noting $b_2^+(-Z) = b_2^-(Z)$ and $\mathrm{ind}_V D(-Z) = -\mathrm{ind}_V D(Z)$ to get

$$\begin{aligned} -w(\Sigma, X, c) &= -\mathrm{ind}_V D(Z) = \mathrm{ind}_V D(-Z) \\ &\leq b_2^+(-Z) - 1 = b_2^-(Z) - 1 = b_2^-(X) + m - 1. \end{aligned}$$

Hence the assertion follows. □

3. Bounding genera of Brieskorn homology 3-spheres

In this section, we calculate w-invariants of Brieskorn homology 3-spheres to give estimates of bounding genera from below, and combining with Matsumoto's result we determine the bounding genera for several examples of Brieskorn homology 3-spheres.

The explicit formula of the w-invariant for the Brieskorn homology 3-spheres is given in a joint work with M. Furuta [7], and more generally, the invariant for the homology 3-spheres of plumbing type [13] is calculated in [6] by using the Kawasaki V-index formula [10]. In fact, the w-invariant of plumbed homology 3-spheres $\Sigma(\Gamma)$ is essentially equal to the $\bar{\mu}$-invariant defined by W. Neumann [12] and L. Siebenmann [17] as follows.

Theorem 3.1 (N. Saveliev [16]; cf. Y. Fukumoto-M. Furuta-M. Ue [8], [5])**.** *Let $\Sigma(\Gamma)$ be a plumbed homology 3-sphere associated to a weighted tree graph Γ. Then there exists a decoration $\hat{\Gamma}$ of Γ and a V-spin structure $\hat{c}$ on the associated plumbed 4-V-manifold $P(\hat{\Gamma})$ such that*

$$w(\Sigma(\Gamma), P(\hat{\Gamma}), \hat{c}) = -\bar{\mu}(\Sigma(\Gamma)).$$

To apply Theorem 1.9 efficiently, we must find a "good" spin 4-V-manifold X to evaluate the w-invariant $w(\Sigma, X, c)$. For Seifert homology 3-spheres $\Sigma = \Sigma(a_1, \ldots, a_n)$, we can take the canonical V-manifold X to be the total space of the D^2-V-bundle over the V-sphere S^2 associated with the Seifert fibration $\Sigma \to S^2$ which can be regarded as an S^1-V-bundle over a V-sphere S^2. If one of the a_i's is even, then X admits a unique V-spin structure c. In this case, we can take a spin resolution $P(\Gamma)$ of X with an even weighted star-shaped graph Γ which satisfies $\Sigma(\Gamma) \cong \Sigma$. Then a decoration $\hat{\Gamma}$ of Γ is obtained by drawing circles enclosing linear arms emanating from the central vertex, and the plumbed V-manifold $P(\hat{\Gamma})$ is diffeomorphic to X with induced V-spin structure $\hat{c}$ isomorphic to c. Then by Theorem 3.1, we have $w(\Sigma, P(\hat{\Gamma}), \hat{c}) = -\bar{\mu}(\Sigma(\Gamma))$. When all a_i's are odd, we must take other choices of X, such as plumbed V-manifolds $P(\hat{\Gamma})$ for some decorated plumbing graph $\hat{\Gamma}$ [16] or "4-dimensional Seifert fibrations" [8].

In the following, we list several results by Y. Matsumoto [11] of estimates on the bounding genera of Brieskorn homology spheres and calculate w-invariants to determine the bounding genera.

Proposition 3.2 ([11, §4, Proposition 4.4])**.** *Let p, q, m be positive integers with $\gcd(p,q) = 1$. Then*

(1) $\|\Sigma(p, q, pqm \pm 1)\| \leq 1$ *for m even;*
(2) *if m is odd and* $\operatorname{Arf}(K(p,q)) = 0$*, then*

$$\|\Sigma(p, q, pqm \pm 1)\| \leq (p-1)(q-1)/2.$$

Remark 3.3. The Arf invariant of the (p,q)-torus knot $K(p,q)$ is as follows [11, §4, Remark]:

$$\operatorname{Arf}(K(p,q)) = \operatorname{Arf}(K(q,p)) = \begin{cases} \left(1-p^2\right)/8 \pmod 2, & p : \text{odd},\ q : \text{even}, \\ 0, & p, q : \text{odd}. \end{cases}$$

Example 3.4. $\|\Sigma(2,3,11)\| \leq 1$, $\|\Sigma(2,7,13)\| \leq 3$.

