PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 4, April 2009, Pages 1489–1497
S 0002-9939(08)09770-0
Article electronically published on November 25, 2008

ON THE LUSTERNIK-SCHNIRELMANN CATEGORY OF SPACES WITH 2-DIMENSIONAL FUNDAMENTAL GROUP

ALEXANDER N. DRANISHNIKOV

(Communicated by Daniel Ruberman)

ABSTRACT. The following inequality

$$\mathrm{cat}_{\mathrm{LS}} X \le \mathrm{cat}_{\mathrm{LS}} Y + \left\lceil \frac{hd(X) - r}{r+1} \right\rceil$$

holds for every locally trivial fibration $f : X \to Y$ between ANE spaces which admits a section and has the r-connected fiber, where $hd(X)$ is the homotopical dimension of X. We apply this inequality to prove that

$$\mathrm{cat}_{\mathrm{LS}} X \le cd(\pi_1(X)) + \left\lceil \frac{\dim X - 1}{2} \right\rceil$$

for every complex X with $cd(\pi_1(X)) \le 2$, where $cd(\pi_1(X))$ denotes the cohomological dimension of the fundamental group of X.

1. INTRODUCTION

In [DKR] we proved that if the Lusternik-Schnirelmann category of a closed n-manifold, $n \ge 3$, equals 2, then the fundamental group of M is free. In the opposite direction we proved that if the fundamental group of an n-manifold is free, then $\mathrm{cat}_{\mathrm{LS}} M \le n - 2$. J. Strom proved that $\mathrm{cat}_{\mathrm{LS}} X \le \frac{2}{3}n$ for every n-complex, $n > 4$, with free fundamental group [St]. Yu. Rudyak conjectured that the coefficient 2/3 in Strom's result could be improved to 1/2. Precisely, he conjectured that the function f defined as $f(n) = \max\{\mathrm{cat}_{\mathrm{LS}} M^n\}$ is asymptotically $\frac{1}{2}n$, where the maximum is taken over all closed n-manifolds with free fundamental group.

In this paper we prove Rudyak's conjecture. Our method gives the same estimate for n-complexes. Moreover, we give the same asymptotic upper bound for $\mathrm{cat}_{\mathrm{LS}}$ of n-complexes with the fundamental group of cohomological dimension ≤ 2. In view of this, the following generalization of Rudyak's conjecture seems to be natural.

Conjecture 1.1. *For every k the function f_k defined as*

$$f_k(n) = \max\{\mathrm{cat}_{\mathrm{LS}} M^n \mid cd(\pi_1(M^n) \le k\}$$

is asymptotically $\frac{1}{2}n$.

The smallest k when it is unknown is 3.

Received by the editors September 25, 2007, and, in revised form, April 27, 2008.
2000 *Mathematics Subject Classification.* Primary 55M30.
The author was supported by NSF grant DMS-0604494.

This paper is organized as follows. Section 2 is an introduction to the Lusternik-Schnirelmann category based on an analogy with dimension theory. Section 3 contains a fibration theorem for $\mathrm{cat}_{\mathrm{LS}}$. In Section 4 this fibration theorem is applied for the proof of Rudyak's conjecture.

2. Kolmogorov-Ostrand's approach to the Lusternik-Schnirelmann category

A subset $A \subset X$ of a topological space X is called *X-contractible* if it can be contracted to a point in X. A cover $\mathcal{U}$ of a topological space X by X-contractible sets is called *X-contractible.* By definition, $\mathrm{cat}_{\mathrm{LS}}\, X \le n$ if there is an X-contractible open cover $\mathcal{U} = \{U_0, \dots, U_n\}$ of X that consists of $n+1$ sets.

We recall [CLOT] that a sequence $\varnothing = O_0 \subset O_1 \subset \cdots \subset O_{n+1} = X$ is called *categorical of length* $n+1$ if each difference $O_{i+1} \setminus O_i$ is contained in an X-contractible open set. It was proven in [CLOT] that $\mathrm{cat}_{\mathrm{LS}}\, X \le n$ if and only if X admits a categorical sequence of length $n+1$.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a family of sets in a topological space X. Formally, it is a function $U : A \to 2^X \setminus \{\varnothing\}$ from the index set to the set of nonempty subsets of X. Thus, it is allowed to have $U_\alpha = U_\beta$ for $\alpha \ne \beta$. The sets U_α in the family $\mathcal{U}$ will be called *elements of* $\mathcal{U}$. The *multiplicity* of $\mathcal{U}$ (or the *order*) at a point $x \in X$, denoted $\mathrm{Ord}_x \mathcal{U}$, is the number of elements of $\mathcal{U}$ that contain x. The *multiplicity* of $\mathcal{U}$ is defined as $\mathrm{Ord}\,\mathcal{U} = \sup_{x \in X} \mathrm{Ord}_x \mathcal{U}$. A family $\mathcal{U}$ is a *cover* of X if $\mathrm{Ord}_x \mathcal{U} \ne 0$ for all x. A cover $\mathcal{U}$ is a *refinement* of another cover $\mathcal{C}$ (*$\mathcal{U}$ refines $\mathcal{C}$*) if for every $U \in \mathcal{U}$ there exists $C \in \mathcal{C}$ such that $U \subset C$. We recall that the *covering dimension* of a topological space X does not exceed n, $\dim X \le n$, if for every open cover $\mathcal{C}$ of X there is an open refinement $\mathcal{U}$ with $\mathrm{Ord}\,\mathcal{U} \le n+1$.

