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LEONHARD EULER AND A q-ANALOGUE OF THE LOGARITHM

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On the 300th anniversary of Euler's birth

ABSTRACT. We study a q-logarithm which was introduced by Euler and give some of its properties. This q-logarithm has not received much attention in the recent literature. We derive basic properties, some of which were already given by Euler in a 1751 paper and in a 1734 letter to Daniel Bernoulli. The corresponding q-analogue of the dilogarithm is introduced. The relation to the values at 1 and 2 of a q-analogue of the zeta function is given. We briefly describe some other q-logarithms that have appeared in the recent literature.

1. INTRODUCTION

In a paper from 1751, Leonhard Euler (1707–1783) introduced the series [8, §6]

$$s = \sum_{k=1}^{\infty} \frac{(1-x)(1-x/a)\cdots(1-x/a^{k-1})}{1-a^k}. \tag{1.1}$$

We will take $q = 1/a$. Then this series is convergent for $|q| < 1$ and $x \in \mathbb{C}$. In this paper we will assume $0 < q < 1$. Then this becomes

$$S_q(x) = -\sum_{k=1}^{\infty} \frac{q^k}{1-q^k}(x;q)_k, \tag{1.2}$$

where $(x;q)_0 = 1$, $(x;q)_k = (1-x)(1-xq)\cdots(1-xq^{k-1})$. This can be written as a basic hypergeometric series

$$S_q(x) = -\frac{q(1-x)}{1-q}\ {}_3\phi_2\left(\begin{matrix} q, q, qx \\ q^2, 0 \end{matrix}\ ; q, q\right).$$

Euler had come across this series much earlier in an attempt to interpolate the logarithm at powers a^k (or q^{-k}); see, e.g., Gautschi's comment [11] discussing Euler's letter to Daniel Bernoulli where Euler introduced the function for $a = 10$. Euler was aware that this interpolation did not work very well; see [11, §§3-4]. The function in (1.2) does not seem to appear in the recent literature, even though it has some nice properties. We will prove some of its properties, some already obtained

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by Euler [8], and indicate why this should be called a q-analogue of the logarithm. A first reason is that for $0 < q < 1$,

$$\lim_{q\to 1}(1-q)S_q(x) = -\sum_{k=1}^{\infty} \lim_{q\to 1} q^k \frac{1-q}{1-q^k}(x;q)_k = -\sum_{k=1}^{\infty} \frac{(1-x)^k}{k} = \log x,$$

which is only a formal limit transition, since interchanging limit and sum seems hard to justify.

In Sections 2–3 we study this q-analogue of the logarithm more closely. In particular, we reprove some of Euler's results. Then we go on to extend the definition in Section 4. Finally, we study the corresponding q-analogue of the dilogarithm in Section 5. It involves also the values at 1 and 2 of a q-analogue of the ζ-function. We give a (incomplete) list of some other q-analogues of the logarithm appearing in the literature in Section 6. The purpose of this note is to draw attention to the q-analogues of the logarithm, dilogarithm and ζ-function for which we expect many interesting results remain to be discovered.

Many results in this paper use the q-binomial theorem [10, §1.3], [1, §10.2]

$$\frac{(ax;q)_\infty}{(x;q)_\infty} = \sum_{j=0}^{\infty} \frac{(a;q)_j}{(q;q)_j} x^j, \qquad |x| < 1. \tag{1.3}$$

We also use the q-exponential functions [10, p. 9], [1, p. 492]

$$e_q(z) = \frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \qquad |z| < 1,$$
$$E_q(z) = (-z;q)_\infty = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n}\, z^n.$$

2. The q-logarithm as an entire function

First of all we will show that the function S_q in (1.2) is an entire function, and as such it is a nicer function than the logarithm, which has a cut along the negative real axis.

Property 2.1. The function S_q defined in (1.2) is an entire function of order zero.

Proof. For $k \in \mathbb{N}$ the q-Pochhammer $(z;q)_k$ is a polynomial of degree k with zeros at $1, 1/q, \ldots, 1/q^{k-1}$. For $|z| \le r$ we have the simple bound

$$|(z;q)_k| \le (1+r)(1+r|q|)\cdots(1+r|q|^{k-1}) = (-r;|q|)_k < (-r;|q|)_\infty,$$

and hence the partial sums are uniformly bounded on the ball $|z| \le r$:

$$\left| -\sum_{k=1}^{n} \frac{q^k}{1-q^k}(z;q)_k \right| \le (-r;|q|)_\infty \sum_{k=1}^{\infty} \frac{|q|^k}{1-|q|^k}.$$

The partial sums therefore are a normal family and are uniformly convergent on every compact subset of the complex plane. The limit of these partial sums is $S_q(z)$ and is therefore an entire function of the complex variable z.

