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ON THE CLASSIFICATION OF SOLUTIONS OF $-\Delta u = e^u$ ON $\mathbb{R}^N$: STABILITY OUTSIDE A COMPACT SET AND APPLICATIONS

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ABSTRACT. In this short paper we prove that, for $3 \le N \le 9$, the problem $-\Delta u = e^u$ on the entire Euclidean space $\mathbb{R}^N$ does not admit any solution stable outside a compact set of $\mathbb{R}^N$. This result is obtained without making any assumption about the boundedness of solutions. Furthermore, as a consequence of our analysis, we also prove the non-existence of finite Morse Index solutions for the considered problem. We then use our results to give some applications to bounded domain problems.

1. INTRODUCTION AND MAIN RESULTS

In this short paper we study and classify solutions $u \in C^2(\mathbb{R}^N)$ of the semilinear partial differential equation

$$-\Delta u = e^u \quad \text{on } \mathbb{R}^N, \qquad N \ge 2, \tag{1.1}$$

which are stable outside a compact set of $\mathbb{R}^N$.

Let us recall that, given a domain $\Omega \subset \mathbb{R}^N$ (possibly unbounded), a solution $u \in C^2(\Omega)$ of the equation $-\Delta u = e^u$ is said to be *stable outside a compact set* $\mathcal{K} \subset \Omega$ if

$$\forall\ \psi \in C_c^1(\Omega \setminus \mathcal{K}) \qquad Q_u(\psi) := \int_{\Omega\setminus\mathcal{K}} |\nabla\psi|^2 - e^u\psi^2 \ \ge 0.$$

In particular, when $\mathcal{K}$ is the empty set, the solution u is said to be stable in Ω.

Our main result is the following.

Theorem 1. *Let $3 \le N \le 9$. Equation* (1.1) *does not admit any C^2 solution stable outside a compact set of $\mathbb{R}^N$.*

The proof of the above result is based on the methods and techniques developed by the second author in [9], [10], [11] as well as on the following integral estimate also proved in [11] (cf. Proposition 5 therein).

Proposition 1. *Assume $N \ge 2$ and let Ω be a domain (possibly unbounded) of $\mathbb{R}^N$. Let $u \in C^2(\Omega)$ be any* **stable** *solution of*

$$-\Delta u = e^u \qquad \text{on } \Omega.$$

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Then, for any integer $m \geq 5$ and any $\alpha \in (0,2)$ we have

$$\int_\Omega e^{(2\alpha+1)u}\psi^{2m} \leq \Big(\frac{m}{2-\alpha}\Big)^{2\alpha+1}\int_\Omega \Big(|\nabla\psi|^2 + |\psi||\Delta\psi|\Big)^{2\alpha+1} \tag{1.2}$$

for all test functions $\psi \in C_c^2(\Omega)$ satisfying $0 \leq \psi \leq 1$ in Ω.

Before turning to the proof of Theorem 1 we would like to make some remarks.

Remark 1. (*i*) Note that Theorem 1 is obtained without making any assumption about the boundedness of solutions. This is an important feature of our result. Indeed, an assumption about the boundedness of solutions (either from below or from above) is a classic (and often necessary) hypothesis, when one studies classification results or non-existence results, for elliptic problems on the entire Euclidean space $\mathbb{R}^N$.

(*ii*) Note that the family of the solutions of (1.1) stable outside a compact set of $\mathbb{R}^N$ includes, as a special case, all finite Morse index solutions (and hence all stable solutions) of (1.1) (see for instance [5], [9], [10], [11]). Thus, for $3 \leq N \leq 9$, the above Theorem 1 also provides a complete classification for this kind of solutions. In this regard we recall that the second author [11] proved that, for $2 \leq N \leq 9$, there is no stable C^2 solution of the equation (1.1) (see also the very recent preprint [8] for a similar result under the stronger additional assumption that u is bounded above). For $N = 2$ and 3 and, again, under the additional assumption that u is bounded above, the non-existence of stable solutions was firstly obtained by the first author [5]. Also, the first author [6] proved that, for $N = 3$, the equation (1.1) has no negative solution of finite Morse index. The results proved in [5], [6] and [8] crucially depend on the assumption that the considered solutions are bounded from above while, as already pointed out, our Theorem 1 is free from this constraint.

