

TOTALLY FREE ARRANGEMENTS OF HYPERPLANES

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ABSTRACT. A central arrangement \mathcal{A} of hyperplanes in an ℓ -dimensional vector space V is said to be *totally free* if a multiarrangement (\mathcal{A}, m) is free for any multiplicity $m : \mathcal{A} \rightarrow \mathbb{Z}_{>0}$. It has been known that \mathcal{A} is totally free whenever $\ell \leq 2$. In this article, we will prove that there does not exist any totally free arrangement other than the obvious ones, that is, a product of one-dimensional arrangements and two-dimensional ones.

1. INTRODUCTION

Let V be an ℓ -dimensional vector space ($\ell \geq 1$) over \mathbb{K} with a coordinate system $\{x_1, \dots, x_\ell\} \subset V^*$. Define $S := \text{Sym}(V^*) \simeq \mathbb{K}[x_1, \dots, x_\ell]$. Let $\text{Der}_{\mathbb{K}}(S)$ be the set of all \mathbb{K} -linear derivations of S to itself. Then $\text{Der}_{\mathbb{K}}(S) = \bigoplus_{i=1}^{\ell} S \cdot \partial_{x_i}$ is a free S -module of rank ℓ . A *central arrangement (of hyperplanes)* in V is a finite collection of linear hyperplanes in V . In this article we assume that every arrangement is central unless otherwise specified. A *multiplicity* m is a function $m : \mathcal{A} \rightarrow \mathbb{Z}_{>0}$ and a pair (\mathcal{A}, m) is called a *multiarrangement*. Fix a linear form α_H ($H \in \mathcal{A}$) in such a way that $\ker(\alpha_H) = H$. The *logarithmic derivation module* $D(\mathcal{A}, m)$ associated with (\mathcal{A}, m) is defined by

$$D(\mathcal{A}, m) := \{\theta \in \text{Der}_{\mathbb{K}}(S) \mid \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \text{ for all } H \in \mathcal{A}\}.$$

In general, $D(\mathcal{A}, m)$ is not necessarily a free S -module. We say that (\mathcal{A}, m) is *free* if $D(\mathcal{A}, m)$ is a free S -module. For a fixed arrangement \mathcal{A} , a multiplicity m on \mathcal{A} is called *free* if a multiarrangement (\mathcal{A}, m) is free. Define

$$\mathcal{NFM}(\mathcal{A}) := \{m : \mathcal{A} \rightarrow \mathbb{Z}_{>0} \mid m \text{ is not a free multiplicity}\}.$$

The following definition was introduced in [4, Definition 5.4].

Definition 1.1. An arrangement \mathcal{A} is called *totally free* if every multiplicity $m : \mathcal{A} \rightarrow \mathbb{Z}_{>0}$ is a free multiplicity, or equivalently $\mathcal{NFM}(\mathcal{A}) = \emptyset$.

When \mathcal{A}_i is an arrangement in V_i ($i = 1, 2$), the *product* $\mathcal{A}_1 \times \mathcal{A}_2$ is an arrangement in $V_1 \oplus V_2$ defined as in [6, Definition 2.13] by

$$\mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\}.$$

Our main theorem is as follows:

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Theorem 1.2. *An arrangement \mathcal{A} is totally free if and only if it has a decomposition*

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_s,$$

where each \mathcal{A}_i is an arrangement in \mathbb{K}^1 or \mathbb{K}^2 .

Ziegler showed in [12, Corollary 7] that (\mathcal{A}, m) is a free multiarrangement whenever $\ell \leq 2$. Note that

$$D(\mathcal{A}_1 \times \mathcal{A}_2, m) \simeq S \cdot D(\mathcal{A}_1, m|_{\mathcal{A}_1}) \oplus S \cdot D(\mathcal{A}_2, m|_{\mathcal{A}_2})$$

holds true as shown in [3, Lemma 1.4]. Thus

$$\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_s$$

is known to be totally free if each \mathcal{A}_i is an arrangement in \mathbb{K}^1 or \mathbb{K}^2 . Theorem 1.2 asserts that the converse is also true. In the next section we will prove Theorem 1.2 in a stronger form: we will show that \mathcal{A} is decomposed into one-dimensional arrangements and two-dimensional ones if $\mathcal{NFM}(\mathcal{A})$ is a finite set.