Let $M(r) = \max_{|z|\le r} |S_q(z)|$. Then

$$M(r) \le (-r;|q|)_\infty \sum_{k=1}^{\infty} \frac{|q|^k}{1-|q|^k}$$

and $(-r;|q|)_\infty = E_{|q|}(r)$ is the maximum of $E_{|q|}(z)$ on the ball $\{|z| \le r\}$. The function E_q is an entire function of order zero, which can be seen from the coefficients a_n of its Taylor series and the formula [2, Theorem 2.2.2]

$$\limsup_{n\to\infty} \frac{n\log n}{\log(1/|a_n|)} \tag{2.1}$$

for the order of $\sum_{n=0}^\infty a_n z^n$. Hence also S_q has order zero. □

Observe that for $0<q<1$ we have

$$M(r) = \max_{|z|\le r} |S_q(z)| = \sum_{k=1}^\infty \frac{q^k}{1-q^k} (-r;q)_k$$

and some simple bounds give

$$(q;q)_\infty \sum_{k=1}^\infty \frac{q^k}{(q;q)_k}(-r;q)_k \le M(r) \le (-r;q)_\infty \sum_{k=1}^\infty \frac{q^k}{1-q^k}.$$

For the lower bound we can use the q-binomial theorem (1.3) to find

$$(-rq;q)_\infty - (q;q)_\infty \le M(r) \le (-r;q)_\infty \sum_{k=1}^\infty \frac{q^k}{1-q^k},$$

which shows that $M(r)$ behaves like $E_q(qr) - C_1 \le M(r) \le C_2 E_q(r)$, where C_1 and C_2 are constants (which depend on q).

Euler [8, §§14-15] essentially also stated the following Taylor expansion.

Property 2.2. The q-logarithm (1.2) has the following Taylor series around $x=0$:

$$S_q(x) = -\sum_{k=1}^\infty \frac{q^k}{1-q^k}\left(1 + q^{k(k-1)/2}\frac{(-x)^k}{(q;q)_k}\right).$$

Proof. Use the q-binomial theorem (1.3) with $x = zq^k$ and $a = q^{-k}$ to find

$$(z;q)_k = \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} q^{j(j-1)/2}(-z)^j, \quad \begin{bmatrix} k \\ j \end{bmatrix} = \frac{(q;q)_k}{(q;q)_j (q;q)_{k-j}}. \tag{2.2}$$

Use this in (1.2), and change the order of summation to find

$$S_q(x) = -\sum_{k=1}^\infty \frac{q^k}{1-q^k} - \sum_{j=1}^\infty q^{j(j-1)/2}(-x)^j \sum_{k=j}^\infty \frac{q^k}{1-q^k}\frac{(q;q)_k}{(q;q)_j(q;q)_{k-j}}.$$

With a new summation index $k = j+\ell$ this becomes

$$S_q(x) = -\sum_{k=1}^\infty \frac{q^k}{1-q^k} - \sum_{j=1}^\infty \frac{q^j}{1-q^j} q^{j(j-1)/2}(-x)^j \sum_{\ell=0}^\infty q^\ell \frac{(q^j;q)_\ell}{(q;q)_\ell}.$$

Now use the q-binomial theorem (1.3) to sum over ℓ to find

$$S_q(x) = -\sum_{k=1}^\infty \frac{q^k}{1-q^k} - \sum_{j=1}^\infty \frac{q^j}{1-q^j} q^{j(j-1)/2}\frac{(-x)^j}{(q;q)_j}.$$

If we combine both series, the required expansion follows. □

This result can be written in terms of basic hypergeometric series as

$$S_q(x) = -\frac{q}{1-q}\ {}_2\phi_1\left(\begin{matrix} q, q \\ q^2 \end{matrix}; q, q\right) - \frac{qx}{(1-q)^2}\ {}_2\phi_2\left(\begin{matrix} q, q \\ q^2, q^2 \end{matrix}; q, q^2 x\right).$$

The growth of the coefficients in this Taylor series again shows that S_q is an entire function of order zero if we use the formula (2.1) for the order of $\sum_{n=0}^{\infty} a_n z^n$; see also [11, §4].

Next we mention the following q-integral representation, where we use Jackson's q-integral (see [10, §1.11])

$$\int_0^a f(t)\, d_q t = (1-q)a \sum_{k=0}^{\infty} f(aq^k)\, q^k, \tag{2.3}$$

defined for functions f whenever the right-hand side converges.