We recall that a family $\mathcal{F}$ of subsets of a topological space X is called *locally finite* if for every $x \in X$ there is a neighborhood U of x which has a nonempty intersection at most with finitely many sets from $\mathcal{F}$. The following proposition makes the LS-category analogous to the covering dimension.

Proposition 2.1. *For a paracompact topological space X, $\mathrm{cat}_{\mathrm{LS}}\, X \le n$ if and only if X admits an X-contractible locally finite open cover $\mathcal{V}$ with $\mathrm{Ord}\,\mathcal{V} \le n+1$.*

Proof. If $\mathrm{cat}_{\mathrm{LS}}\, X \le n$, then by the definition, X admits an open contractible cover that consists of $n+1$ sets and therefore its multiplicity is at most $n+1$.

Let $\mathcal{V}$ be a contractible cover of X of multiplicity $\le n+1$. We construct a categorical sequence $O_0 \subset O_1 \subset \cdots \subset O_{n+1}$ of length $n+1$. We define $O_1 = \{x \in X \mid \mathrm{Ord}_x \mathcal{V} = n+1\}$. Note that

$$O_1 = \bigcup_{\{V_0, \dots, V_n\} \subset \mathcal{V}} V_0 \cap \cdots \cap V_n.$$

Note that this is a disjoint union and every nonempty summand is X-contractible. Thus O_1 is X-contractible. Next, we define $O_2 = \{x \in X \mid \mathrm{Ord}_x \mathcal{V} \ge n\}$. Then

$$O_2 \setminus O_1 = \bigcup_{\{V_0, \dots, V_{n-1}\} \subset \mathcal{V}} (V_0 \cap \cdots \cap V_{n-1} \setminus O_1)$$

is a disjoint union of closed in O_2 subsets. Since $\mathcal{V}$ is locally finite, the family of nonempty summands

$$\{V_0 \cap \cdots \cap V_{n-1} \setminus O_1 \mid V_0, \dots, V_{n-1} \in \mathcal{V},\ V_0 \cap \cdots \cap V_{n-1} \setminus O_1 \ne \varnothing\}$$

is locally finite. We recall that every disjoint locally finite family of closed subsets is discrete. Hence there are open (in O_2 and hence in X) disjoint neighborhoods $W_{V_0,\dots,V_{n-1}}$ of these summands $V_0 \cap \dots \cap V_{n-1} \setminus O_1$. By taking $W_{V_0,\dots,V_{n-1}} \cap V_0$ we may assume that the neighborhood of the summand $V_0 \cap \dots \cap V_{n-1} \setminus O_1$ is contained in V_0. Thus, we may assume that all neighborhoods $W_{V_0,\dots,V_{n-1}}$ are X-contractible. Define $O_3 = \{x \in X \mid Ord_x \mathcal{V} \geq n-1\}$ as the union of $(n-1)$-fold intersections and so on. In general, $O_k = \{x \in X \mid Ord_x \mathcal{V} \geq n-k+2\}$. Similarly,

$$O_{k+1} \setminus O_k = \bigcup_{\{V_0,\dots,V_{n-k}\} \subset \mathcal{V}} (V_0 \cap \dots \cap V_{n-k} \setminus O_k)$$

is a disjoint union of closed in O_{k+1} subsets. Since the family of nonempty summands in this union is locally finite, there are open in O_{k+1}, and hence in X, disjoint neighborhoods of these summands $V_0 \cap \dots \cap V_{n-k} \setminus O_k$ such that each neighborhood lies in some X-contractible set $V \in \mathcal{V}$.

Then O_{n+1} is the union of elements of $\mathcal{V}$ (1-fold intersections) and hence $O_{n+1} = X$. The categorical sequence conditions are satisfied. □

A family $\mathcal{U}$ of subsets of X is called a *k-cover*, $k \in N$ if every subfamily of k elements forms a cover of X.