(*iii*) Theorem 1 is sharp. Indeed, on the one hand, for $N = 2$ the equation (1.1) admits radial solutions stable outside a compact set (cf. Theorem 3 of [11], where all stable solutions outside a compact set of $\mathbb{R}^2$ are classified). On the other hand, for every $N \geq 10$ the equation (1.1) possesses a radial stable solution. The existence of such a solution is a consequence of the analysis performed in [12], as was remarked in [6].

(*iv*) The above theorem answers an open question raised in [11].

The paper is organized as follows. In section 1 we prove our main result, namely Theorem 1. The second and last section is devoted to some applications to bounded domain problems.

Proof of Theorem 1. First we define some smooth compactly supported functions to be used in Proposition 1.

We choose $\varphi \in C_c^2(\mathbb{R})$ satisfying $0 \leq \varphi \leq 1$ everywhere on $\mathbb{R}$ and

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \leq 1, \\ 0 & \text{if } |t| \leq 2. \end{cases}$$

For $s > 0$, we choose a function θ_s such that : $\theta_s \in C_c^2(\mathbb{R}),\ 0 \leq \theta_s \leq 1$ everywhere on $\mathbb{R}$ and

$$\theta_s(t) = \begin{cases} 0 & \text{if } |t| \leq s+1, \\ 1 & \text{if } |t| \geq s+2. \end{cases}$$

The proof of the theorem is by contradiction, and we split it into four steps. Let us suppose that equation (1.1) admits a C^2 solution, which is stable outside a compact set of $\mathbb{R}^N$. Then:

Step 1. *There exists* $R_0 = R_0(u) > 0$ *such that*
a) *for every* $\alpha \in (0,2)$ *and every* $r > R_0 + 3$ *we have*

$$\int_{\{R_0+2<|x|<r\}} e^{(2\alpha+1)u}dx \le A + Br^{N-2(2\alpha+1)}, \tag{1.3}$$

where A *and* B *are positive constants depending on* $\alpha,\ N,$ *and* R_0 *but not on* r.
b) *For every* $\alpha \in (0,2)$ *and every open ball* $B(y,R)$ *such that* $B(y,2R) \subset \{\, x \in \mathbb{R}^N \ :\ |x| > R_0 \,\}$, *we have*

$$\int_{B(y,R)} e^{(2\alpha+1)u}dx \le CR^{N-2(2\alpha+1)}, \tag{1.4}$$

where C *is a positive constant depending on* α, N, R_0 *but not on* R *or on* y.

Since u is stable outside a compact set of $\mathbb{R}^N$ there exists $R_0 > 0$ such that Proposition 1 holds true with $\Omega := \mathbb{R}^N \setminus \overline{B(0,R_0)}$. We fix $m = 5$ and, for every $r > R_0 + 3$, we consider the following test function $\xi_r \in C_c^2(\mathbb{R}^N)$:

$$\xi_r(x) = \begin{cases} \theta_{R_0}(|x|) & \text{if } |x| \le R_0+3, \\ \varphi\left(\frac{|x|}{r}\right) & \text{if } |x| \ge R_0+3, \end{cases}$$

which inserted into (1.2) gives

$$\begin{aligned}
\int_{\{R_0+2<|x|<r\}} e^{(2\alpha+1)u}dx &\le \int_\Omega e^{(2\alpha+1)u}dx \\
&\le \left(\frac{m}{2-\alpha}\right)^{2\alpha+1} \int_\Omega \left(|\nabla \xi_r|^2 + |\xi_r||\Delta\xi_r|\right)^{2\alpha+1} \\
&\le \left(\frac{m}{2-\alpha}\right)^{2\alpha+1} \Big[\int_{\{|x|\le R_0+3\}} \left(|\nabla\theta_{R_0}|^2 + |\theta_{R_0}||\Delta\theta_{R_0}|\right)^{2\alpha+1}dx \\
&\qquad + \int_{\{r\le|x|\le 2r\}} \left(|\nabla \xi_r|^2 + |\xi_r||\Delta\xi_r|\right)^{2\alpha+1}dx\Big] \\
&\le C_1(\alpha,N,\theta_{R_0}) + C_2(\alpha,N,\varphi)r^{N-2(2\alpha+1)},
\end{aligned}$$

for all $r > R_0 + 3$. Hence, the desired integral estimate (1.3) follows.

The integral estimate (1.4) is obtained in the same way by using the test functions $\psi_{R,y}(x) := \varphi\Big(\frac{|x-y|}{R}\Big)$ in Proposition 1.

