A REMARK ON THE REGULARITY OF THE DIV-CURL SYSTEM

IRINA MITREA AND MARIUS MITREA

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Abstract. As a limiting case of the classical Calderón-Zygmund theory, in this note we study the Besov regularity of differential forms \( u \) for which \( du \) and \( \delta u \) have absolutely integrable coefficients in \( \mathbb{R}^n \).

By further refining the work in [3], the following estimates were recently proved in [2]: There exists a constant \( C \) such that for any smooth \( \ell \)-form \( u \) with compact support in \( \mathbb{R}^n, n \geq 2 \), one has

\[
\begin{align*}
\|u\|_{L^{n/(n-1)}} & \leq C[\|du\|_{L^1} + \|\delta u\|_{L^1}] \quad \text{if } \ell \neq 1, n-1, \quad (1) \\
\|u\|_{L^{n/(n-1)}} & \leq C[\|du\|_{L^1} + \|\delta u\|_{H^1}] \quad \text{if } \ell = 1, \quad (2) \\
\|u\|_{L^{n/(n-1)}} & \leq C[\|du\|_{H^1} + \|\delta u\|_{L^1}] \quad \text{if } \ell = n-1. \quad (3)
\end{align*}
\]

Here \( H^1 \) denotes the standard Hardy space. The cases \( \ell = 0 \) and \( \ell = 1 \) correspond, respectively, to the classical Gagliardo-Nirenberg inequality and to an estimate obtained by J. Bourgain and H. R. Brezis in [1]. In light of (1)–(3), it is of interest to establish similar estimates which emphasize the size of \( u \) in Besov spaces \( B^p_q, 0 < p, q \leq \infty, s \in \mathbb{R} \) (see, e.g., [4] for definitions and basic properties). In this regard, we shall prove the following result.

**Proposition 1.** Assume that \( n > 2 \). Then for each \( 0 < \theta < 1 \) there exists \( C > 0 \) such that

\[
\begin{align*}
(4) & \quad \|u\|_{B^{-n/2}_1} \leq C[\|du\|_{L^1(\mathbb{R}^n, \Lambda^{\ell+1})} + \|\delta u\|_{L^1(\mathbb{R}^n, \Lambda^{\ell-1})}] \quad \text{if } \ell \neq 1, n-1, \\
(5) & \quad \|u\|_{B^{-n/2}_1} \leq C[\|du\|_{L^1(\mathbb{R}^n, \Lambda^{\ell+1})} + \|\delta u\|_{H^1(\mathbb{R}^n, \Lambda^{\ell-1})}] \quad \text{if } \ell = 1, \\
(6) & \quad \|u\|_{B^{-n/2}_1} \leq C[\|du\|_{H^1(\mathbb{R}^n, \Lambda^{\ell+1})} + \|\delta u\|_{L^1(\mathbb{R}^n, \Lambda^{\ell-1})}] \quad \text{if } \ell = n-1
\end{align*}
\]

for any smooth \( \ell \)-form \( u \) with support in a fixed compact subset of \( \mathbb{R}^n \).

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In the above, $\Lambda^\ell$ stands for the $\ell$-th exterior power of $\mathbb{R}^n$ (i.e., $\ell$-differential forms) and, given a space of distributions $X$ in $\mathbb{R}^n$, we set $X(\mathbb{R}^n, \Lambda^\ell) := X \otimes \Lambda^\ell$.

The proof of (13) employs the potential theoretic integral representation formula
\begin{equation}
 u = d\Pi(\delta u) + \delta\Pi(du),
\end{equation}
where $\Pi$ is the Newtonian potential in $\mathbb{R}^n$ (i.e., the operator of formal convolution with the standard radial fundamental solution for the Laplacian in $\mathbb{R}^n$). Given that $d\Pi$ and $\delta\Pi$ are smoothing operators of order one on the scale of both Besov and Triebel-Lizorkin spaces (the latter containing $H^1$), the key issue which arises in this scenario is determining the amount of regularity exhibited on this scale by a closed (respectively, co-closed) differential form with absolutely integrable coefficients.

**Proposition 2.** If $\ell < n$ and $F = \sum F_i dx^i \in L^1(\mathbb{R}^n, \Lambda^\ell)$, $n \geq 2$, is a differential form such that $dF = 0$, then
\begin{equation}
 F \in B_{-\theta}^{n\theta'}(\mathbb{R}^n, \Lambda^\ell)
\end{equation}
for each $\theta \in (0,1)$. Furthermore, if $\ell > 0$ and $G \in L^1(\mathbb{R}^n, \Lambda^\ell)$ satisfies $\delta G = 0$, then
\begin{equation}
 G \in B_{\theta}^{n\theta'}(\mathbb{R}^n, \Lambda^\ell)
\end{equation}
whenever $0 < \theta < 1$.

**Proof of Proposition 1.** The estimate (14) follows from (17), given the regularity exhibited by $F := du$ and $G := \delta u$, respectively. The estimates (15) and (16) follow as well, using the mapping properties of the Newtonian potential on Triebel-Lizorkin spaces.

