LEONHARD EULER
AND A $q$-ANALOGUE OF THE LOGARITHM

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On the 300th anniversary of Euler’s birth

Abstract. We study a $q$-logarithm which was introduced by Euler and give
some of its properties. This $q$-logarithm has not received much attention in
the recent literature. We derive basic properties, some of which were already
given by Euler in a 1751 paper and in a 1734 letter to Daniel Bernoulli. The
corresponding $q$-analogue of the dilogarithm is introduced. The relation to the
values at 1 and 2 of a $q$-analogue of the zeta function is given. We briefly
describe some other $q$-logarithms that have appeared in the recent literature.

1. Introduction

In a paper from 1751, Leonhard Euler (1707–1783) introduced the series [8, §6]

\begin{equation}
    s = \sum_{k=1}^{\infty} \frac{(1-x)(1-x/a)\cdots(1-x/a^{k-1})}{1-a^k}.
\end{equation}

We will take $q = 1/a$. Then this series is convergent for $|q| < 1$ and $x \in \mathbb{C}$. In this
dpaper we will assume $0 < q < 1$. Then this becomes

\begin{equation}
    S_q(x) = -\sum_{k=1}^{\infty} \frac{q^k}{1-q^k}(x; q)_k,
\end{equation}

where $(x; q)_0 = 1$, $(x; q)_k = (1-x)(1-xq)\cdots(1-xq^{k-1})$. This can be written as
a basic hypergeometric series

\[ S_q(x) = -\frac{q(1-x)}{1-q} \, _3\phi_2 \left( \begin{array}{c} q, qx \\ q^2, 0 \end{array} ; q, q \right). \]

Euler had come across this series much earlier in an attempt to interpolate the
logarithm at powers $a^k$ (or $q^{-k}$); see, e.g., Gautschi’s comment [11] discussing
Euler’s letter to Daniel Bernoulli where Euler introduced the function for $a = 10$.
Euler was aware that this interpolation did not work very well; see [11, §§3–4]. The
function in [12] does not seem to appear in the recent literature, even though it has
some nice properties. We will prove some of its properties, some already obtained.
by Euler [8], and indicate why this should be called a $q$-analogue of the logarithm. A first reason is that for $0 < q < 1$,

$$\lim_{q \to 1} (1 - q)S_q(x) = -\sum_{k=1}^{\infty} q^{k} \frac{1 - q}{1 - q^{k}} (x; q)_{k} = -\sum_{k=1}^{\infty} \frac{(1 - x)^k}{k} = \log x,$$

which is only a formal limit transition, since interchanging limit and sum seems hard to justify.

In Sections 2 and 3 we study this $q$-analogue of the logarithm more closely. In particular, we reprove some of Euler’s results. Then we go on to extend the definition in Section 4. Finally, we study the corresponding $q$-analogue of the dilogarithm in Section 5. It involves also the values at 1 and 2 of a $q$-analogue of the $\zeta$-function. We give a (incomplete) list of some other $q$-analogues of the logarithm appearing in the literature in Section 6. The purpose of this note is to draw attention to the $q$-analogues of the logarithm, dilogarithm and $\zeta$-function for which we expect many interesting results remain to be discovered.

Many results in this paper use the $q$-binomial theorem [10, §1.3], [11, §10.2]

$$\frac{(ax; q)_{\infty}}{(x; q)_{\infty}} = \sum_{j=0}^{\infty} \frac{(a; q)_{j}}{(q; q)_{j}} x^{j}, \quad |x| < 1. \tag{1.3}$$

We also use the $q$-exponential functions [10, p. 9], [11, p. 492]

$$e_q(z) = \frac{1}{(z; q)_{\infty}} = \sum_{n=0}^{\infty} \frac{z^{n}}{(q; q)_{n}}, \quad |z| < 1,$n

$$E_q(z) = (-z; q)_{\infty} = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q; q)_{n}} z^{n}.$$

2. The $q$-logarithm as an entire function

First of all we will show that the function $S_q$ in (1.2) is an entire function, and as such it is a nicer function than the logarithm, which has a cut along the negative real axis.

**Property 2.1.** The function $S_q$ defined in (1.2) is an entire function of order zero.

**Proof.** For $k \in \mathbb{N}$ the $q$-Pochhammer $(z; q)_{k}$ is a polynomial of degree $k$ with zeros at $1, 1/q, \ldots, 1/q^{k-1}$. For $|z| \leq r$ we have the simple bound

$$|(z; q)_{k}| \leq (1 + r)(1 + r|q|) \cdots (1 + r|q|^{k-1}) = (-r; |q|)_{k} < (-r; |q|)_{\infty},$$

and hence the partial sums are uniformly bounded on the ball $|z| \leq r$:

$$\left| -\sum_{k=1}^{n} \frac{q^{k}}{1 - q^{k}} (z; q)_{k} \right| \leq (-r; |q|)_{\infty} \sum_{k=1}^{\infty} \frac{|q|^{k}}{1 - |q|^{k}}.$$

The partial sums therefore are a normal family and are uniformly convergent on every compact subset of the complex plane. The limit of these partial sums is $S_q(z)$ and is therefore an entire function of the complex variable $z$.

