A REMARK ON THE REGULARITY
OF THE DIV-CURL SYSTEM

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ABSTRACT. As a limiting case of the classical Calderón-Zygmund theory, in
this note we study the Besov regularity of differential forms \( u \) for which \( \mathbf{d}u \) and \( \delta u \) have absolutely integrable coefficients in \( \mathbb{R}^n \).

By further refining the work in [3], the following estimates were recently proved
in [2]: There exists a constant \( C \) such that for any smooth \( \ell \)-form \( u \) with compact
support in \( \mathbb{R}^n, n \geq 2 \), one has

\[
\| u \|_{L^\theta(n/(n-1))} \leq C \left[ \| \delta u \|_{L^1} + \| \delta u \|_{H^1} \right] \quad \text{if } \ell \neq 1, n - 1, \tag{1}
\]

\[
\| u \|_{L^\theta(n/(n-1))} \leq C \left[ \| \mathbf{d}u \|_{L^1} + \| \delta u \|_{H^1} \right] \quad \text{if } \ell = 1, \tag{2}
\]

\[
\| u \|_{L^\theta(n/(n-1))} \leq C \left[ \| \delta u \|_{H^1} + \| \delta u \|_{L^1} \right] \quad \text{if } \ell = n - 1. \tag{3}
\]

Here \( H^1 \) denotes the standard Hardy space. The cases \( \ell = 0 \) and \( \ell = 1 \) cor-
respond, respectively, to the classical Gagliardo-Nirenberg inequality and to an
estimate obtained by J. Bourgain and H. R. Brezis in [1].

In light of (1)–(3), it is of interest to establish similar estimates which emphasize
the size of \( u \) in Besov spaces \( B_{p,q}^{s} \), \( 0 < p, q \leq \infty, s \in \mathbb{R} \) (see, e.g., [4] for definitions
and basic properties). In this regard, we shall prove the following result.

**Proposition 1.** Assume that \( n > 2 \). Then for each \( 0 < \theta < 1 \) there exists \( C > 0 \)
such that

\[
\| u \|_{B_{1-\theta}^{n/(n-1)}(\mathbb{R}^n, \Lambda^\ell)} \leq C \left[ \| \mathbf{d}u \|_{L^1(\mathbb{R}^n, \Lambda^{\ell+1})} + \| \delta u \|_{L^1(\mathbb{R}^n, \Lambda^{\ell-1})} \right] \quad \text{if } \ell \neq 1, n - 1, \tag{4}
\]

\[
\| u \|_{B_{1-\theta}^{n/(n-1)}(\mathbb{R}^n, \Lambda^\ell)} \leq C \left[ \| \delta u \|_{L^1(\mathbb{R}^n, \Lambda^{\ell+1})} + \| \delta u \|_{H^1(\mathbb{R}^n, \Lambda^{\ell-1})} \right] \quad \text{if } \ell = 1, \tag{5}
\]

\[
\| u \|_{B_{1-\theta}^{n/(n-1)}(\mathbb{R}^n, \Lambda^\ell)} \leq C \left[ \| \delta u \|_{H^1(\mathbb{R}^n, \Lambda^{\ell+1})} + \| \delta u \|_{L^1(\mathbb{R}^n, \Lambda^{\ell-1})} \right] \quad \text{if } \ell = n - 1. \tag{6}
\]

for any smooth \( \ell \)-form \( u \) with support in a fixed compact subset of \( \mathbb{R}^n \).

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In the above, $\Lambda^\ell$ stands for the $\ell$-th exterior power of $\mathbb{R}^n$ (i.e., $\ell$-differential forms) and, given a space of distributions $X$ in $\mathbb{R}^n$, we set $X(\mathbb{R}^n, \Lambda^\ell) := X \otimes \Lambda^\ell$.