Property 2.3. For every $x \in \mathbb{C}$ we have

$$S_q(x) = -\frac{q(1-x)}{1-q} \int_0^1 G_q(qx, qt)\, d_q t,$$

with

$$G_q(x,t) = \sum_{k=0}^{\infty} t^k (x;q)_k = {}_2\phi_1\left(\begin{matrix} x, q \\ 0 \end{matrix}; q, t\right) = \frac{1}{1-t}\ {}_1\phi_1\left(\begin{matrix} q \\ qt \end{matrix}; q, xt\right).$$

Since $\int_0^a f(t)\, d_q t \to \int_0^a f(t)\, dt$ when $q \to 1$ and $G_q(x,t) \to 1/(1-t(1-x))$ when $q \to 1$ for $x > 0$, we see (at least formally) that Property 2.3 is a q-analogue of the integral representation

$$\log(x) = -\int_0^1 \frac{1-x}{1-t(1-x)}\, dt, \qquad x \notin (-\infty, 0]$$

for the logarithm.

Proof. Observe that

$$\frac{1-q}{1-q^{k+1}} = (1-q)\sum_{p=0}^{\infty} q^{(k+1)p} = \int_0^1 t^k\, d_q t.$$

Inserting this in the definition (1.2) of S_q and interchanging summations, which is justified by the absolute convergence of the double sum, give the result. The identity between the basic hypergeometric series representing $G_q(x,t)$ is the case $c = 0$ of [10, (III.4)]. □

Note that, as in the proof of Property 2.2, one can show that

$$G_q(x,t) = \sum_{j=0}^{\infty} \frac{(-xt)^j q^{j(j-1)/2}}{(t;q)_{j+1}}. \tag{2.4}$$

3. The q-difference equation

The function S_q satisfies a simple q-difference equation:

Property 3.1. The q-logarithm (1.2) satisfies

$$S_q(x/q) - S_q(x) = 1 - (x;q)_{\infty}. \tag{3.1}$$

Proof. Recall the q-difference operator

$$D_q f(x) = \frac{f(qx) - f(x)}{x(q-1)}.$$

Then a simple exercise is

$$D_{1/q}(x;q)_k = -\frac{1-q^k}{1-q}(x;q)_{k-1}.$$

Use this in (1.2) to find

$$D_{1/q}S_q(x) = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}\frac{1-q^k}{1-q}(x;q)_{k-1} = \frac{q}{1-q}\sum_{k=0}^{\infty} q^k (x;q)_k.$$

Observe that $(x;q)_{k+1} - (x;q)_k = (x;q)_k[1 - xq^k - 1] = -xq^k(x;q)_k$. Summing we find $-x\sum_{k=0}^{n} q^k(x;q)_k = (x;q)_{n+1} - (x;q)_0$, and when $n \to \infty$,

$$\sum_{k=0}^{\infty} q^k(x;q)_k = \frac{1-(x;q)_\infty}{x}.$$

If we use this result, then

$$D_{1/q}S_q(x) = \frac{q}{1-q}\frac{1-(x;q)_\infty}{x},$$

which is (3.1). □

In order to see how this is related to the classical derivative of $\log x$, one may rewrite this as

$$D_q((1-q)S_q(x)) = \frac{1}{x} - \frac{(qx;q)_\infty}{x}.$$

This q-difference equation can already be found in [8, §6], where Euler writes $s = S_q(x)$ and $t = S_q(x/q)$ and gives the relation

$$1 + s - t = (1-x)\left(1 - \frac{x}{a}\right)\left(1 - \frac{x}{a^2}\right)\left(1 - \frac{x}{a^3}\right)\left(1 - \frac{x}{a^4}\right)\left(1 - \frac{x}{a^5}\right)\cdots,$$

where $q = 1/a$.

As a corollary one has [8, §7].

Property 3.2. For every positive integer n one has $S_q(q^{-n}) = n$.

Proof. Use (3.1) with $x = q^{-n+1}$ to find $S_q(q^{-n}) - S_q(q^{-n+1}) = 1$, since $(x;q)_\infty$ vanishes whenever $x = q^{-n}$ for $n \geq 0$. The result then follows by induction and $S_q(1) = 0$. □

It is this property, which is quite similar to $\log_a a^n = n$, where $\log_a$ is the logarithm with base a, which gives S_q the flavor of a q-logarithm and which made Euler consider this function as an interpolation of the logarithm; see [11, §1]. Observe that this interpolation property can be stated as follows: $-\log q\, S_q(x)$ approximates $\log x$ as $q \uparrow 1$ and for fixed q this approximation is perfect if $x = q^{-n}$ $(n = 1, 2, \ldots)$.

Another interesting value is

$$S_q(0) = -\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} = -\zeta_q(1),$$

which is a q-analogue of the harmonic series, where the q-analogue of the ζ-function is defined by

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{n^{s-1} q^n}{1-q^n}.$$

It has been proved (see Erdős [7], Borwein [3, 4], Van Assche [27]) that this quantity is irrational whenever $q = 1/p$ with p an integer ≥ 2. For the specific argument 1 this coincides, up to a factor, with the value at 1 of the q-ζ-function considered by Ueno and Nishizawa [26].