Let $M(r) = \max_{|z| \leq r} |S_q(z)|$. Then

$$M(r) \leq (-r; |q|)_{\infty} \sum_{k=1}^{\infty} \frac{|q|^{k}}{1 - |q|^{k}}.$$
and \((-r; q)_\infty = E_{|q|}(r)\) is the maximum of \(E_{|q|}(z)\) on the ball \(\{|z| \leq r\}\). The function \(E_q\) is an entire function of order zero, which can be seen from the coefficients \(a_n\) of its Taylor series and the formula \([2, Theorem 2.2.2]\)

\[
(2.1) \quad \lim_{n \to \infty} \sup \frac{n \log n}{\log(1/|a_n|)}
\]

for the order of \(\sum_{n=0}^{\infty} a_n z^n\). Hence also \(S_q\) has order zero. \(\square\)

Observe that for \(0 < q < 1\) we have

\[
M(r) = \max_{|z| \leq r} |S_q(z)| = \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} (-r; q)_k
\]

and some simple bounds give

\[
(q; q)_\infty \sum_{k=1}^{\infty} \frac{q^k}{(q; q)_k} (-r; q)_k \leq M(r) \leq (-r; q)_\infty \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k}.
\]

For the lower bound we can use the \(q\)-binomial theorem \([1.3]\) to find

\[
(-rq; q)_\infty - (q; q)_\infty \leq M(r) \leq (-r; q)_\infty \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k},
\]

which shows that \(M(r)\) behaves like \(E_q(qr) - C_1 \leq M(r) \leq C_2 E_q(r)\), where \(C_1\) and \(C_2\) are constants (which depend on \(q\)).

Euler \([8, \S\S 14-15]\) essentially also stated the following Taylor expansion.

**Property 2.2.** The \(q\)-logarithm \([1.2]\) has the following Taylor series around \(x = 0\):

\[
S_q(x) = -\sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} \left(1 + q^{k(k-1)/2} \frac{(-x)^k}{(q; q)_k}\right).
\]

**Proof.** Use the \(q\)-binomial theorem \([1.3]\) with \(x = z q^k\) and \(a = q^{-k}\) to find

\[
(2.2) \quad (z; q)_k = \sum_{j=0}^{k} \binom{k}{j} q^{j(j-1)/2} (-z)^j, \quad \binom{k}{j} = \frac{(q; q)_k}{(q; q)_j (q; q)_{k-j}}.
\]

Use this in \([1.2]\), and change the order of summation to find

\[
S_q(x) = -\sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} - \sum_{j=1}^{\infty} q^j (q; q)_j \sum_{k=j}^{\infty} \frac{q^k}{1 - q^k} \frac{(q; q)_k}{(q; q)_j (q; q)_{k-j}}.
\]

With a new summation index \(k = j + \ell\) this becomes

\[
S_q(x) = -\sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} - \sum_{j=1}^{\infty} \frac{q^j}{1 - q^j} q^{j(j-1)/2} (-x)^j \sum_{\ell=0}^{\infty} \frac{q^\ell (q^2; q)_{\ell}}{(q; q)_{\ell}}.
\]

Now use the \(q\)-binomial theorem \([1.3]\) to sum over \(\ell\) to find

\[
S_q(x) = -\sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} - \sum_{j=1}^{\infty} \frac{q^j}{1 - q^j} q^{j(j-1)/2} (-x)^j \frac{(q^2; q)}{(q; q)_{j}}.
\]

If we combine both series, the required expansion follows. \(\square\)
This result can be written in terms of basic hypergeometric series as

\[ S_q(x) = -\frac{q}{1-q} \phi_1\left(\frac{q, q}{q^2}; q, q\right) - \frac{q x}{(1-q)^2} \phi_2\left(\frac{q, q}{q^2}; q, q^2 x\right). \]

The growth of the coefficients in this Taylor series again shows that \( S_q \) is an entire function of order zero if we use the formula (2.1) for the order of \( \sum_{n=0}^{\infty} a_n z^n \); see also [11 §4].

Next we mention the following \( q \)-integral representation, where we use Jackson’s \( q \)-integral (see [10 §1.11])

\[
\int_{0}^{a} f(t) \, dq \, t = (1-q)a \sum_{k=0}^{\infty} f(aq^k) \, q^k,
\]

defined for functions \( f \) whenever the right-hand side converges.