Example. Let

$$U = \bigcup_{i \in \mathbb{Z}} (mi, m(i+1)-1)$$

be the union of disjoint intervals in $\mathbf{R}$ of length $m-1$ with the distance 1 between any two consecutive intervals. Let $\mathcal{U} = \{T_r U \mid r = 0, \dots, m-1\}$ be the family of translates $T_r U = \{x + r \mid x \in U\}$ of U. Clearly, $\mathcal{U}$ is a 3-cover of $\mathbf{R}$ that consists of m subsets.

If we take the intervals of length $m-k$ and the distance k,

$$U = \bigcup_{i \in \mathbb{Z}} (mi, m(i+1)-k),$$

then $\mathcal{U} = \{T_r U \mid r = 0, \dots, m-1\}$ is a $(k+2)$-cover that consists of m subsets. The proof can be derived from the following:

Proposition 2.2. *A family $\mathcal{U}$ that consists of m subsets of X is an $(n+1)$-cover of X if and only if $\operatorname{Ord}_x \mathcal{U} \geq m-n$ for all $x \in X$.*

Proof. If $\operatorname{Ord}_x \mathcal{U} < m-n$ for some $x \in X$, then $n+1 = m-(m-n)+1$ elements of $\mathcal{U}$ do not cover x.

If $n+1$ elements of $\mathcal{U}$ do not cover some x, then $\operatorname{Ord}_x \mathcal{U} \leq m-(n+1) < m-n$. □

Inspired by the work of Kolmogorov on Hilbert's 13th problem, Ostrand gave the following characterization of the covering dimension [Os].

Theorem 2.3 (Ostrand)**.** *A metric space X is of dimension $\leq n$ if and only if for each open cover $\mathcal{C}$ of X and each integer $m \geq n+1$, there exist m disjoint families of open sets $\mathcal{U}_1, \dots, \mathcal{U}_m$ such that their union $\bigcup \mathcal{U}_i$ is an $(n+1)$-cover of X and it refines $\mathcal{C}$.*

Let $\mathcal{U}$ be a family of subsets in X and let $A \subset X$. We denote by $\mathcal{U}|_A = \{U \cap A \mid U \in \mathcal{U}\}$ the *restriction* of $\mathcal{U}$ to A.

Definition 2.4. Let $f : X \to Y$ be a map. An open cover $\mathcal{U} = \{U_0, U_1, \dots, U_n\}$ of X is called *uniformly f-contractible* if for every $y \in Y$ there is a neighborhood V such that the restriction $\mathcal{U}|_{f^{-1}(V)}$ of $\mathcal{U}$ to the preimage $f^{-1}(V)$ consists of X-contractible sets.

We will use uniformly f-contractible covers to give in the next section an alternative extension of the Lusternik-Schnirelmann category to mappings. The standard extension $\mathrm{cat}_{\mathrm{LS}}(f)$ [CLOT] satisfies the equalities $\mathrm{cat}_{\mathrm{LS}}(1_X) = \mathrm{cat}_{\mathrm{LS}} X$ and $\mathrm{cat}_{\mathrm{LS}}(c) = 0$, where 1_X and $c : X \to *$ are the identity map and the constant map respectively. Our extension $\mathrm{cat}_{\mathrm{LS}}{}^*$ satisfies the opposite: $\mathrm{cat}_{\mathrm{LS}}{}^*(c) = \mathrm{cat}_{\mathrm{LS}} X$. Also it satisfies $\mathrm{cat}_{\mathrm{LS}}{}^*(1_X) = 0$ for locally contractible spaces (see §3).

Theorem 2.5. *Let $\mathcal{U} = \{U_0, \dots, U_n\}$ be an open cover of a normal topological space X. Then for any $m = n, n+1, \dots, \infty$ there is an open $(n+1)$-cover of X, $\mathcal{U}_m = \{U_0, \dots, U_m\}$ such that for $k > n$, $U_k = \bigcup_{i=0}^n V_i$ is a disjoint union with $V_i \subset U_i$.*

In particular, if $\mathcal{U}$ is X-contractible, the cover $\mathcal{U}_m$ is X-contractible. If $\mathcal{U}$ is uniformly f-contractible for some $f : X \to Z$, the cover $\mathcal{U}_m$ is uniformly f-contractible.

Proof. We construct the family $\mathcal{U}_m$ by induction on m. For $m = n$ we take $\mathcal{U}_m = \mathcal{U}$.