Step 2. *Let* $\eta > 0$. *Then there exist* $R_1 = R_1(N,\eta,u) > R_0$ *such that*

$$\int_{|x|>R_1} e^{\frac{N}{2}u}dx \le \eta^{\frac{N}{2}}. \tag{1.5}$$

Let $\alpha_1 := \frac{N-2}{4} \in (0,2)$. By part a) of Step 1 we infer that, for all $r > R_0 + 3$,

$$\int_{\{R_0+2<|x|<r\}} e^{\frac{N}{2}u}dx = \int_{\{R_0+2<|x|<r\}} e^{(2\alpha_1+1)u}dx \le A + Br^{N-2(2\alpha_1+1)};$$

hence, $\int_{|x|>R_0+2} e^{\frac{N}{2}u}dx < +\infty$, which immediately yields (1.5).

Step 3.

$$\lim_{|x|\to+\infty} |x|^2 e^{u(x)} = 0.$$

Set $\epsilon = \frac{1}{10}$ and observe that $\frac{N}{2-\epsilon} \in (1,5)$. Thus, there exist $\alpha_2 = \alpha_2(N) \in (0,2)$ such that $2\alpha_2 + 1 = \frac{N}{2-\epsilon}$. Here we have used the assumption $3 \le N \le 9$.

Next we fix $\eta > 0$ and observe that $w = e^u$ satisfies

$$-\Delta w - e^u w \le 0 \qquad \text{on } B(y, 2R).$$

According to a well-known Harnack inequality (cf. [14], [13]) for positive subsolutions of the linear equation $-\Delta w - e^u w = 0$, we have, for any $t > 1$,

$$\|w\|_{L^\infty(B(y,R))} \le C_{ST} R^{-\frac{N}{t}} \Big(\|w\|_{L^t(B(y,2R))} \Big), \tag{1.6}$$

where C_{ST} is a positive constant depending on N and also on

$$R^\epsilon \|e^u\|_{L^{\frac{N}{2-\epsilon}}(B(y,2R))}. \tag{1.7}$$

In order to apply the above result we consider points $y \in \mathbb{R}^N$ such that $|y| > 10R_1$ and set $R = \frac{|y|}{4}$, $t = \frac{N}{2} > 1$. Here $R_1 > R_0$ is defined by (1.5) of Step 2. This choice yields

$$B(y, 2R) \subset \{\, x \in \mathbb{R}^N \ :\ |x| > R_1 \,\} \subset \{\, x \in \mathbb{R}^N \ :\ |x| > R_0 \,\},$$

$$\int_{|x| \ge R_1} e^{\frac{N}{2} u} dx < \eta^{\frac{N}{2}}$$

and

$$R^\epsilon \|e^u\|_{L^{\frac{N}{2-\epsilon}}(B(y,2R))} = R^\epsilon \Big(\int_{B(y,2R)} e^{\frac{N}{2-\epsilon} u} \Big)^{\frac{2-\epsilon}{N}}$$

$$= R^\epsilon \Big(\int_{B(y,2R)} e^{(2\alpha_2+1)u} \Big)^{\frac{2-\epsilon}{N}} \le R^\epsilon \Big[C R^{N-2(2\alpha_2+1)} \Big]^{\frac{2-\epsilon}{N}} \le C' R^\epsilon R^{2-\epsilon} R^{-2} = C',$$

where in the latter we have used part (b) of Step 1.

This proves that the constant C_{ST} in (1.6) is independent of both y and R. Actually it depends only on N and R_0.

Now, using $t = \frac{N}{2}$ in (1.6) and Step 2, we are led to

$$|e^{u(y)}| \le C_{ST} R^{-2} \|w\|_{L^{\frac{N}{2}}(B(y,2R))} \le 16 C_{ST} |y|^{-2} \|w\|_{L^{\frac{N}{2}}(B(y,2R))} \le 16 C_{ST} |y|^{-2} \eta,$$

which proves the claim.