**Property 2.3.** For every \( x \in \mathbb{C} \) we have

\[ S_q(x) = \frac{q(1-x)}{1-q} \int_{0}^{1} G_q(qx, qt) \, dq \, t, \]

with

\[ G_q(x, t) = \sum_{k=0}^{\infty} t^k(x; q)_k = \phi_1\left(\frac{x, q}{0}; q, t\right) = \frac{1}{1-t} \phi_1\left(\frac{q}{qt}; q, xt\right). \]

Since \( \int_{0}^{a} f(t) \, dq \, t \to \int_{0}^{a} f(t) \, dt \) when \( q \to 1 \) and \( G_q(x, t) \to 1/(1-t(1-x)) \) when \( q \to 1 \) for \( x > 0 \), we see (at least formally) that Property 2.3 is a \( q \)-analogue of the integral representation

\[ \log(x) = -\int_{0}^{1} \frac{1-x}{1-t(1-x)} \, dt, \quad x \notin (-\infty, 0] \]

for the logarithm.

**Proof.** Observe that

\[ \frac{1-q}{1-q^{k+1}} = (1-q) \sum_{p=0}^{\infty} q^{(k+1)p} = \int_{0}^{1} t^k \, dq \, t. \]

Inserting this in the definition (1.2) of \( S_q \) and interchanging summations, which is justified by the absolute convergence of the double sum, give the result. The identity between the basic hypergeometric series representing \( G_q(x, t) \) is the case \( c = 0 \) of [10 (III.4)]. \( \square \)

Note that, as in the proof of Property 2.2, one can show that

\[
G_q(x, t) = \sum_{j=0}^{\infty} \frac{(-xt)^j q^{j(j-1)/2}}{(t; q)_{j+1}}.
\]

**3. The \( q \)-difference equation**

The function \( S_q \) satisfies a simple \( q \)-difference equation:

**Property 3.1.** The \( q \)-logarithm (1.2) satisfies

\[ (3.1) \quad S_q(x/q) - S_q(x) = 1 - (x; q)_\infty. \]
Proof. Recall the $q$-difference operator

$$D_q f(x) = \frac{f(qx) - f(x)}{x(q - 1)}.$$  

Then a simple exercise is

$$D_{1/q}(x; q)_k = -\frac{1-q^k}{1-q}(x; q)_{k-1}.$$  

Use this in (1.2) to find

$$D_{1/q}S_q(x) = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \frac{1-q^k}{1-q}(x; q)_{k-1} = \frac{q}{1-q} \sum_{k=0}^{\infty} q^k(x; q)_k.$$  

Observe that $(x; q)_k - (x; q) = (x; q)_k [1 - xq^k - 1] = -xq^k(x; q)_k$. Summing we find

$$-x \sum_{k=0}^{n} q^k(x; q)_k = (x; q)_{n+1} - (x; q)_0,$$

and when $n \to \infty$,

$$\sum_{k=0}^{\infty} q^k(x; q)_k = \frac{1 - (x; q)_{\infty}}{x}.$$  

If we use this result, then

$$D_{1/q}S_q(x) = \frac{q}{1-q} \frac{1 - (x; q)_{\infty}}{x},$$  

which is (3.1). \qed

In order to see how this is related to the classical derivative of $\log x$, one may rewrite this as

$$D_q((1-q)S_q(x)) = \frac{1}{x} - \frac{(qx; q)_{\infty}}{x}.$$  

This $q$-difference equation can already be found in [8, §6], where Euler writes $s = S_q(x)$ and $t = S_q(x/q)$ and gives the relation

$$1 + s - t = (1-x) \left(1 - \frac{x}{a}\right) \left(1 - \frac{x}{a^2}\right) \left(1 - \frac{x}{a^3}\right) \left(1 - \frac{x}{a^4}\right) \cdots,$$

where $q = 1/a$.

As a corollary one has [8, §7].

Property 3.2. For every positive integer $n$ one has $S_q(q^{-n}) = n$.

Proof. Use (3.1) with $x = q^{-n+1}$ to find $S_q(q^{-n}) - S_q(q^{-n+1}) = 1$, since $(x; q)_{\infty}$ vanishes whenever $x = q^{-n}$ for $n \geq 0$. The result then follows by induction and $S_q(1) = 0$. \qed

It is this property, which is quite similar to $\log_a a^n = n$, where $\log_a$ is the logarithm with base $a$, which gives $S_q$ the flavor of a $q$-logarithm and which made Euler consider this function as an interpolation of the logarithm; see [11, §1]. Observe that this interpolation property can be stated as follows: $-\log q S_q(x)$ approximates $\log x$ as $q \uparrow 1$ and for fixed $q$ this approximation is perfect if $x = q^{-n}$ ($n = 1, 2, \ldots$).