The values of $S_q(q^n)$ for $n \in \mathbb{N}$ are distinctly different, and for these values we do not get the same flavor as the logarithm.

Property 3.3. For every positive integer n one has

$$S_q(q^n) = -n + (q;q)_\infty \sum_{k=0}^{n-1} \frac{1}{(q;q)_k}. \tag{3.2}$$

Proof. Choose $x = q^{k+1}$ in (3.1). Then $S_q(q^k) - S_q(q^{k+1}) = 1 - (q^{k+1};q)_\infty$. Summing and using the telescoping property give

$$S_q(q^0) - S_q(q^n) = \sum_{k=0}^{n-1} \big(S_q(q^k) - S_q(q^{k+1})\big) = n - \sum_{k=0}^{n-1} (q^{k+1};q)_\infty.$$

By Property 3.2 we have $S_q(1) = 0$. Now $(q^{k+1};q)_\infty = (q;q)_\infty/(q;q)_k$ gives the required expression (3.2). □

In order to see how this approximates $\log x$, one may reformulate this as

$$-\log q \; S_q(q^n) = \log q^n - \log q \sum_{k=0}^{n-1} (q^{k+1};q)_\infty.$$

In [8, §10] Euler writes $s = S_q(q^n)$, $t = S_q(q^{n-1})$, $u = S_q(q^{n-2})$, and he writes the recursion

$$s = \frac{2t - u + aq^n(1-t)}{1 - aq^n},$$

where $q = 1/a$. In contemporary notation we write $y_n = S_q(q^n)$ and obtain the recurrence relation

$$y_n(1 - q^{n-1}) - (2 - q^{n-1}) y_{n-1} + y_{n-2} = q^{n-1}.$$

One can verify that this recurrence relation indeed holds for $y_n = S_q(q^n)$ given in (3.2). More generally one in fact has

$$(1-qx) S_q(q^2 x) - (2 - qx) S_q(qx) + S_q(x) = qx,$$

which is a second order non-homogeneous q-difference equation for S_q.

Note that the explicit evaluation $S(q^{-n}) = n$, $n \in \mathbb{N}$, gives the following summation formulas:

$$\sum_{k=1}^{n} \frac{(q^{-n};q)_k}{1-q^k} q^k = -n, \qquad \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2} (-1)^{k-1} q^{-nk}}{(1-q^k)\,(q;q)_k} = n + \sum_{k=1}^{\infty} \frac{q^k}{1-q^k}, \tag{3.3}$$

using the definition of $S_q(x)$ and the Taylor expansion in Property 2.2. Similarly, the evaluation at q^n, $n \in \mathbb{N}$, given in (3.2) gives the summation formulas

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{(q^n;q)_k}{1-q^k} q^k = n - \sum_{k=0}^{n-1} (q^{k+1};q)_\infty, \\ &\sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1} q^{nk}}{(1-q^k)\,(q;q)_k} = -n + \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} + \sum_{k=0}^{n-1} (q^{k+1};q)_\infty. \end{aligned} \tag{3.4}$$

Note that all infinite series are absolutely convergent and that for $n = 0$ the results in (3.3) and (3.4) coincide. The first sums become trivial, and the second sum gives the following expansion for $\zeta_q(1)$:

$$\zeta_q(1) = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} = \sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1}}{(1-q^k)\,(q;q)_k}. \tag{3.5}$$

Using (3.5) in Property 2.2 gives the expansion

$$S_q(x) = -\sum_{k=1}^{\infty} \frac{q^{k(k+1)/2}(-1)^{k-1}(1-x^k)}{(1-q^k)\,(q;q)_k},$$

so that in particular

$$-\frac{dS_q}{dx}(1) = \lim_{x\to 1} \frac{S_q(x)}{1-x} = -\sum_{k=1}^{\infty} \frac{k\, q^{k(k+1)/2}(-1)^{k-1}}{(1-q^k)\,(q;q)_k}.$$

4. An extension of the q-logarithm and Lambert series

If we have the definition of $S_q(x)$ resembling Lambert series, it is natural to look for the extension

$$F_q(x,t) = -\sum_{k=1}^{\infty} (x;q)_k \frac{t^k}{1-t^k}, \tag{4.1}$$

which is a Lambert series; see [15, §58.C]. Since $|(x;q)_k| \le (-|x|;|q|)_k \le (-r;|q|)_\infty$ for x in $\{x \in \mathbb{C} \mid |x| \le r\}$, the convergence in (4.1) is uniform on compact sets in x and on compact subsets of the open unit disk in t. Also since the series $-\sum_{k=1}^{\infty} (x;q)_k t^k$ is absolutely convergent for $|t| < 1$ uniformly in x in compact sets, it follows by [15, Satz 259] that F_q is analytic for $(x,t) \in \mathbb{C} \times \{t \in \mathbb{C} \mid |t| < 1\}$. Observe that $S_q(x) = F_q(x,q)$.