Recall that the *intersection lattice* $L(\mathcal{A})$ is the set $\{X = H_1 \cap \cdots \cap H_s \mid H_i \in \mathcal{A}, s \geq 0\}$ with the reverse inclusion ordering as in [6, Definition 2.1]. Then Theorem 1.2 implies:

Corollary 1.3. *Whether an arrangement \mathcal{A} is totally free or not depends only on its intersection lattice $L(\mathcal{A})$.*

Let \mathcal{A} be a nonempty central arrangement and $H_0 \in \mathcal{A}$. Define the *deletion* \mathcal{A}' and the *restriction* \mathcal{A}'' as in [6, Definition 1.14]:

$$\mathcal{A}' := \mathcal{A} \setminus \{H_0\}, \quad \mathcal{A}'' := \{H_0 \cap H \mid H \in \mathcal{A}'\}.$$

Because of the characterization in Theorem 1.2, the total freeness is stable under deletion and restriction:

Corollary 1.4. *Any subarrangement or restriction of a totally free arrangement is also totally free.*

A multiarrangement was introduced and studied by Ziegler in [12]. The third author proved in [9] and [10] that the freeness of a simple arrangement is closely related with the freeness of Ziegler’s canonical restriction. Recently the first and second authors and Wakefield developed a general theory of free multiarrangements and introduced the concept of free multiplicity in [3] and [4]. Several papers, including [1], [2], [5] and [11], studied the set of free multiplicities for a fixed arrangement \mathcal{A} . The main theorem (Theorem 1.2) in this article shows that the set of free multiplicities (or $\mathcal{NFM}(\mathcal{A})$) imposes strong restrictions on the original arrangement \mathcal{A} .

2. PROOF OF THEOREM 1.2

First we review in Theorem 2.1 a necessary condition for a given multiarrangement to be free.

Let (\mathcal{A}, m) be a multiarrangement. When (\mathcal{A}, m) is free, there exists a homogeneous basis $\theta_1, \dots, \theta_\ell$ for $D(\mathcal{A}, m)$. The set $\exp(\mathcal{A}, m)$ of *exponents* is defined by $\exp(\mathcal{A}, m) := (\deg \theta_1, \dots, \deg \theta_\ell)$, where $\deg(\theta_i) := \deg \theta_i(\alpha)$ for some linear form α with $\theta_i(\alpha) \neq 0$.

Define $L(\mathcal{A})_2 := \{X \in L(\mathcal{A}) \mid \text{codim}_V(X) = 2\}$ and $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subset H\}$. For $X \in L(\mathcal{A})_2$ the multiarrangement $(\mathcal{A}_X, m|_{\mathcal{A}_X})$ is free with exponents

$(d_1^X, d_2^X, 0, \dots, 0)$. Define the *second local mixed product* $LMP_2(\mathcal{A}, m)$ as in [3, Definition 4.3] by

$$LMP_2(\mathcal{A}, m) := \sum_{X \in L(\mathcal{A})_2} d_1^X d_2^X.$$

If \mathcal{B} is a subarrangement of \mathcal{A} , then it is easy to see that

$$LMP_2(\mathcal{A}, m) \geq LMP_2(\mathcal{B}, m|_{\mathcal{B}}).$$

Next assume that (\mathcal{A}, m) is free with exponents (d_1, \dots, d_ℓ) . Define the *second global mixed product* $GMP_2(\mathcal{A}, m)$ as in [3, Definition 4.5] by

$$GMP_2(\mathcal{A}, m) := \sum_{1 \leq i < j \leq \ell} d_i d_j.$$

Theorem 2.1. *If a multiarrangement (\mathcal{A}, m) is free, then $GMP_2(\mathcal{A}, m) = LMP_2(\mathcal{A}, m)$.*

In fact, Theorem 2.1 is true for any GMP_k and LMP_k ($1 \leq k \leq \ell$); see [3, Corollary 4.6].

An arrangement \mathcal{A} is said to be *reducible* if $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ for certain arrangements \mathcal{A}_i in V_i ($i = 1, 2$). We say \mathcal{A} is *irreducible* if it is not reducible.