Step 4. *End of the proof.*

By Step 3, there exist $R_2 > 0$ such that the function $v = v(|x|)$, defined as the average of the solution u over spheres of radii $|x| > 0$, satisfies

$$-\Delta v \le \frac{1}{2r^2} \qquad \forall\, r > R_2.$$

Hence, the radial function v satisfies

$$v'(r) \ge \frac{C(N)}{r^{N-1}} - \frac{1}{2(N-2)r} \qquad \forall\, r > R_2,$$

and thus

$$v'(r) \ge -\frac{1}{r} \qquad \forall\, r > R_3,$$

for some $R_3 > R_2$. Integrating the latter and taking the exponential we get

$$r^2 e^{v(r)} \ge Cr \qquad \forall\, r > R_3, \tag{1.8}$$

where C is a positive constant independent of r.

To conclude the proof of Theorem 1, we observe that (1.8) contradicts what we proved in Step 3. Indeed, by Jensen's inequality, we have

$$\max_{|x|=r}(|x|^2 e^{u(x)}) = r^2 \max_{|x|=r} e^{u(x)} \geq r^2 \frac{1}{|\{|x|=r\}|} \int_{\{|x|=r\}} e^u \geq r^2 e^{v(r)} \geq cr \qquad \forall\, r > R_3,$$

which clearly contradicts the claim of Step 3.

2. Some applications to bounded domain problems

We conclude the present work with two applications of our main results to bounded domain problems.

Theorem 2. *Assume $3 \leq N \leq 9$ and let Ω be a bounded smooth domain of $\mathbb{R}^N$. Assume that $(u_n)_{n\in\mathbf{N}}$ is a sequence of smooth solutions of the problem*

$$\begin{cases} -\Delta u = e^u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Then the sequence $(u_n)_{n\in\mathbf{N}}$ is bounded in $L^\infty(\Omega)$ if and only if the sequence of their Morse indices is bounded in $\mathbb{R}$.

This result extends, up to dimension $N = 9$, a previous result established by E.N. Dancer [6] in dimension $N = 3$. (See also [1] and [10] for similar results for the case of non-linearities of power-like type.)

The proof of the above Theorem 2 is the same as the one of Theorem 2.1 of [6]. The only difference is that the use of Theorem 1.1 of [6] is replaced by the use of Theorem 1 here.

Our next result concerns positive solutions of the problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $\lambda \geq 0$, Ω is a smooth bounded domain of $\mathbb{R}^N$, with $3 \leq N \leq 9$, $f \in C^1(\mathbb{R},\mathbb{R})$, $f(t) > 0$ for $t > 0$ and either $f(0) > 0$ or $f(0) = 0$ and $f'(0) > 0$. Furthermore, we assume that f is a real analytic function in a neighborhood of $[0, +\infty)$ such that $\lim_{t\to+\infty} \frac{f'(t)}{e^t} = C > 0$, and we let λ_1 be the first eigenvalue of $-\Delta$ on Ω for the Dirichlet boundary conditions. In [4], by making use of analytic bifurcation theory [3], it is proven that there exist an unbounded connected arc of positive solutions $\widehat{T}$ of the problem (3.2) such that $(0, \lambda_1(f'(0))^{-1}) \in \widehat{T}$ (or $(0,0) \in \widehat{T}$ if $f(0) > 0$) and $\|u(s)\|_{C^1} + |\lambda(s)| \to +\infty$ as $s \to +\infty$ (where $\widehat{T} = \{\, (u(s), \lambda(s)) \,:\, s \geq 0 \,\}$) and $-\Delta - \lambda(s) f'(u(s)) Id$ (plus the boundary condition) is an invertible operator except at isolated points. Thus, one can apply the implicit function theorem on $\widehat{T}$ except at the above-mentioned isolated points. This enables us to prove the following.

Theorem 3. *Assume that the above conditions hold and let S be a bounded subset of $C^0[0,1] \times \mathbb{R}$. Then the set $\hat{T} \setminus S$ contains infinitely many bifurcation points.*

The proof of the above result is a modification of the one of Theorem 2.2 of [6]. (One has to note that the use of Theorem 2.1 of [6] is replaced here by the use of Theorem 1 and that the blow-up argument performed in [5] still applies in the present situation.) For these reasons we omit the details.

Remark 2. Theorem 3 generalizes a result of Joseph and Lundgren [12], which was proved for the special case $f(t) = e^t$ and Ω a Euclidean open ball (the so-called Gelfand problem on an open ball). Our method is completely different from the one used in [12], and our results hold true for much more general domains as well as for much more general non-linearities f.

(ii) The results proven in [12] show that Theorem 3 does not hold for an open ball whenever $N \geq 10$. (The method of moving planes shows that all solutions must be radial in this case.)

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