The general theory of Lambert series then gives the power series of F in powers of t:

$$F_q(x,t) = \sum_{\ell=1}^{\infty} \Big(\sum_{k|\ell} (x;q)_k \Big) t^\ell \implies S_q(x) = \sum_{\ell=1}^{\infty} \Big(\sum_{k|\ell} (x;q)_k \Big) q^\ell.$$

We are mainly interested in the power series development with respect to x.

Property 4.1. For $|t| < 1$ one has

$$F_q(x,t) = -\sum_{k=1}^{\infty} \frac{t^k}{1-t^k} - \sum_{\ell=1}^{\infty} x^\ell (-1)^\ell q^{\ell(\ell-1)/2} \left(\sum_{n=1}^{\infty} t^{n\ell} \frac{(t^n q^{\ell+1};q)_\infty}{(t^n;q)_\infty} \right).$$

In case $t = q$, Property 4.1 reduces to Property 2.2, and this is equivalent to the summation formula

$$\sum_{n=1}^{\infty} q^{n\ell} \frac{(q^{\ell+n+1};q)_\infty}{(q^n;q)_\infty} = \frac{q^\ell}{(1-q^\ell)\,(q;q)_\ell} \implies \sum_{n=1}^{\infty} \frac{(q;q)_{n-1}}{(q^{\ell+1};q)_n} q^{n\ell} = \frac{q^\ell}{1-q^\ell} \tag{4.2}$$

for $\ell \in \mathbb{N}$, $\ell \geq 1$. This can be obtained as a special case of the q-Gauss sum [10, (1.5.1)].

Proof. The proof is along the same lines as the proof of Property 2.2. We find similarly

$$F_q(x,t) = -\sum_{k=1} \frac{t^k}{1-t^k} - \sum_{j=1}^{\infty} q^{j(j-1)/2}(-xt)^j \sum_{\ell=0}^{\infty} \frac{(q^{j+1};q)_\ell}{(q;q)_\ell} \frac{t^\ell}{1-t^{j+\ell}}$$

and we write

$$\begin{aligned}\sum_{\ell=0}^{\infty} \frac{(q^{j+1};q)_\ell}{(q;q)_\ell} \frac{t^\ell}{1-t^{j+\ell}} &= \sum_{\ell=0}^{\infty} \frac{(q^{j+1};q)_\ell}{(q;q)_\ell} t^\ell \sum_{p=0}^{\infty} t^{p(j+\ell)} \\ &= \sum_{p=0}^{\infty} t^{jp} \sum_{\ell=0}^{\infty} \frac{(q^{j+1};q)_\ell}{(q;q)_\ell} t^{\ell(1+p)} = \sum_{p=0}^{\infty} t^{jp} \frac{(t^{1+p}q^{j+1};q)_\infty}{(t^{1+p};q)_\infty}\end{aligned}$$

using the q-binomial theorem again and the absolute convergence of the double sum, which justifies the interchange of summations. Using this and replacing $n = p+1$ we get the result. □

Consider the case $t = q^2$. Following the line of proof of Property 2.2 we write

$$-\sum_{k=1}^{\infty} \frac{q^{2k}(x;q)_k}{1-q^{2k}} = -\sum_{k=1}^{\infty} \frac{q^{2k}}{1-q^{2k}} - \sum_{j=1}^{\infty} \frac{(-1)^j q^{j(j-1)/2} x^j}{(q;q)_j} \sum_{\ell=0}^{\infty} \frac{(q;q)_{\ell+j}\, q^{2\ell+2j}}{(q;q)_\ell\,(1-q^{2\ell+2j})},$$

and we can write the inner sum over ℓ as

$$\sum_{\ell=0}^{\infty} \frac{(q;q)_{\ell+j-1}\, q^{2\ell+2j}}{(q;q)_\ell\,(1+q^{\ell+j})} = \frac{(q;q)_{j-1} q^{2j}}{1+q^j} \sum_{\ell=0}^{\infty} \frac{(q^j;q)_\ell(-q^j;q)_\ell}{(q;q)_\ell(-q^{j+1};q)_\ell} q^{2\ell}.$$

Using Property 4.1 for $t = q^2$ then gives

$$\sum_{n=1}^{\infty} q^{2nj} \frac{(q^{2n+j+1};q)_\infty}{(q^{2n};q)_\infty} = \frac{q^{2j}}{(1-q^{2j})} \sum_{\ell=0}^{\infty} \frac{(q^j;q)_\ell(-q^j;q)_\ell}{(q;q)_\ell(-q^{j+1};q)_\ell} q^{2\ell}. \tag{4.3}$$

This can also be proved directly using the q-binomial theorem and geometric series. We can rewrite (4.3) in standard basic hypergeometric series form (see [10]) as the quadratic transformation

$$\frac{(1-q^{2j})}{(q^2;q)_{j+1}}\, {}_3\phi_2\left(\begin{matrix} q^2, q^2, q^3 \\ q^{j+3}, q^{j+4} \end{matrix}; q^2, q^2\right) = {}_2\phi_1\left(\begin{matrix} q^j, -q^j \\ -q^{j+1} \end{matrix}; q, q^2\right). \tag{4.4}$$

Analogous to Property 2.3 and using the notation of Property 2.3, we have the following.