As an application of Theorem 1.9 in this case, we have the following:

Proposition 3.5. *Let p, q be coprime positive integers and m be a positive odd integer.*

(1) *If $w(p, q, pq \pm 1) > 0$, then*

$$|\Sigma(p, q, pqm \pm 1)| \geq w(p, q, pq \pm 1) + 1;$$

and

(2) *if $w(p, q, pq \pm 1) < 0$, then*

$$|\Sigma(p, q, pqm \pm 1)| \geq -w(p, q, pq \pm 1).$$

Proof. Let k be a non-negative integer such that $m = 2k+1$. Let X be the disk V-bundle over S^2 associated with the Seifert fibration $\Sigma(p, q, pqm \pm 1)$. Then $b_2^+(X) = 0$ and $b_2^-(X) = 1$. Since one of the $p, q, pqm \pm 1$ is even, X admits a unique V-spin structure. By Theorem 3.1 and a formula of W. Neumann [12], $\bar{\mu}(\Sigma(p,q,r)) = \bar{\mu}(\Sigma(p,q,r+2kpq))$ for any non-negative integer k, we have

$$\begin{aligned} w(p,q,pqm \pm 1) &= w(p,q,pq(2k+1) \pm 1) = w(p,q,pq \pm 1 + 2kpq) \\ &= -\bar{\mu}(\Sigma(p,q,pq \pm 1 + 2kpq)) = -\bar{\mu}(\Sigma(p,q,pq \pm 1)) \\ &= w(p,q,pq \pm 1). \end{aligned}$$

Hence the assertion follows. □

The following is an application of Proposition 3.5.

Proposition 3.6. $|\Sigma(2,7,14m-1)| = 3$ *for any positive odd integer* m.

Proof. The w-invariant of $\Sigma(2,7,13)$ is $w(2,7,13) = 2 > 0$ and hence by Proposition 3.5, we have

$$|\Sigma(2,7,14m-1)| \geq w(2,7,13) - 0 + 1 = 2 - 0 + 1 = 3.$$

On the other hand, by Matsumoto's estimate

$$|\Sigma(2,7,14m-1)| = |\Sigma(2,7,14 \cdot (2k-1) - 1)| \leq \frac{(2-1)(7-1)}{2} = 3.$$

Therefore $|\Sigma(2,7,14m-1)| = 3$ for any odd m. □

The following two propositions are cases where we could not determine the bounding genera.

Proposition 3.7. $2 \leq |\Sigma(2,7,14m+1)| \leq 3$ *for any positive odd integer* m.

Proof. The w-invariant of $\Sigma(2,7,15)$ is $w(2,7,15) = -2 < 0$ and hence by Proposition 3.5, we have $|\Sigma(2,7,14m+1)| \geq -(-2) - 1 + 1 = 2$. On the other hand, by Matsumoto's estimate $|\Sigma(2,7,14m+1)| \leq (2-1)(7-1)/2 = 3$. Hence the assertion follows. □

Proposition 3.8. $2 \leq |\Sigma(3,5,15m+1)| \leq 4,\ 3 \leq |\Sigma(3,5,15m-1)| \leq 4$ *for any positive odd integer* m.

Proof. The w-invariant of $\Sigma(3,5,16)$ is calculated to be $w(3,5,16) = -2 < 0$ and hence by Proposition 3.5, $|\Sigma(3,5,15m+1)| \geq (-2) - 1 + 1 = 2$ for m odd. On the other hand, the w-invariant of $\Sigma(3,5,14)$ is calculated to be $w(3,5,14) = 2 > 0$ and hence by Proposition 3.5 $|\Sigma(3,5,15m-1)| \geq 3$ for m odd. Since $\mathrm{Arf}(K(3,5)) = 0$, Proposition 3.2.2 can be applied to obtain $|\Sigma(3,5,15m+1)| \leq (3-1)(5-1)/2 = 4$ for any m odd. □

The above estimate is sharpened by Matsumoto for small m's.

Proposition 3.9 ([11, §4, Proposition 4.5])**.** *Suppose that* $\mathrm{Arf}(K(p,q)) = 0$. *Let* m *be an odd integer such that* $0 < m \leq \lfloor p/2 \rfloor \lfloor q/2 \rfloor + 1$. *Then* $\|\Sigma(p,q,pqm \pm 1)\| \leq (p-1)(q-1)/2 - 1$.

This proposition enables us to determine the bounding genera in the following case.

Proposition 3.10. $|\Sigma(2,7,14m+1)| = 2$ *for* $m = 1, 3$.

Proof. By Proposition 3.7 we have $|\Sigma(2,7,14m+1)| \geq 2$ for any odd m. On the other hand, by Matsumoto's estimates (Proposition 3.9), we have $\|\Sigma(2,7,14m\pm 1)\| \leq (2-1)(7-1)/2-1 = 2$ for $m=1,3$ and therefore $|\Sigma(2,7,14m+1)| = 2$ for $m=1,3$. $\square$

The following is a case where we could not determine the bounding genera even if we use the sharpened estimate.