Another interesting value is

$$S_q(0) = -\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} = -\zeta_q(1),$$

where $\zeta_q(s)$ is Euler’s $q$-logarithm.
which is a $q$-analogue of the harmonic series, where the $q$-analogue of the ζ-function is defined by

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{n^{s-1}q^n}{1-q^n}.$$  

It has been proved (see Erdős [7], Borwein [3, 4], Van Assche [27]) that this quantity is irrational whenever $q = 1/p$ with $p$ an integer $\geq 2$. For the specific argument 1 this coincides, up to a factor, with the value at 1 of the $q$-ζ-function considered by Ueno and Nishizawa [26].

The values of $S_q(q^n)$ for $n \in \mathbb{N}$ are distinctly different, and for these values we do not get the same flavor as the logarithm.

**Property 3.3.** For every positive integer $n$ one has

$$S_q(q^n) = -n + (q; q)_\infty \sum_{k=0}^{n-1} \frac{1}{(q; q)_k}.$$  

**Proof.** Choose $x = q^{k+1}$ in (3.1). Then $S_q(q^k) - S_q(q^{k+1}) = 1 - (q^{k+1}; q)_\infty$. Summing and using the telescoping property give

$$S_q(q^0) - S_q(q^n) = \sum_{k=0}^{n-1} (S_q(q^k) - S_q(q^{k+1})) = n - \sum_{k=0}^{n-1} (q^{k+1}; q)_\infty.$$  

By Property 3.2 we have $S_q(1) = 0$. Now $(q^{k+1}; q)_\infty = (q; q)_\infty/(q; q)_k$ gives the required expression (3.2).

In order to see how this approximates $\log x$, one may reformulate this as

$$-\log q \ S_q(q^n) = \log q^n - \log q \sum_{k=0}^{n-1} (q^{k+1}; q)_\infty.$$  

In [8, §10] Euler writes $s = S_q(q^n)$, $t = S_q(q^{n-1})$, $u = S_q(q^{n-2})$, and he writes the recursion

$$s = \frac{2t - u + aq^n(1 - t)}{1 - aq^n},$$  

where $q = 1/a$. In contemporary notation we write $y_n = S_q(q^n)$ and obtain the recurrence relation

$$y_n(1 - q^{n-1}) - (2 - q^{n-1})y_{n-1} + y_{n-2} = q^{n-1}.$$  

One can verify that this recurrence relation indeed holds for $y_n = S_q(q^n)$ given in (3.2). More generally one in fact has

$$(1 - qx)S_q(q^2x) - (2 - qx)S_q(qx) + S_q(x) = qx,$$

which is a second order non-homogeneous $q$-difference equation for $S_q$.

Note that the explicit evaluation $S(q^{-n}) = n, n \in \mathbb{N}$, gives the following summation formulas:

$$\sum_{k=1}^{\infty} \frac{q^{n-k}}{1-q^k} q^k = -n, \quad \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1}q^{-nk}}{(1-q^k)(q; q)_k} = n + \sum_{k=1}^{\infty} \frac{q^k}{1-q^k},$$
using the definition of $S_q(x)$ and the Taylor expansion in Property 2.2. Similarly, the evaluation at $q^n$, $n \in \mathbb{N}$, given in (3.2) gives the summation formulas

$$
\sum_{k=1}^{\infty} \frac{(q^n; q)_k}{1 - q^k} q^k = n - \sum_{k=0}^{n-1} (q^{k+1}; q)_\infty,
$$

(3.4)

$$
\sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1} q^{nk}}{(1 - q^k) (q; q)_k} = -n + \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} + \sum_{k=0}^{n-1} (q^{k+1}; q)_\infty.
$$

Note that all infinite series are absolutely convergent and that for $n = 0$ the results in (3.3) and (3.4) coincide. The first sums become trivial, and the second sum gives the following expansion for $\zeta_q(1)$:

$$
\zeta_q(1) = \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} = \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1}}{(1 - q^k) (q; q)_k}.
$$

Using (3.5) in Property 2.2 gives the expansion

$$
S_q(x) = - \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1}(1 - x^k)}{(1 - q^k) (q; q)_k},
$$

so that in particular

$$
\frac{dS_q}{dx}(1) = \lim_{x \to 1} \frac{S_q(x)}{1 - x} = - \sum_{k=1}^{\infty} \frac{k q^{k(k+1)/2}(-1)^{k-1}}{(1 - q^k) (q; q)_k}.
$$