The proof of (4)–(6) employs the potential theoretic integral representation formula
\begin{equation}
(7) \quad u = d\Pi(\delta u) + \delta \Pi(du),
\end{equation}
where $\Pi$ is the Newtonian potential in $\mathbb{R}^n$ (i.e., the operator of formal convolution with the standard radial fundamental solution for the Laplacian in $\mathbb{R}^n$). Given that $d\Pi$ and $\delta \Pi$ are smoothing operators of order one on the scale of both Besov and Triebel-Lizorkin spaces (the latter containing $H^1$), the key issue which arises in this scenario is determining the amount of regularity exhibited on this scale by a closed (respectively, co-closed) differential form with absolutely integrable coefficients.

**Proposition 2.** If $\ell < n$ and $F = \sum_{i} F_i dx_i \in L^1(\mathbb{R}^n, \Lambda^\ell)$, $n \geq 2$, is a differential form such that $dF = 0$, then
\begin{equation}
(8) \quad F \in B_{-\theta}^{\frac{n}{\ell}}(\mathbb{R}^n, \Lambda^\ell)
\end{equation}
for each $\theta \in (0, 1)$. Furthermore, if $\ell > 0$ and $G \in L^1(\mathbb{R}^n, \Lambda^\ell)$ satisfies $\delta G = 0$, then
\begin{equation}
(9) \quad G \in B_{\theta}^{\frac{n}{\ell}}(\mathbb{R}^n, \Lambda^\ell)
\end{equation}
whenever $0 < \theta < 1$.

**Proof of Proposition 1.** The estimate (8) follows from (7), given the regularity exhibited by $F := du$ and $G := \delta u$, respectively. The estimates (5) and (8) follow as well, using the mapping properties of the Newtonian potential on Triebel-Lizorkin spaces.

**Proof of Proposition 2.** Of course, it suffices to establish (8), as (9) follows from this and Hodge duality. With this goal in mind, let $W^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$, $s \in \mathbb{R}$, stand for the $L^p$-based Sobolev space of order $s$. We shall first show that
\begin{equation}
(10) \quad F \in W^{-1, \frac{n}{\ell-1}}(\mathbb{R}^n, \Lambda^\ell).
\end{equation}
Then (10) and complex interpolation give
\begin{equation}
(11) \quad F \in \left[W^{-1, \frac{n}{\ell-1}}(\mathbb{R}^n, \Lambda^\ell), L^1(\mathbb{R}^n, \Lambda^\ell)\right]_{\theta} = \left[W^{1-n}(\mathbb{R}^n, \Lambda^\ell), L^\infty(\mathbb{R}^n, \Lambda^\ell)\right]_{\theta}
\end{equation}
for any $0 \leq \theta \leq 1$. The last equality is a consequence of Calderón’s duality theorem for the complex interpolation method (see, e.g., p. 72 in [4]). Now, since
\begin{equation}
(12) \quad B_0^{\infty, 1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \quad \text{and} \quad B_1^{n,2}(\mathbb{R}^n) \hookrightarrow W^{1,n}(\mathbb{R}^n)
\end{equation}
(here $n \geq 2$ is used), it follows that
\begin{equation}
(13) \quad B_0^{\frac{n}{\ell}, \frac{n}{\ell-1}}(\mathbb{R}^n) = \left[B_0^{\infty, 1}(\mathbb{R}^n), B_1^{n,2}(\mathbb{R}^n)\right]_{\theta} \hookrightarrow \left[L^\infty(\mathbb{R}^n), W^{1,n}(\mathbb{R}^n)\right]_{\theta}
\end{equation}
for every $\theta \in [0, 1]$. In particular,
\begin{equation}
(14) \quad \left[W^{1,n}(\mathbb{R}^n), L^\infty(\mathbb{R}^n)\right]_{\theta} \hookrightarrow \left(B_0^{\frac{n}{\ell}, \frac{n}{\ell-1}}(\mathbb{R}^n)\right)' = B_{-\theta}^{\frac{n}{\ell}}(\mathbb{R}^n),
\end{equation}
and (8) follows readily from this.
The proof of (10) follows closely the presentation in [2]. The departure point is the observation that, given $N \in \mathbb{N}$, $p > N$ and $\lambda > 0$, there exists $C > 0$ such that any function $\Phi \in C_0^\infty(\mathbb{R}^N)$ can be written in the form
\begin{equation}
\Phi = \Phi^1 + \Phi^2 \quad \text{with} \quad \lambda^{1+N/p} \|\Phi^1\|_{L^\infty(\mathbb{R}^N)} + \lambda^{N/p} \|\nabla\Phi^2\|_{L^\infty(\mathbb{R}^N)} \leq C \|\nabla\Phi\|_{L^p(\mathbb{R}^N)}.
\end{equation}
See Lemma 2 in [2], where a slightly more general result is proved.