Let $\mathcal{U}_{m-1} = \{U_0, \dots, U_{m-1}\}$ be the corresponding family for $m > n$. By Proposition 2.2, $\mathrm{Ord}_x \mathcal{U} \geq m - n$. Consider $Y = \{x \in X \mid \mathrm{Ord}_x \mathcal{U} = m - n\}$. Clearly, it is a closed subset of X. If $Y = \emptyset$, then by Proposition 2.2, $\mathcal{U}$ is an n-cover and we can add $U_m = U_0$ to obtain a desired $(n+1)$-cover. Assume that $Y \neq \emptyset$. We show that for every $i \leq n$, the set $Y \cap U_i$ is closed in X. Let x be a limit point of $Y \cap U_i$ that does not belong to U_i. Let $U_{i_1}, \dots, U_{i_{m-n}}$ be the elements of the cover $\mathcal{U}$ that contain $x \in Y$. The limit point condition implies that $(U_{i_1} \cap \dots \cap U_{i_{m-n}}) \cap (Y \cap U_i) \neq \emptyset$. Then $\mathrm{Ord}_y \mathcal{U} = m - n + 1$ for all $y \in Y \cap U_i \cap U_{i_0} \cap \dots \cap U_{i_{m-n}}$, a contradiction.

We define recursively $F_0 = Y \cap U_0$ and $F_{i+1} = Y \cap U_{i+1} \setminus (\bigcup_{k=0}^i U_k)$. Note that $\{F_i\}_{i=0}^n$ is a disjoint finite family of closed subsets with $\bigcup_{i=0}^n F_i = Y$. Since X is normal, we can fix disjoint open neighborhoods V_i of F_i with $V_i \subset U_i$. We define $U_m = \bigcup_{i=0}^n V_i$. In view of Proposition 2.2, $U_0, \dots, U_{m-1}, U_m$ is an $(n+1)$-cover.

Clearly, if all U_i are X-contractible, $i \leq n$, then U_m is X-contractible. If all U_i are uniformly f-contractible, for some $f : X \to Z$, then U_m is uniformly f-contractible. $\square$

Corollary 2.6. *For a normal topological space X, $\mathrm{cat}_{\mathrm{LS}} X \leq n$ if and only if for any $m > n$, X admits an open $(n+1)$-cover by m X-contractible sets.*

This corollary is a $\mathrm{cat}_{\mathrm{LS}}$-analog of Ostrand's theorem. It also can be found in [CLOT] with further reference to [Cu].

3. Fibration theorems for $\mathrm{cat}_{\mathrm{LS}}$

Definition 3.1. The $*$-category $\mathrm{cat}_{\mathrm{LS}}{}^* f$ of a map $f : X \to Y$ is the minimal n, if it exists, such that there is a uniformly f-contractible open cover $\mathcal{U} = \{U_0, U_1, \dots, U_n\}$ of X.

Note that $\mathrm{cat}_{\mathrm{LS}}{}^* c = \mathrm{cat}_{\mathrm{LS}} X$ for a constant map $c : X \to pt$. More generally, $\mathrm{cat}_{\mathrm{LS}}{}^* \pi = \mathrm{cat}_{\mathrm{LS}} X$ for the projection $\pi : X \times Y \to Y$.

Theorem 3.2. *The inequality $\mathrm{cat}_{\mathrm{LS}} X \leq \dim Y + \mathrm{cat}_{\mathrm{LS}}{}^* f$ holds true for any continuous map of a normal space.*

Proof. The requirements to the spaces in the theorem are that the Ostrand theorem holds true for Y; i.e. they are fairly general (say, Y is normal).

Let $\dim Y = n$ and $\mathrm{cat_{LS}}^* f = m$. Let $\mathcal{U} = \{U_0, \dots, U_m\}$ be a uniformly f-contractible cover of X. For $y \in Y$ denote by V_y a neighborhood of y from the definition of the uniform f-contractibility. In view of Theorem 2.3 there is a refinement $\mathcal{V} = \mathcal{V}_0 \cup \dots \cup \mathcal{V}_{n+m}$ of the cover $\{V_y \mid y \in Y\}$ of Y such that each family $\mathcal{V}_i$ is disjoint and $\mathcal{V}$ is an $(n+1)$-cover. Let $V_i = \bigcup \mathcal{V}_i$.