4. AN EXTENSION OF THE $q$-LOGARITHM AND LAMBERT SERIES

If we have the definition of $S_q(x)$ resembling Lambert series, it is natural to look for the extension

$$
F_q(x, t) = - \sum_{k=1}^{\infty} (x; q)_k \frac{t^k}{1 - t^k},
$$

which is a Lambert series; see [13, §58.C]. Since $|(x; q)_k| \leq (-|x|; |q|)_k \leq (-r; |q|)_\infty$ for $x \in \mathbb{C}$ with $|x| \leq r$, the convergence in (4.1) is uniform on compact sets in $x$ and on compact subsets of the open unit disk in $t$. Also since the series $- \sum_{k=1}^{\infty} (x; q)_k t^k$ is absolutely convergent for $|t| < 1$ uniformly in $x$ in compact sets, it follows by [13, Satz 259] that $F_q$ is analytic for $(x, t) \in \mathbb{C} \times \{t \in \mathbb{C} \mid |t| < 1\}$. Observe that $S_q(x) = F_q(x, q)$.

The general theory of Lambert series then gives the power series of $F$ in powers of $t$:

$$
F_q(x, t) = \sum_{\ell=1}^{\infty} \left( \sum_{k|\ell} (x; q)_k \right) t^\ell \implies S_q(x) = \sum_{\ell=1}^{\infty} \left( \sum_{k|\ell} (x; q)_k \right) q^\ell.
$$

We are mainly interested in the power series development with respect to $x$.

**Property 4.1.** For $|t| < 1$ one has

$$
F_q(x, t) = - \sum_{k=1}^{\infty} \frac{t^k}{1 - t^k} - \sum_{\ell=1}^{\infty} x^\ell (-1)^\ell q^{\ell(\ell-1)/2} \left( \sum_{n=1}^{\infty} n^\ell \frac{(t^n q^{\ell+1}; q)_\infty}{(t^n; q)_\infty} \right).
$$
In case \( t = q \), Property 4.1 reduces to Property 2.2 and this is equivalent to the summation formula

\[
(4.2) \quad \sum_{n=1}^{\infty} q^{n\ell} \frac{(q^{\ell+n+1}; q)_\infty}{(q^\ell; q)_\infty} = \frac{q^\ell}{(1-q^\ell) (q; q)_\ell} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{(q; q)_{n-1}}{(q^{\ell+1}; q)_n} q^{n\ell} = \frac{q^\ell}{1-q^\ell}
\]

for \( \ell \in \mathbb{N}, \ell \geq 1 \). This can be obtained as a special case of the \( q \)-Gauss sum \([10, (1.5.1)]\).

**Proof.** The proof is along the same lines as the proof of Property 2.2. We find similarly

\[
F_q(x, t) = -\sum_{k=1}^{\infty} \frac{t^k}{1-t^k} - \sum_{j=1}^{\infty} q^j (j-1)/2 (-xt)^j \sum_{\ell=0}^{\infty} \frac{(q^j+1; q)_\ell}{(q; q)_\ell} \frac{t^\ell}{1-t^{j+\ell}}
\]

and we write

\[
\sum_{\ell=0}^{\infty} \frac{(q^j+1; q)_\ell}{(q; q)_\ell} \frac{t^\ell}{1-t^{j+\ell}} = \sum_{\ell=0}^{\infty} \frac{(q^j+1; q)_\ell}{(q; q)_\ell} t^\ell \sum_{p=0}^{\infty} t^{p(j+\ell)} = \sum_{p=0}^{\infty} t^{j+p} \frac{(q^{j+1}; q)_\ell}{(q; q)_\ell} t^{(1+p)(1+\ell)}
\]

using the \( q \)-binomial theorem again and the absolute convergence of the double sum, which justifies the interchange of summations. Using this and replacing \( n = p + 1 \) we get the result.

Consider the case \( t = q^2 \). Following the line of proof of Property 2.2, we write

\[
-\sum_{k=1}^{\infty} \frac{q^{2k}(x; q)_k}{1-q^{2k}} = -\sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} - \sum_{j=1}^{\infty} \frac{(-1)^j q^{j(j-1)/2} x^j}{(q; q)_j} \sum_{\ell=0}^{\infty} \frac{(q; q)_{\ell+j} q^{2\ell+2j}}{(q; q)_\ell (1-q^{2\ell+2j})}
\]

and we can write the inner sum over \( \ell \) as

\[
\sum_{\ell=0}^{\infty} \frac{(q; q)_{\ell+j-1} q^{2\ell+2j}}{(q; q)_\ell (1+q^{2\ell})} = \frac{(q; q)_j}{1+q^j} \sum_{\ell=0}^{\infty} \frac{(q^j; q)_\ell (-q^j; q)_\ell}{(q; q)_\ell (-q^{j+1}; q)_\ell} q^{2\ell}.
\]