Lemma 2.2. *Let \mathcal{A} be an irreducible arrangement in \mathbb{K}^ℓ with $\ell \geq 3$. Then there exists a subarrangement \mathcal{B} with $|\mathcal{B}| = \ell + 1$ such that:*

$$\text{codim}_V H_i \cap H_j \cap H_k = 3$$

whenever H_i, H_j, H_k are three hyperplanes in \mathcal{B} . Moreover the arrangement \mathcal{B} is not free.

Proof. Let $H_0 \in \mathcal{A}$. Let \mathcal{A}' and \mathcal{A}'' be the deletion and the restriction, respectively. Then either \mathcal{A}' or \mathcal{A}'' is irreducible by Tutte [8] (see also [7, Theorem 4.3.1]). When $|\mathcal{A}| = \ell + 1$, the arrangement \mathcal{A} itself satisfies the condition. We will prove by an induction on $|\mathcal{A}|$. If \mathcal{A}' is irreducible, then \mathcal{A}' contains $\ell + 1$ hyperplanes satisfying the condition. So we may assume that \mathcal{A}'' is irreducible. Let $\mathcal{A}' = \{H_1, H_2, \dots, H_{n-1}\}$ and $\overline{H}_i := H_i \cap H_0$ for $1 \leq i \leq n - 1$.

Suppose $\ell = 3$. Since \mathcal{A}'' is irreducible, $3 \leq |\mathcal{A}''| \leq |\mathcal{A}'|$.

Case 1. If $3 \leq |\mathcal{A}''| = |\mathcal{A}'|$, then $\overline{H}_1, \overline{H}_2, \dots, \overline{H}_{n-1}$ are distinct. Note that $H_1 \cap H_2 \cap \dots \cap H_{n-1} = \{\mathbf{0}\}$ because \mathcal{A} is irreducible. Thus we may assume $H_1 \cap H_2 \cap H_3 = \{\mathbf{0}\}$. So $\{H_0, H_1, H_2, H_3\}$ satisfies the condition.

Case 2. If $3 \leq |\mathcal{A}''| < |\mathcal{A}'|$, then we may assume that $\overline{H}_1, \overline{H}_2, \overline{H}_3$ are distinct and $\overline{H}_3 = \overline{H}_4$. Note that $H_3 \cap H_4 \subset H_0$. Thus

$$H_1 \cap H_2 \cap H_3 \cap H_4 = \overline{H}_1 \cap \overline{H}_2 \cap \overline{H}_3 = \{\mathbf{0}\}.$$

Therefore either $H_1 \cap H_2 \cap H_3 = \{\mathbf{0}\}$ or $H_1 \cap H_2 \cap H_4 = \{\mathbf{0}\}$. So we may conclude that either $\{H_0, H_1, H_2, H_3\}$ or $\{H_0, H_1, H_2, H_4\}$ satisfies the condition.

Suppose $\ell \geq 4$. Then, by the induction assumption, there exists a subarrangement $\{\overline{H}_1, \overline{H}_2, \dots, \overline{H}_\ell\}$ of \mathcal{A}'' satisfying the condition. Then the subarrangement $\{H_0, H_1, \dots, H_\ell\}$ of \mathcal{A} satisfies the condition.

Suppose that the arrangement \mathcal{B} is free. Then the sum of the exponents is equal to $\ell + 1$. Thus one has $GMP_2 \leq \binom{\ell}{2}((\ell + 1)/\ell)^2$. We also have $LMP_2 = \binom{\ell+1}{2}$. Since

$$LMP_2 - GMP_2 \geq \binom{\ell + 1}{2} - \binom{\ell}{2}((\ell + 1)/\ell)^2 = (\ell + 1)/2\ell > 0,$$

this contradicts Theorem 2.1. □

Recall

$$\mathcal{NFM}(\mathcal{A}) = \{m : \mathcal{A} \rightarrow \mathbb{Z}_{>0} \mid m \text{ is not a free multiplicity}\}.$$