We apply Theorem 2.5 to extend the family $\mathcal{U}$ to a uniformly f-contractible $(m+1)$-cover $\{U_0, \dots, U_{n+m}\}$. Consider the family $\mathcal{W} = \{f^{-1}(V_i) \cap U_i\}_{0 \le i \le n+m}$. Note that it is X-contractible. Thus, in order to get the inequality $\mathrm{cat_{LS}}\, X \le n + m$ it suffices to show that $\mathcal{W}$ is a cover of X. Since $\mathcal{V}$ is an $(n+1)$-cover, by Proposition 2.2, every $y \in Y$ is covered by $m+1$ elements of $\mathcal{V}$, $V_{i_0}, \dots, V_{i_m}$. Since $\{U_0, \dots, U_{n+m}\}$ is an $(m+1)$-cover, the family $U_{i_0}, \dots, U_{i_m}$ covers X. Therefore the family $f^{-1}(V_{i_0}) \cap U_{i_0}, \dots, f^{-1}(V_{i_m}) \cap U_{i_m}$ covers the fiber $f^{-1}(y)$. Since $y \in Y$ is arbitrary, $\mathcal{W}$ covers all X. □

Corollary 3.3 (Corollary 9.35 [CLOT], [OW])**.** *Let $p : X \to Y$ be a closed map of ANE. If each fiber $p^{-1}(y)$ is contractible in X, then $\mathrm{cat_{LS}}\, X \le \dim Y$.*

Proof. In this case $\mathrm{cat_{LS}}^* p = 0$. Indeed, since X is an ANE, a contraction of $p^{-1}(y)$ to a point can be extended to a neighborhood U. Since the map p is closed there is a neighborhood V of y such that $p^{-1}(V) \subset U$. □

We recall that the *homotopical dimension* of a space X, $hd(X)$, is the minimal dimension of a CW-complex homotopy equivalent to X [CLOT].

Proposition 3.4. *Let $p : E \to X$ be a fibration with $(n-1)$-connected fiber where $n = hd(X)$. Then p admits a section.*

Proof. Let $h : Y \to X$ be a homotopy equivalence with the homotopy inverse $g : X \to Y$, where Y is a CW-complex of dimension n. Since the fiber of p is $(n-1)$-connected, the map h admits a lift $h' : Y \to E$. Let H be a homotopy connecting $h \circ g$ with 1_X. By the homotopy lifting property there is a lift $H' : X \times I \to E$ of H with $H|_{X \times \{0\}} = h' \circ g$. Then the restriction $H|_{X \times \{1\}}$ is a section. □

We introduce a fiberwise version of Ganea's fibration. First we recall that the k-th *Ganea's fibration* $p_k : E_k(Z, z_0) \to Z$ over a path connected space Z with a fixed base point z_0 is the fiberwise join product of $k+1$ copies of Serre's path fibrations $p_0 : PZ \to Z$. We recall that PZ consists of paths ϕ in Z with the initial point z_0 and p_0 takes ϕ to $\phi(1)$. Note that p_0 is a Hurewicz fibration and since the fiberwise join of Hurewicz fibrations is a Hurewicz fibration, so are all p_k [Sv]. Also we note that the fiber of p_0 is the loop space ΩZ and therefore, the fiber of p_k is the join product $*^{k+1}\Omega Z$ of $k+1$ copies of ΩZ (see [CLOT] for more details).

Theorem 3.5 (Ganea, Švarc)**.** *For a path connected normal space X with a non-degenerate base point, $\mathrm{cat_{LS}}(X) \le k$ if and only if the Ganea fibration $p_k : E_k(Z, z_0) \to Z$ has a section.*

The proof can be found in [CLOT], [Sv].

The Ganea construction can be done simultaneously for all possible choices of the base points z_0. Namely, for the path fibration we consider the map $\tilde{p}_0 : C(I, Z) \to Z \times Z$ defined on all paths in Z as $\tilde{p}_0(\phi) = (\phi(1), \phi(0))$. It is easy to check that $\tilde{p}_0$ is a Hurewicz fibration. Therefore the (iterated) fiberwise join of $\tilde{p}_0$ with itself is a

Hurewicz fibration. Let $\tilde{p}_k : \tilde{E}_k \to Z \times Z$ denote the fiberwise join of $k+1$ copies of $\tilde{p}_0$. We call $\tilde{p}_k$ *the extended Ganea fibration*. Note that for every $z_0 \in Z$, the preimage $\tilde{p}_k^{-1}(Z \times \{z_0\})$ is homeomorphic to $E_k(Z, z_0)$ and the restriction of $\tilde{p}_k$ to $\tilde{p}_k^{-1}(Z \times \{z_0\})$ is the Ganea fibration p_k with the base point z_0.