Using Property 4.1 for \( t = q^2 \) then gives

\[
(4.3) \quad \sum_{n=1}^{\infty} q^{2nj} \frac{(q^{2n+j+1}; q)_\infty}{(q^{2n}; q)_\infty} = \frac{q^{2j}}{(1-q^{2j})} \sum_{\ell=0}^{\infty} \frac{(q^j; q)_\ell (-q^j; q)_\ell}{(q; q)_\ell (-q^{j+1}; q)_\ell} q^{2\ell}.
\]

This can also be proved directly using the \( q \)-binomial theorem and geometric series. We can rewrite (4.3) in standard basic hypergeometric series form (see [10]) as the quadratic transformation

\[
(4.4) \quad \frac{(1-q^{2j})}{(q^2; q)_{j+1}} \phi_2 \left( \begin{array}{c} q^2, q^3, q^3 \ q^2 \ 1 \end{array} \right)_{j+3} \phi_1 \left( \begin{array}{c} q^j, -q^j \ -q^j+1 \ q, q \end{array} \right)_3.
\]

Analogous to Property 2.3 and using the notation of Property 2.2, we have the following.

**Property 4.2.** For \( |p| < 1 \) one has

\[
F_q(x, p) = -\frac{p(1-x)}{(1-p)} \int_0^1 G(qx, pt) \, d\nu_t.
\]
5. A $q$-analogue of the dilogarithm

Euler’s dilogarithm is defined by the first equality in

$$\text{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t} \, dt = -\int_{1-x}^{1} \frac{\log(t)}{1-t} \, dt = \frac{\pi^2}{6} - \text{Li}_2(1-x)$$

for $0 \leq x \leq 1$; see [18], [14] for more information and references. Here we use $\text{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$. In particular, $x \frac{d\text{Li}_2}{dx} = -\log(1-x)$, and the definition by the series can be extended to complex $x$ being absolutely convergent for $|x| \leq 1$.

We define the $q$-dilogarithm by

$$\text{Li}_2(x; q) = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}(x; q)_k.$$ (5.1)

We have \(\lim_{q \downarrow 1} (1-q)^2 \text{Li}_2(x; q) = \sum_{k=1}^{\infty} (1-x)^k/k^2 = \text{Li}_2(1-x).\) In this case we can justify the interchange of the limit and summation using dominated convergence. We assume $0 < q < 1$, and we first observe that $|(x; q)_k| \leq 1$ for $|1-x| \leq 1$. Next we use

$$\frac{1-q^k}{1-q} = \sum_{j=0}^{k-1} q^j = q^{(k-1)/2} \begin{cases} \sum_{j=0}^{\frac{k-1}{2}} \left(q^{j+\frac{1}{2}} + q^{-j-\frac{1}{2}}\right), & k \text{ even} \\ 1 + \sum_{j=0}^{k-1} \left(q^{j+1} + q^{-j-1}\right), & k \text{ odd}, \end{cases}$$

and $x + 1/x \geq 2$ for $x \in [0,1]$ then gives

$$\frac{1-q^k}{1-q} \geq kq^{(k-1)/2},$$

so that

$$q^k \frac{(1-q)^2}{(1-q^k)^2} \leq \frac{1}{k^2}.$$ Combining both estimates gives

$$\left|\frac{q^k}{(1-q^k)^2}(x; q)_k\right| \leq \frac{1}{k^2}$$

for $|1-x| \leq 1$, and dominated convergence is established.

We list some properties of the $q$-dilogarithm. In the following we use $\zeta_q(2) = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}$ as an analogue of $\frac{1}{6}\pi^2$. This is equal to the $q$-$\zeta$-function

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{n^{s-1}q^n}{1-q^n}$$

for $s = 2$ since

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty} \sum_{k=1}^{n} q^{nk} = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}$$

(see, e.g., [21] Part VIII, Chapter 1, problem 75). This quantity was considered by Zudilin [28], [29], Krattenthaler et al. [17], Postelmans and Van Assche [22], who studied its irrationality when $1/q$ is an integer $\geq 2$. Note that this no longer corresponds to Ueno and Nishizawa [26], who essentially have $\sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^{2k})^2}$ as the value at 2 for their $q$-$\zeta$-function.
Property 5.1. $\text{Li}_2(\cdot; q)$ is an entire function of order zero. Moreover, we have the special values

$$\text{Li}_2(1; q) = 0, \quad \text{Li}_2(0; q) = \zeta(q)(2), \quad \text{Li}_2(q^{-n}; q) = -\sum_{k=1}^{n} \frac{k}{1 - q^k},$$

and $(1 - q)(1 - x)\left(D_q\text{Li}_2(\cdot; q)\right)(x) = S_q(x)$ and

$$\text{Li}_2(x; q) = \zeta(q)(2) + \frac{1}{1 - q} \int_{0}^{x} S_q(t) \frac{1}{1 - t} \, dt.$$ 