Property 4.2. For $|p| < 1$ one has

$$F_q(x,p) = \frac{-p(1-x)}{(1-p)} \int_0^1 G(qx, pt)\, d_p t.$$

5. A q-analogue of the dilogarithm

Euler's dilogarithm is defined by the first equality in

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t}\,dt = -\int_{1-x}^{1} \frac{\log(t)}{1-t}\,dt = \frac{\pi^2}{6} - \mathrm{Li}_2(1-x)$$

for $0 \leq x \leq 1$; see [18], [14] for more information and references. Here we use $\mathrm{Li}_2(1) = \zeta(2) = \frac{\pi^2}{6}$. In particular, $x\frac{d\mathrm{Li}_2}{dx} = -\log(1-x)$, and the definition by the series can be extended to complex x being absolutely convergent for $|x| \leq 1$.

We define the q-dilogarithm by

$$\mathrm{Li}_2(x;q) = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}(x;q)_k. \tag{5.1}$$

We have $\lim_{q\uparrow 1}(1-q)^2\mathrm{Li}_2(x;q) = \sum_{k=1}^{\infty}(1-x)^k/k^2 = \mathrm{Li}_2(1-x)$. In this case we can justify the interchange of the limit and summation using dominated convergence. We assume $0 < q < 1$, and we first observe that $|(x;q)_k| \leq 1$ for $|1-x| \leq 1$. Next we use

$$\frac{1-q^k}{1-q} = \sum_{j=0}^{k-1} q^j = q^{(k-1)/2} \begin{cases} \sum_{j=0}^{\frac{k}{2}-1}\left(q^{j+\frac{1}{2}} + q^{-j-\frac{1}{2}}\right), & k \text{ even},\\ 1 + \sum_{j=0}^{\frac{k-1}{2}-1}\left(q^{j+1} + q^{-j-1}\right), & k \text{ odd}, \end{cases}$$

and $x + 1/x \geq 2$ for $x \in [0,1]$ then gives

$$\frac{1-q^k}{1-q} \geq kq^{(k-1)/2},$$

so that

$$q^k\frac{(1-q)^2}{(1-q^k)^2} \leq \frac{1}{k^2}.$$

Combining both estimates gives

$$\left|\frac{q^k}{(1-q^k)^2}(x;q)_k\right| \leq \frac{1}{k^2}$$

for $|1-x| \leq 1$, and dominated convergence is established.

We list some properties of the q-dilogarithm. In the following we use $\zeta_q(2) = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}$ as an analogue of $\frac{1}{6}\pi^2$. This is equal to the q-ζ-function

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{n^{s-1}q^n}{1-q^n}$$

for $s = 2$ since

$$\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} nq^{nk} = \sum_{k=1}^{\infty} \frac{q^k}{(1-q^k)^2}$$

(see, e.g., [21, Part VIII, Chapter 1, problem 75]). This quantity was considered by Zudilin [28, 29], Krattenthaler et al. [17], Postelmans and Van Assche [22], who studied its irrationality when $1/q$ is an integer ≥ 2. Note that this no longer corresponds to Ueno and Nishizawa [26], who essentially have $\sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^k)^2}$ as the value at 2 for their q-ζ-function.

Property 5.1. $\mathrm{Li}_2(\,\cdot\,;q)$ is an entire function of order zero. Moreover, we have the special values

$$\mathrm{Li}_2(1;q)=0,\quad \mathrm{Li}_2(0;q)=\zeta_q(2),\quad \mathrm{Li}_2(q^{-n};q)=-\sum_{k=1}^{n}\frac{k}{1-q^k},$$

and $(1-q)(1-x)\big(D_q\mathrm{Li}_2(\,\cdot\,;q)\big)(x)=S_q(x)$ and

$$\mathrm{Li}_2(x;q)=\zeta_q(2)+\frac{1}{1-q}\int_0^x\frac{S_q(t)}{1-t}\,d_qt.$$

Moreover, the q-dilogarithm has the Taylor expansion

$$\mathrm{Li}_2(x;q)=\zeta_q(2)+\sum_{j=1}^{\infty}\frac{(-1)^jq^{j(j+1)/2}\,x^j}{(1-q^j)^2}\,{}_2\phi_1\left(\begin{matrix}q^j,q^j\\ q^{j+1}\end{matrix};q,q\right).$$