Now let $f : X \to Y$ be a locally trivial bundle with a path connected fiber Z and let f admit a section $s : Y \to X$. We define a space

$$E_0 = \{\phi \in C(I, X) \mid s(f\phi(I)) = \{\phi(0)\}\}$$

to be the space of all paths ϕ in X with the initial point $s(y)$ for some $y \in Y$ such that the image of ϕ is contained in the fiber $f^{-1}(y)$. The topology in E_0 is inherited from $C(I, X)$. We define a map $\xi_0 : E_0 \to X$ by the formula $\xi_0(\phi) = \phi(1)$. Then $\xi_k : E_k \to X$ is defined as the fiberwise join of $k+1$ copies of ξ_0. Formally, we define E_k inductively as a subspace of the join $E_0 * E_{k-1}$:

$$E_k = \bigcup\{\phi * \psi \in E_0 * E_{k-1} \mid \xi_0(\phi) = \xi_{k-1}(\psi)\},$$

which is the union of all intervals $[\phi, \psi] = \phi * \psi$ with the endpoints $\phi \in E_0$ and $\psi \in E_{k-1}$ such that $\xi_0(\phi) = \xi_{k-1}(\psi)$. There is a natural projection $\xi_k : E_k \to X$ that takes all points of each interval $[\phi, \psi]$ to $\phi(0)$.

Note that when $f : X = Z \times Y \to Y$ is a trivial bundle and a section $s : Y \to X$ is defined by a point $z_0 \in Z$, then $E_k = E_k(Z, z_0) \times Y$ and $\xi_k = p_k \times 1_Y$ where $p_k : E_k \to Z$ is the Ganea fibration.

Lemma 3.6. *Let $f : X \to Y$ be a locally trivial bundle between paracompact spaces with a path connected fiber Z and with a section $s : Y \to X$. Then*

i. For each k the map $\xi_k : E_k \to X$ is a Hurewicz fibration.

ii. The fiber of ξ_k is precisely the join of $k+1$ copies of the space of paths from $sf(x)$ to x which is homeomorphic to $*^{k+1}\Omega Z$.

iii. ξ_k has a section if and only if X has an open cover $\mathcal{U} = \{U_0, \dots, U_k\}$ by sets, each of which admits a fiberwise deformation into $s(Y)$.

Proof. i. In view of Dold's theorem [Do] it suffices to show that ξ_k is a Hurewicz fibration over $f^{-1}(U)$ for all $U \in \mathcal{U}$ for some locally finite cover of X. We consider a cover $\mathcal{U}$ such that f admits a trivialization over U for all $U \in \mathcal{U}$, i.e., fiberwise homeomorphisms $h_U : f^{-1}(U) \to U \times Z$. Then the section s defines a map $\sigma_U = \pi_2 \circ h_U \circ s : U \to Z$ where $\pi_2 : U \times Z \to Z$ is the projection to the second factor. If the map σ_U were constant, the fibration ξ_k over $f^{-1}(U) \cong U \times Z$ would be a Hurewicz fibration being homeomorphic to the product $1_U \times p_k$. In the general case the fibration ξ_k over $f^{-1}(U)$ is obtained as the pull-back of the extended Ganea fibration $\tilde{p}_k : \tilde{E}_k \to Z \times Z$ under the map $(\sigma_U \times 1_Z) \circ h_U : f^{-1}(U) \to Z \times Z$. Hence it is a Hurewicz fibration.

ii. We note that the map ξ_k over the fiber $(f^{-1}(x), s(x))$ coincides with the Ganea fibration p_k for Z. Therefore, the fiber of ξ_k coincides with the fiber of p_k; i.e., it is $*^{k+1}\Omega Z$.

iii. Note that when $Y = pt$, iii turns into the Ganea-Švarc theorem. Thus, iii can be viewed as a fiberwise version of the Ganea-Švarc theorem.

Suppose ξ_k has a section $\sigma : X \to E_k$. For each $x \in X$ the element $\sigma(x)$ of $*^{k+1}\Omega F$ can be presented as the $(k+1)$-tuple

$$\sigma(x) = ((\phi_0, t_0), \dots, (\phi_k, t_k)) \mid \sum t_i = 1, t_i \geq 0).$$

We use the notation $\sigma(x)_i = t_i$. Clearly, $\sigma(x)_i$ is a continuous function.

A section $\sigma : X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \dots, U_k\}$ of X as follows:

$$U_i = \{x \in X \mid \sigma(x)_i > 0\}.$$

By the construction of U_i for $i \le n$ for every $x \in U_i$ there is a canonical path connecting x with $sf(x)$. We use these paths to contract a fiberwise deformation of U_i into $s(Y)$.

The other direction of iii is not used in the paper. Since the proof of it is similar to that for the Ganea-Švarc theorem, we leave it to the reader. □

We recall that $\lceil x \rceil$ denotes the smallest integer n such that $x \le n$.