Moreover, the $q$-dilogarithm has the Taylor expansion

$$\text{Li}_2(x; q) = \zeta(q)(2) + \sum_{j=1}^{\infty} \left(\frac{1}{1 - q^{j+1}x^j}\right) \frac{1}{1 - q^j} \phi_1 \left(\frac{q^j, q^j}{q^{j+1}; q, q}\right).$$

Here the $2\phi_1$-series is defined by

$$2\phi_1 \left(\frac{q^j, q^j}{q^{j+1}; q, q}\right) = \sum_{k=0}^{\infty} \left(\frac{q^j}{q^{j+1}}\right)^{\ell} \frac{\left(\frac{q^j}{q^{j+1}}\right)^{\ell}}{\left(\frac{q^j}{q^{j+1}}\right)^{\ell+1} q^\ell}.$$ 

Unfortunately, this series cannot be summed using the (non-terminating) $q$-Chu-Vandermonde sum.

Note that after multiplying the integral representation for $\text{Li}_2(x; q)$ by $(1 - q)^2$, we can take a formal limit $q \uparrow 1$ to get

$$\text{Li}_2(1 - x) = \frac{\pi^2}{6} + \int_{0}^{x} \frac{\log(t)}{1 - t} \, dt = -\int_{0}^{1 - x} \frac{\log(1 - t)}{t} \, dt,$$ 

so that we recover the integral representation for the dilogarithm.

Proof. The proof of $\text{Li}_2(\cdot; q)$ being an entire function of order zero is derived as in Property 2.1. Since $(q x; q)_k - (x; q)_k = x(1 - q^k)(q x; q)_{k-1}$ we obtain

$$(5.2) \quad \text{Li}_2(q x; q) - \text{Li}_2(x; q) = \frac{x}{1 - x} \sum_{k=1}^{\infty} \frac{q^k}{1 - q^k} = \frac{-x}{1 - x} S_q(x).$$

This implies $(1 - q)(1 - x)\left(D_q\text{Li}_2(\cdot; q)\right)(x) = S_q(x)$.

Using (5.2) for $x = q^{-n}$, $n \in \mathbb{N}$, and $\text{Li}_2(1; q) = 0$, $S(q^{-n}) = n$, we find the value for $\text{Li}_2(q^{-n}; q)$. Iterating (5.2) we get

$$\text{Li}_2(x; q) = \sum_{k=0}^{N} \frac{x q^k}{1 - x q^k} S_q(x q^k) + \text{Li}_2(x q^{N+1}; q),$$

and by letting $N \rightarrow \infty$ we get the convergent series expansion

$$\text{Li}_2(x; q) = \text{Li}_2(0; q) + \sum_{k=0}^{\infty} \frac{x q^k}{1 - x q^k} S_q(x q^k) = \zeta(q)(2) + \frac{1}{1 - q} \int_{0}^{x} S_q(t) \frac{1}{1 - t} \, dt.$$ 

Finally, the Taylor expansion proceeds as in the proof of Property 2.2 and we find

$$\text{Li}_2(x; q) = \sum_{k=0}^{\infty} \frac{q^k}{(1 - q^k)^2} + \sum_{j=1}^{\infty} \frac{(-x)^j q^{j(j-1)/2}}{(q; q)_j} \sum_{\ell=0}^{j} \frac{(q; q)_{j+\ell} q^{j+\ell}}{(q; q)_j (1 - q^{j+\ell})^2}.$$
The inner sum over $\ell$ can be rewritten as
\[
\frac{q^j}{1-q^2} \sum_{\ell=0}^{\infty} \frac{\phi(q^j) \ell \phi(q^{j+1})}{\phi(q^j) \ell \phi(q^{j+1})} q^\ell,
\]
and this gives the result. □

The evaluation of the $q$-dilogarithm gives the following summation (cf. (3.3)):
\[
\sum_{k=1}^{n} \frac{(q^{-n}; q)kq^k}{(1-q)^2} = -\sum_{k=1}^{n} \frac{k}{1-q^k}.
\]
In particular, for $n = 0$ we obtain an alternating series representation for $\zeta_q(2)$:
\[
\zeta_q(2) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} q^{2j(j+1)/2}}{(1-q^j)^2} 2\phi_1 \left( q^j, q^j \right) q^{j+1}.
\]
If we write $\text{Li}_2(x; q) = \sum_{n=0}^{\infty} a_n x^n$, $S_q(x) = \sum_{n=0}^{\infty} b_n x^n$ temporarily, then (5.2) implies that $q^n a_n - a_n$ equals the coefficient, say $c_n$, of $x^n$ in $-S_q(x) x/(1-x)$. If we use $-x/(1-x) = \sum_{k=1}^{\infty} -x^k$, it follows that $c_n = -\sum_{p=0}^{n-1} b_p$. Note that the relation is trivial in case $n = 0$, and for integer $n \geq 1$ we find from the explicit Taylor expansions for $S_q(x)$ and $\text{Li}_2(x; q)$ the relation
\[
\frac{(-1)^{n-1} q^n(q+1)/2}{(1-q^n)} 2\phi_1 \left( q^n, q^n \right) = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} + \sum_{j=1}^{n-1} \frac{(-1)^{j} q^{j(j+1)/2}}{(1-q^j) (q; q)_j}.
\]
Note that this relation gives an explicit expression for the remainder if approximating $\zeta_q(1)$ with the alternating series as in (3.5). Of course, we get the same result if we use the Taylor expansion of $S_q$ as in Property 2.2 in the integral representation for the $q$-dilogarithm in Property 6.1.