Proposition 2.3. *If \mathcal{A} is an irreducible arrangement in \mathbb{K}^ℓ with $\ell \geq 3$, then $\mathcal{NFM}(\mathcal{A})$ is an infinite set.*

Proof. Suppose that $\mathcal{NFM}(\mathcal{A})$ is a finite set. Choose a subarrangement \mathcal{B} of \mathcal{A} with $|\mathcal{B}| = \ell + 1$ satisfying the condition in Lemma 2.2. Consider the multiplicity m defined by

$$m(H) = \begin{cases} 1 & \text{if } H \notin \mathcal{B}, \\ k & \text{if } H \in \mathcal{B}, \end{cases}$$

for every positive integer k . Since $\mathcal{NFM}(\mathcal{A})$ is a finite set, the multiarrangement (\mathcal{A}, m) is free whenever k is sufficiently large. Note that $|L(\mathcal{B})_2| = \binom{\ell+1}{2}$. By the definition of LMP_2 ,

$$LMP_2(\mathcal{A}, m) \geq LMP_2(\mathcal{B}, m|_{\mathcal{B}}) = |L(\mathcal{B})_2|k^2 = \binom{\ell + 1}{2}k^2.$$

Let $|\mathcal{A}| = n$. Then

$$\sum_{d \in \text{exp}(\mathcal{A}, m)} d = (k - 1)(\ell + 1) + n$$

and thus

$$GMP_2(\mathcal{A}, m) \leq \binom{\ell}{2} \left\{ \frac{(k - 1)(\ell + 1) + n}{\ell} \right\}^2 = \frac{(\ell + 1)^2(\ell - 1)}{2\ell}k^2 + Ak + B$$

with some constants A and B . By Theorem 2.1 we have

$$\binom{\ell + 1}{2}k^2 \leq LMP_2(\mathcal{A}, m) = GMP_2(\mathcal{A}, m) \leq \frac{(\ell + 1)^2(\ell - 1)}{2\ell}k^2 + Ak + B$$

whenever k is sufficiently large. This is a contradiction because

$$\binom{\ell + 1}{2} - \frac{(\ell + 1)^2(\ell - 1)}{2\ell} = \frac{\ell + 1}{2\ell} > 0.$$

□

We now prove the following theorem, which is stronger than Theorem 1.2.

Theorem 2.4. *The following four conditions for a central arrangement \mathcal{A} are equivalent:*

- (1) \mathcal{A} is totally free; i.e., $\mathcal{NFM}(\mathcal{A})$ is empty.
- (2) $\mathcal{NFM}(\mathcal{A})$ is a finite set.

(3) \mathcal{A} has a decomposition

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_s,$$

where each \mathcal{A}_i is an arrangement in \mathbb{K}^1 or \mathbb{K}^2 .

(4) Every subarrangement of \mathcal{A} is free.

Proof. The implications (1) \Rightarrow (2), (3) \Rightarrow (4) and (3) \Rightarrow (1) are obvious. Thus it is enough to prove that (2) \Rightarrow (3) and (4) \Rightarrow (3).

(2) \Rightarrow (3): Suppose that $\mathcal{NFM}(\mathcal{A})$ is a finite set. Decompose \mathcal{A} into

$$\mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_s$$

such that each \mathcal{A}_i is irreducible. Since

$$D(\mathcal{A}, m) \simeq S \cdot D(\mathcal{A}_1, m|_{\mathcal{A}_1}) \oplus S \cdot D(\mathcal{A}_2, m|_{\mathcal{A}_2}) \oplus \cdots \oplus S \cdot D(\mathcal{A}_s, m|_{\mathcal{A}_s})$$

holds by [3, Lemma 1.4], each \mathcal{A}_i is an irreducible arrangement and $\mathcal{NFM}(\mathcal{A}_i)$ is a finite set. Thus Proposition 2.3 shows that each arrangement \mathcal{A}_i is in \mathbb{K}^1 or \mathbb{K}^2 .

(4) \Rightarrow (3): Decompose \mathcal{A} into irreducible arrangements. Then each of the irreducible arrangements satisfies the assumption (4). Therefore we may assume that \mathcal{A} is irreducible from the beginning. Then, by Lemma 2.2, we may conclude $\ell \leq 2$. \square

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