Proposition 3.11. $2 \leq |\Sigma(3,5,15m+1)| \leq 3$ *and* $|\Sigma(3,5,15m-1)| = 3$ *for* $m=1,3$.

Proof. By Proposition 3.8 we have $|\Sigma(3,5,15m+1)| \geq 2$ and $|\Sigma(3,5,15m-1)| \geq 3$. On the other hand, for m odd with $m \leq \lfloor 3/2 \rfloor \lfloor 5/2 \rfloor + 1 = 3$, we have the inequality $|\Sigma(3,5,15m\pm 1)| \leq (3-1)(5-1)/2-1 = 3$ for $m=1,3$ by Matsumoto's estimates (Proposition 3.9). $\square$

In the case where the w-invariant vanishes for Brieskorn homology 3-spheres $\Sigma(p,q,r)$ such as $\Sigma(2,3,12k-1)$ for any integers k, we can apply the Fintushel-Stern invariant $R(p,q,r)$ [3]. The explicit formula of the invariant is given in terms of the trigonometric sums by using the Kawasaki V-index formula. W. Neumann and D. Zagier [14] derived the useful expression $R(\alpha_1,\dots,\alpha_n) = 2b-3$ by using the "b-invariant" of the Seifert fibration $\Sigma(\alpha_1,\dots,\alpha_n)$ where b satisfies $b + \sum_{i=1}^n \beta_i/\alpha_i = 1/\prod_{i=1}^n \alpha_i$ and $0 < \beta_i < \alpha_i$ with $\beta_i \alpha/\alpha_i \equiv -1 \pmod{\alpha_i}$. By using this expression we have the following:

Proposition 3.12. *Let p,q,r be pairwise coprime positive integers. If $R(p,q,r) > 0$, then*

$$|\Sigma(p,q,r+kpq)| \geq 1$$

for any non-negative integers k.

Proof. The b-invariants of $\Sigma(p,q,r)$ and $\Sigma(p,q,r+pq)$ coincide, and hence by the formula of W. Neumann and D. Zagier [14] we have $R(p,q,r) = R(p,q,r+pq)$. Therefore if $R(p,q,r) > 0$, then $R(p,q,r+kpq) > 0$, and by the theorem of R. Fintushel and R. Stern [3], $\Sigma(p,q,r+kpq)$ cannot be the boundary of an acyclic 4-manifold for any non-negative integer k. $\square$

As an application of Proposition 3.12, we have the following:

Proposition 3.13. $|\Sigma(2,3,12k-1)| = 1$ *for any non-negative integer k.*

Proof. By Matsumoto's estimate $|\Sigma(2,3,12k\pm 1)| \leq 1$ for any integer k. The w-invariant of $\Sigma(2,3,11)$ is $w(2,3,11) = 0$; hence we cannot apply Proposition 3.5. However, the Fintushel-Stern invariant $R(2,3,11) = 1 > 0$, and hence by Proposition 3.12 we have $|\Sigma(2,3,12k-1)| \geq 1$ for any integer k, and the assertion follows. Note that in the case $\Sigma(2,3,12k+1)$, the w-invariant of $\Sigma(2,3,13)$ is $w(2,3,13) = 0$ and the Fintushel-Stern invariant is $R(2,3,13) = -1 < 0$, and hence we cannot apply Proposition 3.5 nor Proposition 3.12. In fact, it is known that $\Sigma(2,3,13)$ [1], [4] and $\Sigma(2,3,25)$ [2] bound contractible smooth manifolds. $\square$

Matsumoto also gave estimates for the so-called Casson series of Brieskorn homology spheres.

Proposition 3.14 (Casson's series [11, §5, Proposition 5.1]). *Let p,q,r be odd integers satisfying $qr+rp+pq=-1$. Then $\|\Sigma(|p|,|q|,|r|)\| \leq 1$.*

In fact, we have the following:

Proposition 3.15. *Let p, q, r be odd integers satisfying $qr + rp + pq = -1$. Then* $|\Sigma(|p|, |q|, |r|)| = 1$.

Proof. By the equality $qr+rp+pq = -1$, we see that the b-invariant of $\Sigma(|p|, |q|, |r|)$ is 2, and hence by the formula of W. Neumann and D. Zagier [14], we have $R(|p|, |q|, |r|) = 2 \cdot 2 - 3 = 1 > 0$ and therefore $|\Sigma(|p|, |q|, |r|)| \geq 1$. Hence the assertion follows. $\square$

Example 3.16. $|\Sigma(2n+1, 4n+1, 4n+3)| = 1$ for any positive integer n.