The classical dilogarithm satisfies many interesting properties, such as a simple functional equation, a five-term recursion, a characterisation by these first two properties, explicit evaluation at certain special points, etc.; see [18], [14], for more information and references. It would be interesting to see if these interesting properties have appropriate analogues for the $q$-analogue of the dilogarithm discussed here.

6. Other $q$-LOGARITHMS

In the physics literature (see e.g. [25]), one defines $\ln_q(x) = \frac{x^{1-q} - 1}{1-q}$. There are no $q$-series, $q$-Pochhammer symbols, $q$-difference relations, etc. The choice of the letter $q$ and the fact that $\lim_{q \to 1} \ln_q(x) = \log x$ are not sufficient motivation to call this a $q$-analogue. It just shows that the logarithmic function is somewhere between the constant function and powers $x^\alpha - 1$ for $\alpha > 0$. This $q$-analogue of the logarithm plays a role in statistical mechanics and, as pointed out in [13], was also introduced by Euler in 1779; see [13] for more information and references.

Borwein [4], Zudilin et al. [19], and Van Assche [27] consider
\[
\ln_q(1 + z) = \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{1-q^k}, \quad |z| < |q|,
\]
They prove that \( \log_q(1 + z) \) is irrational for \( z = \pm 1 \) and \( q \) an integer greater than 2. For \( z = -1 \) one has a \( q \)-analogue of the harmonic series, and this is essentially the generating function of \( d_n = \sum_{k|n} 1 \), i.e. the number of divisors of \( n \). A similar formula, but now for \( 0 < q < 1 \),
\[
\log_q(z) = \sum_{k=1}^{\infty} \frac{z^k}{1 - q^k} = \frac{z e'_{q}(z)}{e_{q}(z)}, \quad |z| < 1,
\]
has been considered as a \( q \)-analogue of the logarithm by Kirillov \[14\] and Koornwinder \[16\]. This \( q \)-analogue is well adapted to non-commutative algebras; see \[14, \S 2.5, \text{Ex. 11}\], \[16, \text{Prop. 6.1}\], since \( \log_q(x + y - xy) = \log_q(x) + \log_q(y) \) for \( xy = qyx \). The corresponding \( q \)-analogue of the dilogarithm, provisionally denoted by \( \widetilde{\text{Li}}_2(x; q) \), is defined by
\[
\widetilde{\text{Li}}_2(x; q) = \sum_{k=1}^{\infty} \frac{x^k}{k(1 - q^k)} = \log(e_{q}(z)) \iff \log_q(z) = z \widetilde{\text{Li}}_2'(z; q).
\]
Zudilin \[29\] considers a similar \( q \)-logarithm but a different \( q \)-dilogarithm,
\[
L_1(x; q) = \sum_{n=1}^{\infty} \frac{(xq)^n}{1 - q^n}, \quad L_2(x; q) = \sum_{n=1}^{\infty} \frac{n(xq)^n}{1 - q^n},
\]
and mainly studies simultaneous rational approximation to \( L_1 \) and \( L_2 \) in order to obtain quantitative linear independence over \( \mathbb{Q} \) for certain values of these functions.

Other \( q \)-logarithms are defined as inverses of \( q \)-exponential functions; see Nelson and Gartley \[20\] for two different cases viewed from complex function theory, and Chung et al. \[5\], where implicitly \( q \)-commuting variables are used. Fock and Goncharov \[9, 12\] introduce a \( q \)-logarithm of \( \ln(e^z + 1) \) by an integral. The corresponding \( q \)-dilogarithm is essentially Ruijsenaars’ hyperbolic \( \Gamma \)-function; see \[23, \text{II.A}\]. For other \( q \)-logarithms based on Jacobi theta functions, see Sauloy \[24\] and Duval \[6\], where the \( q \)-logarithms play a role in difference Galois theory in constructing the analogue of a unipotent monodromy representation.

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