Here the ${}_2\phi_1$-series is defined by

$${}_2\phi_1\left(\begin{matrix}q^j,q^j\\ q^{j+1}\end{matrix};q,q\right)=\sum_{\ell=0}^{\infty}\frac{(q^j;q)_\ell(q^j;q)_\ell}{(q;q)_\ell(q^{j+1};q)_\ell}\,q^\ell=\sum_{\ell=0}^{\infty}\frac{(q^j;q)_\ell(1-q^j)}{(q;q)_\ell(1-q^{j+\ell})}\,q^\ell.$$

Unfortunately, this series cannot be summed using the (non-terminating) q-Chu-Vandermonde sum.

Note that after multiplying the integral representation for $\mathrm{Li}_2(x;q)$ by $(1-q)^2$, we can take a formal limit $q\uparrow 1$ to get

$$\mathrm{Li}_2(1-x)=\frac{\pi^2}{6}+\int_0^x\frac{\log(t)}{1-t}\,dt=-\int_0^{1-x}\frac{\log(1-t)}{t}\,dt,$$

so that we recover the integral representation for the dilogarithm.

Proof. The proof of $\mathrm{Li}_2(\,\cdot\,;q)$ being an entire function of order zero is derived as in Property 2.1. Since $(qx;q)_k-(x;q)_k=x(1-q^k)(qx;q)_{k-1}$ we obtain

$$\mathrm{Li}_2(qx;q)-\mathrm{Li}_2(x;q)=\frac{x}{1-x}\sum_{k=1}^{\infty}\frac{q^k(x;q)_k}{1-q^k}=\frac{-x}{1-x}\,S_q(x). \tag{5.2}$$

This implies $(1-q)(1-x)\big(D_q\mathrm{Li}_2(\,\cdot\,;q)\big)(x)=S_q(x)$.

Using (5.2) for $x=q^{-n}$, $n\in\mathbb{N}$, and $\mathrm{Li}_2(1;q)=0$, $S(q^{-n})=n$, we find the value for $\mathrm{Li}_2(q^{-n};q)$. Iterating (5.2) we get

$$\mathrm{Li}_2(x;q)=\sum_{k=0}^{N}\frac{xq^k}{1-xq^k}S_q(xq^k)+\mathrm{Li}_2(xq^{N+1};q),$$

and by letting $N\to\infty$ we get the convergent series expansion

$$\mathrm{Li}_2(x;q)=\mathrm{Li}_2(0;q)+\sum_{k=0}^{\infty}\frac{xq^k}{1-xq^k}S_q(xq^k)=\zeta_q(2)+\frac{1}{1-q}\int_0^x\frac{S_q(t)}{1-t}\,d_qt.$$

Finally, the Taylor expansion proceeds as in the proof of Property 2.2, and we find

$$\mathrm{Li}_2(x;q)=\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)^2}+\sum_{j=1}^{\infty}\frac{(-x)^jq^{j(j-1)/2}}{(q;q)_j}\sum_{\ell=0}^{\infty}\frac{(q;q)_{j+\ell}\,q^{j+\ell}}{(q;q)_\ell\,(1-q^{j+\ell})^2}.$$

The inner sum over ℓ can be rewritten as

$$\frac{q^j (q;q)_{j-1}}{1-q^j} \sum_{\ell=0}^{\infty} \frac{(q^j;q)_\ell (q^j;q)_\ell}{(q;q)_\ell (q^{j+1};q)_\ell}\, q^\ell,$$

and this gives the result. □

The evaluation of the q-dilogarithm gives the following summation (cf. (3.3)):

$$\sum_{k=1}^{n} \frac{(q^{-n};q)_k q^k}{(1-q^k)^2} = -\sum_{k=1}^{n} \frac{k}{1-q^k} = \sum_{j=1}^{\infty} \frac{q^j}{(1-q^j)^2} + \sum_{j=1}^{\infty} \frac{(-1)^j q^{j(j+1)/2}\, q^{-nj}}{(1-q^j)^2}\, {}_2\phi_1\left(\begin{matrix} q^j, q^j \\ q^{j+1} \end{matrix}; q, q\right). \tag{5.3}$$

In particular, for $n=0$ we obtain an alternating series representation for $\zeta_q(2)$:

$$\zeta_q(2) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1} q^{j(j+1)/2}}{(1-q^j)^2}\, {}_2\phi_1\left(\begin{matrix} q^j, q^j \\ q^{j+1} \end{matrix}; q, q\right).$$