Theorem 3.7. *Suppose that a locally trivial fibration $f : X \to Y$ with an r-connected fiber F admits a section. Then*

$$\mathrm{cat_{LS}}^* f \le \left\lceil \frac{hd(X) - r}{r+1} \right\rceil.$$

Moreover,

$$\mathrm{cat_{LS}}\, X \le \mathrm{cat_{LS}}\, Y + \left\lceil \frac{hd(X) - r}{r+1} \right\rceil.$$

Proof. Let $\mathrm{cat_{LS}}\, Y = m$ and $hd(X) = n$.

Let $s : Y \to X$ be a section. By Lemma 3.6 i-ii ξ_k is a Hurewicz fibration with the fiber the join product $*^{k+1}\Omega F$ of $k+1$ copies of the loop space ΩF. Thus, it is $(k + (k+1)r - 1)$-connected. By Proposition 3.4 there is a section $\sigma : X \to E_k$ whenever $k(r+1) + r \ge n$. The smallest such k is equal to $\lceil \frac{n-r}{r+1} \rceil$.

By Lemma 3.6 iii a section $\sigma : X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \dots, U_k\}$ by the sets fiberwise contractible to $s(Y)$. Let $\mathcal{U}_{m+k} = \{U_0, \dots, U_{m+k}\}$ be an extension of $\mathcal{U}$ to a $(k+1)$-cover of X from Theorem 2.5.

Let $\mathcal{V} = \{V_0, \dots, V_{m+k}\}$ be an open Y-contractible $(m+1)$-cover of Y. We show that the sets $W_i = f^{-1}(V_i) \cap U_i$ are contractible in X for all i. By Theorem 2.5 U_i is fiberwise contractible into $s(Y)$ for $i \le m+k$. Hence we can contract $f^{-1}(V_i) \cap U_i$ to $s(V_i)$ in X. Then we apply a contraction of $s(V_i)$ to a point in $s(Y)$.

Similarly as in the proof of Theorem 3.2 we show that $\{W_i\}_{i=0}^{m+k}$ is a cover of X. Since $\mathcal{V}$ is an $(m+1)$-cover, by Proposition 2.2 every $y \in Y$ is covered by at least $k+1$ elements $V_{i_0}, \dots, V_{i_k}$ of $\mathcal{V}$. By the construction $U_{i_0}, \dots, U_{i_k}$ is a cover of X. Hence $W_{i_0}, \dots, W_{i_k}$ covers $f^{-1}(y)$. □

4. The Lusternik-Schnirelmann category of complexes with low dimensional fundamental groups

Theorem 4.1. *For every complex X with $cd(\pi_1(X)) \le 2$ the following inequality holds true:*

$$\mathrm{cat_{LS}}\, X \le cd(\pi_1(X)) + \left\lceil \frac{hd(X) - 1}{2} \right\rceil.$$

Proof. Let $\pi = \pi_1(X)$ and let $\tilde{X}$ denote the universal cover of X. We consider Borel's construction

$$\begin{array}{ccccc} \tilde{X} & \longleftarrow & \tilde{X} \times E\pi & \longrightarrow & E\pi \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{g} & \tilde{X} \times_\pi E\pi & \xrightarrow{f} & B\pi. \end{array}$$

We claim that there is a section $s : B\pi \to \tilde{X} \times_\pi E\pi$ of f. By the condition $cd\pi \leq 2$ we may assume that $B\pi$ is a complex of dimension ≤ 3. Note that f is a locally trivial bundle with the fiber $\tilde{X}$. Since the fiber of f is simply connected, there is a lift of the 2-skeleton. The condition $cd\pi \leq 2$ implies $H^3(B\pi, E) = 0$ for every π-module. Thus, we have no obstruction for the lift of the 3-skeleton (see, for example, [Po], [Th] for the basics of obstruction theory with twisted coefficients).

We apply Theorem 3.7 to obtain the inequality

$$\mathrm{cat}_{\mathrm{LS}} X \leq \mathrm{cat}_{\mathrm{LS}}(B\pi) + \left\lceil \frac{hd(\tilde{X} \times_\pi E\pi) - 1}{2} \right\rceil.$$

Since g is a fibration with the homotopy trivial fiber, the space $\tilde{X} \times_\pi E\pi$ is homotopy equivalent to X. Thus, $hd(\tilde{X} \times_\pi E\pi) = hd(X)$. Note that the results of Eilenberg and Ganea [EG] in view of the Stallings-Swan theorem [Sta], [Swan] imply that $\mathrm{cat}_{\mathrm{LS}} B\pi = cd\pi$ for all groups π. □

Corollary 4.2. *For every complex X with free fundamental group,*

$$\mathrm{cat}_{\mathrm{LS}} X \leq 1 + \left\lceil \frac{\dim X - 1}{2} \right\rceil.$$

Note that this estimate is sharp on $X = S^1 \times \mathbb{C}P^n$.