**Proof of Proposition 2.** Of course, it suffices to establish (18), as (19) follows from this and Hodge duality. With this goal in mind, let $W^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$, $s \in \mathbb{R}$, stand for the $L^p$-based Sobolev space of order $s$. We shall first show that
\begin{equation}
 F \in W^{-1, \frac{n\theta}{n-1}}(\mathbb{R}^n, \Lambda^\ell).
\end{equation}

Then (20) and complex interpolation give
\begin{equation}
 F \in \left[ W^{-1, \frac{n\theta}{n-1}}(\mathbb{R}^n, \Lambda^\ell) , L^1(\mathbb{R}^n, \Lambda^\ell) \right]_{\theta} = \left[ W^{1-n}(\mathbb{R}^n, \Lambda^\ell) , L^\infty(\mathbb{R}^n, \Lambda^\ell) \right]_{\theta}'
\end{equation}
for any $0 \leq \theta \leq 1$. The last equality is a consequence of Calderón’s duality theorem for the complex interpolation method (see, e.g., p. 72 in [4]). Now, since
\begin{equation}
 B_{0}^{\infty, 1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \quad \text{and} \quad B_{1}^{n,2}(\mathbb{R}^n) \hookrightarrow W^{1,n}(\mathbb{R}^n)
\end{equation}
(here $n \geq 2$ is used), it follows that
\begin{equation}
 B_{0}^{\frac{n}{n-1}, \infty}(\mathbb{R}^n) = \left[ B_{0}^{\infty, 1}(\mathbb{R}^n), B_{1}^{n,2}(\mathbb{R}^n) \right]_{\theta} \hookrightarrow \left[ L^\infty(\mathbb{R}^n), W^{1,n}(\mathbb{R}^n) \right]_{\theta}
\end{equation}
for every $\theta \in [0,1]$. In particular,
\begin{equation}
 \left[ W^{1,n}(\mathbb{R}^n), L^\infty(\mathbb{R}^n) \right]_{\theta}' \hookrightarrow \left( B_{0}^{\frac{n}{n-1}, \infty}(\mathbb{R}^n) \right)_{\theta}' = B_{-\theta}^{\frac{n\theta'}{n-1}}(\mathbb{R}^n),
\end{equation}
and (21) follows readily from this.
The proof of (10) follows closely the presentation in [2]. The departure point is the observation that, given $N \in \mathbb{N}$, $p > N$ and $\lambda > 0$, there exists $C > 0$ such that any function $\Phi \in C^\infty_0(\mathbb{R}^N)$ can be written in the form

$$\Phi = \Phi^1 + \Phi^2 \quad \text{with} \quad \lambda^{-1 + N/p} \|\Phi^1\|_{L^\infty(\mathbb{R}^N)} + \lambda^{N/p} \|\nabla\Phi^2\|_{L^\infty(\mathbb{R}^N)} \leq C \|\nabla\Phi\|_{L^p(\mathbb{R}^N)}.$$  

See Lemma 2 in [2], where a slightly more general result is proved.

Note that (10) follows as soon as we show that

$$0 = dF = \sum_{j=1}^n dx_j \wedge \partial_{x_j} F;$$

hence, after taking the interior product with $dx_1$,

$$0 = dx_1 \wedge \left( \sum_{j=2}^n dx_j \wedge \partial_{x_j} F \right) + dx_1 \wedge (dx_1 \wedge \partial_{x_1} F) + \sum_{j \neq 1} \partial_{x_j} F \, dx^j.$$
In particular,

\[(25) \quad \sum_{I \neq 1} \partial_{x_I} F_I dx^I = - \sum_{j=2}^n dx_j \lor (dx_j \land \partial_{x_j} F).\]

As a consequence of the Fundamental Theorem of Calculus, the identity \[(25)\] and integration by parts, we may then write

\[
J^2(t) = \sum_{I \neq 1} \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} (\partial_{x_I} F_I)(x_1, x') G_I^2(t, x') dx' dx_1
\]
\[
= -\sum_{j=2}^n \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} (dx_j \lor (dx_j \land \partial_{x_j} F(x_1, x')), G^2(t, x')) dx' dx_1
\]
\[
= \sum_{j=2}^n \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} (dx_j \lor (dx_j \land F(x_1, x')), \partial_{x_j} G^2(t, x')) dx' dx_1.
\]

From this and \[(24)\] we conclude that

\[(27) \quad |J^2(t)| \leq CA^{-1+1/n} \|F\|_{L^1(\mathbb{R}^n)} \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.\]

Together, \[(22)\] and \[(27)\] yield

\[(28) \quad |J(t)| \leq C \left( \lambda^{-1+1/n} \|F\|_{L^1(\mathbb{R}^n)} + \lambda^{1/n} \|F(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})} \right) \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.\]

To minimize the right-hand side of the above inequality in the parameter \(\lambda > 0\), for each \(t \in \mathbb{R}\) choose \(\lambda = \lambda(t) := (n-1) \|F\|_{L^1(\mathbb{R}^n)} \|F(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})}^{-1}\). This leads to

\[(29) \quad |J(t)| \leq C \|F\|_{L^1(\mathbb{R}^n)}^{1/n} \|F(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})}^{1-1/n} \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.
\]

Integrating in \(t \in \mathbb{R}\) and using Hölder’s inequality with exponents \(n/(n-1)\) and \(n\) readily give \((5)\). \(\square\)

**Remark.** Note that \((1)-(3)\) are simple consequences of the integral representation formula \(\text{(7)}\), the regularity result \((10)\) along with its Hodge dual, and the mapping properties of the Newtonian potential.

A natural conjecture is that the estimates \((4)-(6)\) hold with the stronger norm \(\|u\|_{W^{1-\theta} \frac{n}{n-\theta} (\mathbb{R}^n, A^t)} \) in place of \(\|u\|_{B_{1-\theta} \frac{n}{n-\theta} (\mathbb{R}^n, A^t)} \). Then the estimates \((1)-(3)\) would follow directly from this strengthened version of \((4)-(6)\) and the embedding

\[(30) \quad W^{1-\theta} \frac{n}{n-\theta} (\mathbb{R}^n) \hookrightarrow L^{n/(n-1)} (\mathbb{R}^n).
\]

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**References**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22904
E-mail address: im3p@virginia.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MISSOURI, COLUMBIA, MISSOURI 65211
E-mail address: marius@math.missouri.edu