If we write $\mathrm{Li}_2(x;q) = \sum_{n=0}^{\infty} a_n x^n$, $S_q(x) = \sum_{n=0}^{\infty} b_n x^n$ temporarily, then (5.2) implies that $q^n a_n - a_n$ equals the coefficient, say c_n, of x^n in $-S_q(x)x/(1-x)$. If we use $-x/(1-x) = \sum_{k=1}^{\infty} -x^k$, it follows that $c_n = -\sum_{p=0}^{n-1} b_p$. Note that the relation is trivial in case $n=0$, and for integer $n \geq 1$ we find from the explicit Taylor expansions for $S_q(\,\cdot\,)$ and $\mathrm{Li}_2(\,\cdot\,;q)$ the relation

$$\frac{(-1)^{n-1} q^{n(n+1)/2}}{(1-q^n)}\, {}_2\phi_1\left(\begin{matrix} q^n, q^n \\ q^{n+1} \end{matrix}; q, q\right) = \sum_{k=1}^{\infty} \frac{q^k}{1-q^k} + \sum_{j=1}^{n-1} \frac{(-1)^j q^{j(j+1)/2}}{(1-q^j)\,(q;q)_j}.$$

Note that this relation gives an explicit expression for the remainder if approximating $\zeta_q(1)$ with the alternating series as in (3.5). Of course, we get the same result if we use the Taylor expansion of S_q as in Property 2.2 in the integral representation for the q-dilogarithm in Property 5.1.

The classical dilogarithm satisfies many interesting properties, such as a simple functional equation, a five-term recursion, a characterisation by these first two properties, explicit evaluation at certain special points, etc.; see [18], [14] for more information and references. It would be interesting to see if these interesting properties have appropriate analogues for the q-analogue of the dilogarithm discussed here.

6. Other q-logarithms

In the physics literature (see e.g. [25]), one defines $\ln_q(x) = \frac{x^{1-q}-1}{1-q}$. There are no q-series, q-Pochhammer symbols, q-difference relations, etc. The choice of the letter q and the fact that $\lim_{q\to 1} \ln_q(x) = \log x$ are not sufficient motivation to call this a q-analogue. It just shows that the logarithmic function is somewhere between the constant function and powers $x^\alpha - 1$ for $\alpha > 0$. This q-analogue of the logarithm plays a role in statistical mechanics and, as pointed out in [13], was also introduced by Euler in 1779; see [13] for more information and references.

Borwein [4], Zudilin et al. [19], and Van Assche [27] consider

$$\ln_q(1+z) = \sum_{k=1}^{\infty} \frac{(-1)^k z^k}{1-q^k}, \qquad |z| < |q|,$$

with $|q|>1$. They prove that $\ln_q(1+z)$ is irrational for $z=\pm 1$ and q an integer greater than 2. For $z=-1$ one has a q-analogue of the harmonic series, and this is essentially the generating function of $d_n=\sum_{k|n} 1$, i.e. the number of divisors of n. A similar formula, but now for $0<q<1$,

$$\log_q(z)=\sum_{k=1}^{\infty}\frac{z^n}{1-q^n}=\frac{z\,e_q'(z)}{e_q(z)}, \qquad |z|<1,$$

has been considered as a q-analogue of the logarithm by Kirillov [14] and Koornwinder [16]. This q-analogue is well adapted to non-commutative algebras; see [14, §2.5, Ex. 11], [16, Prop. 6.1], since $\log_q(x+y-xy)=\log_q(x)+\log_q(y)$ for $xy=qyx$. The corresponding q-analogue of the dilogarithm, provisionally denoted by $\widetilde{\mathrm{Li}_2}(x;q)$, is defined by

$$\widetilde{\mathrm{Li}_2}(x;q)=\sum_{k=1}^{\infty}\frac{z^k}{k\,(1-q^k)}=\log(e_q(z)) \implies \log_q(z)=z\,\widetilde{\mathrm{Li}_2}'(z;q).$$

Zudilin [29] considers a similar q-logarithm but a different q-dilogarithm,

$$L_1(x;q)=\sum_{n=1}^{\infty}\frac{(xq)^n}{1-q^n}, \quad L_2(x;q)=\sum_{n=1}^{\infty}\frac{n(xq)^n}{1-q^n},$$

and mainly studies simultaneous rational approximation to L_1 and L_2 in order to obtain quantitative linear independence over $\mathbb{Q}$ for certain values of these functions.

Other q-logarithms are defined as inverses of q-exponential functions; see Nelson and Gartley [20] for two different cases viewed from complex function theory, and Chung et al. [5], where implicitly q-commuting variables are used. Fock and Goncharov [9, 12] introduce a q-logarithm of $\ln(e^z+1)$ by an integral. The corresponding q-dilogarithm is essentially Ruijsenaars' hyperbolic Γ-function; see [23, II.A]. For other q-logarithms based on Jacobi theta functions, see Sauloy [24] and Duval [6], where the q-logarithms play a role in difference Galois theory in constructing the analogue of a unipotent monodromy representation.

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