Corollary 4.3. *For every 3-dimensional complex X with free fundamental group,* $\mathrm{cat}_{\mathrm{LS}} X \leq 2$.

This corollary can also be derived from the fact that in the case of a free fundamental group every 2-complex is homotopy equivalent to the wedge of circles and 2-spheres [KR].

It is unclear whether the estimate $\mathrm{cat}_{\mathrm{LS}} X \leq 2 + \lceil \frac{\dim X - 1}{2} \rceil$ is sharp for complexes with $cd(\pi_1(X)) = 2$. It is sharp if the answer to the following question is affirmative.

Question 4.4. Does there exists a 4-complex K with free fundamental group and with $\mathrm{cat}_{\mathrm{LS}}(K \times S^1) = 4$?

Indeed, for $X = K \times S^1$ we would have the equality $4 = 2 + \lceil \frac{5-1}{2} \rceil$. Note that $cd(\pi_1(X)) = 2$.

Acknowledgement

The author is thankful to the referee for valuable remarks.

References

CLOT. Cornea, O.; Lupton, G.; Oprea, J.; Tanré, D.: Lusternik-Schnirelmann category. Mathematical Surveys and Monographs, **103**. American Mathematical Society, Providence, RI, 2003. MR1990857 (2004e:55001)

Cu. Cuvilliez, M.: LS-catégorie et k-monomorphisme. Thèse, Université Catholique de Louvain, 1998.

Do. Dold, A.: Partitions of unity in the theory of fibrations. *Ann. of Math. (2)* **78** (1963), 223-255. MR0155330 (27:5264)

DKR. Dranishnikov, A; Katz, M.; Rudyak, Y.: Small values of the Lusternik-Schnirelmann category for manifolds. *Geometry and Topology* **12** (2008), issue 3, 1711-1727. MR2421138

EG. Eilenberg, S.; Ganea, T.: On the Lusternik-Schnirelmann category of abstract groups. *Ann. of Math. (2)* **65** (1957), 517-518. MR0085510 (19:52d)

KR. Katz, M.; Rudyak, Y.: Lusternik-Schnirelmann category and systolic category of low-dimensional manifolds. *Communications on Pure and Applied Mathematics* **59** (2006), 1433-1456. MR2248895 (2007h:53055)

OW. Oprea, J.; Walsh, J.: Quotient maps, group actions and Lusternik-Schnirelmann category, *Topology Appl.* **117** (2002), 285-305. MR1874091 (2002i:55001)

Os. Ostrand, Ph.: Dimension of metric spaces and Hilbert's problem 13, *Bull. Amer. Math. Soc.* **71** (1965), 619-622. MR0177391 (31:1654)

Po. Postnikov, M.: Classification of the continuous mappings of an arbitrary n-dimensional polyhedron into a connected topological space which is aspherical in dimensions greater than unity and less than n. (Russian) *Doklady Akad. Nauk SSSR (N.S.)* **67** (1949), 427-430. MR0033522 (11:451c)

Sta. Stallings, J.: Groups of dimension 1 are locally free. *Bull. Amer. Math. Soc.* **74** (1968), 361-364. MR0223439 (36:6487)

St. Strom, J.: Lusternik-Schnirelmann category of spaces with free fundamental group. *Algebr. Geom. Topol.* **7** (2007), 1805-1808. MR2366179 (2008k:55007)

Sv. Švarc, A.: The genus of a fibered space. *Trudy Moskov. Mat. Obšč.* **10** (1961), 217-272; The genus of a fibre space. *Trudy Moskov. Mat. Obšč.* **11** (1962), 99-126; in Amer. Math. Soc. Transl. Series 2, vol. **55**, 1966. MR0154284 (27:4233), MR0151982 (27:1963)

Swan. Swan, R.: Groups of cohomological dimension one. *J. Algebra* **12** (1969), 585-610. MR0240177 (39:1531)

Th. Thomas, E.: Seminar on fiber spaces. Lecture Notes in Mathematics, **13**, Springer-Verlag, Berlin, New York, 1966. MR0203733 (34:3582)

Department of Mathematics, University of Florida, 358 Little Hall, Gainesville, Florida 32601-8105

E-mail address: `dranish@math.ufl.edu`