We have the following estimates of the bounding genera for Brieskorn homology 3-spheres in [11, §5, Proposition 5.5]. The upper bounds are given by Matsumoto.

Proposition 3.17 (cf. [11, §5, Proposition 5.5])**.**

q	Brieskorn $\mathbb{Z}HS^3$	$\|\Sigma\| \leq$	$w(\Sigma)$	$R(\Sigma)$	$\|\Sigma\|$
3	$\Sigma(2,3,12k \pm 1)$	≤ 1	0	$-1(+), 1(-)$	$\geq 0(+), = 1(-)$
	$\Sigma(2,3,12k \pm 5)$	$+\infty$	± 1	$1(+), -1(-)$	$+\infty$
5	$\Sigma(2,5,20k \pm 1)$	≤ 1	0	$-1(+), 1(-)$	$\geq 0(+), = 1(-)$
	$\Sigma(2,5,20k \pm 3)$	$+\infty$	± 1	$1(+), -1(-)$	$+\infty$
	$\Sigma(2,5,20k \pm 7)$	≤ 1	0	$-1(+), 1(-)$	$\geq 0(+), = 1(-)$
	$\Sigma(2,5,20k \pm 9)$	$+\infty$	± 1	$1(+), -1(-)$	$+\infty$
7	$\Sigma(2,7,28k \pm 1)$	≤ 1	0	$-1(+), 1(-)$	$\geq 0(+), = 1(-)$
	$\Sigma(2,7,28k \pm 3)$	$+\infty$	∓ 1	$-1(+), 1(-)$	$+\infty$
	$\Sigma(2,7,28k \pm 5)$	≤ 1	0	$-1(+), 1(-)$	$\geq 0(+), = 1(-)$
	$\Sigma(2,7,28k \pm 9)$	≤ 1	0	$1(+), -1(-)$	$= 1(+), \geq 0(-)$
	$\Sigma(2,7,28k \pm 11)$	$+\infty$	± 1	$1(+), -1(-)$	$+\infty$
	$\Sigma(2,7,28k \pm 13)$	≤ 3	± 2	$1(+), -1(-)$	$= 3(+), \geq 2(-)$
9	$\Sigma(2,9,36k \pm 1)$	≤ 1	0	$-1(+), 1(-)$	$\geq 0(+), = 1(-)$
	$\Sigma(2,9,36k \pm 5)$	$+\infty$	± 1	$1(+), -1(-)$	$+\infty$
	$\Sigma(2,9,36k \pm 7)$	≤ 2	0	$1(+), -1(-)$	$= 1(+), \geq 0(-)$
	$\Sigma(2,9,36k \pm 11)$	≤ 1	0	$-1(+), 1(-)$	$\geq 0(+), = 1(-)$
	$\Sigma(2,9,36k \pm 13)$	$+\infty$	∓ 1	$-1(+), 1(-)$	$+\infty$
	$\Sigma(2,9,36k \pm 17)$	≤ 4	± 2	$1(+), -1(-)$	$\geq 3(+), \geq 2(-)$

Remark 3.18 ([11, §5, Remark]). Matsumoto improved the estimates for several Brieskorn homology 3-spheres in the above lists. We give estimates below for them.

q	Brieskorn $\mathbb{Z}HS^3$	$\|\Sigma\| \leq$	$w(\Sigma)$	$R(\Sigma)$	$\|\Sigma\|$
3	$\Sigma(2,3,12k+1)$, $k=1,2$	0, $k=1,2$	0	-1	$= 0$
	$\Sigma(2,3,12k-1)$	$\leq 1_{\mathrm{cr}}$, $\forall k > 0$	0	1	$= 1$
5	$\Sigma(2,5,7)$	0	0	-1	$= 0$
	$\Sigma(2,5,21)$	0	0	-1	$= 0$
7	$\Sigma(2,7,28k-13)$, $k=1,2$	≤ 2, $k=1,2$	-2	-1	$= 2$
	$\Sigma(2,7,13)$	$\leq 3_{\mathrm{cr}}$	2	1	$= 3$
	$\Sigma(2,7,15)$	$\leq 2_{\mathrm{cr}}$	-2	-1	$= 2$
	$\Sigma(2,7,19)$	0	0	-1	$= 0$
9	$\Sigma(2,9,11)$	0	0	-1	$= 0$
	$\Sigma(2,9,36k-17)$, $1 \leq k \leq 3$	$\leq 3, 1 \leq k \leq 3$	-2	-1	≥ 2

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