Note that (10) follows as soon as we show that
\begin{equation}
\lambda^{1/n} \|G^1(t, \cdot)\|_{L^\infty(\mathbb{R}^{n-1})} + \lambda^{1/n} \|\nabla G^2(t, \cdot)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.
\end{equation}

It suffices to show that there exists $C > 0$ such that
\begin{equation}
\sum_{I \neq I_0} \int_{\mathbb{R}^{n-1}} F_I(t, x') G_I(t, x') \, dx' \leq C \|F\|_{L^1} \|\nabla G\|_{L^n}, \quad \forall j \in \{1, \ldots, n\}.
\end{equation}

In the proof of (13) there is no loss of generality in assuming that $j = 1$. Next, set
\begin{equation}
J(t) := \sum_{I \neq I_0} \int_{\mathbb{R}^{n-1}} F_I(t, x') G_I(t, x') \, dx', \quad t \in \mathbb{R}.
\end{equation}

For each $t \in \mathbb{R}$ and $\lambda > 0$ to be specified later, use the decomposition (15) with $p = n$ and $N = n - 1$ in order to write $G(t, \cdot) = G^1(t, \cdot) + G^2(t, \cdot)$ where
\begin{equation}
\lambda^{-1/n} \|G^1(t, \cdot)\|_{L^\infty(\mathbb{R}^{n-1})} + \lambda^{-1/n} \|\nabla G^2(t, \cdot)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.
\end{equation}

Accordingly, write $J(t) = J^1(t) + J^2(t)$ where
\begin{equation}
J^j(t) := \sum_{I \neq I_0} \int_{\mathbb{R}^{n-1}} F_I(t, x') G^j_I(t, x') \, dx', \quad j = 1, 2,
\end{equation}

and note that
\begin{equation}
|J^1(t)| \leq C \lambda^{1/n} \|F(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})} \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.
\end{equation}

To estimate $J^2(t)$ we first note that
\begin{equation}
0 = dF = \sum_{j=1}^n dx_j \wedge \partial_{x_j} F;
\end{equation}

hence, after taking the interior product with $dx_1$,
\begin{align}
0 &= dx_1 \vee \left( \sum_{j=2}^n dx_j \wedge \partial_{x_j} F \right) + dx_1 \vee (dx_1 \wedge \partial_{x_1} F) \\
&= dx_1 \vee \left( \sum_{j=2}^n dx_j \wedge \partial_{x_j} F \right) + \sum_{I \neq I_0} \partial_{x_1} F_I dx_I.
\end{align}
In particular,

\[
\sum_{I \neq 1} \partial_{x_I} F_t dx^I = - \sum_{j=2}^n dx_j \vee (dx_j \wedge \partial_{x_j} F).
\]

As a consequence of the Fundamental Theorem of Calculus, the identity (25) and integration by parts, we may then write

\[
J^2(t) = \sum_{I \neq 1} \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} (\partial_{x_I} F_t)(x_1, x') G_I^2(t, x') \, dx' \, dx_1
\]

\[
= -\sum_{j=2}^n \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} \langle dx_j \vee (dx_j \wedge \partial_{x_j} F(x_1, x')), G^2(t, x') \rangle \, dx' \, dx_1
\]

\[
= \sum_{j=2}^n \int_{-\infty}^t \int_{\mathbb{R}^{n-1}} \langle dx_j \vee (dx_j \wedge F(x_1, x')), \partial_{x_j} G^2(t, x') \rangle \, dx' \, dx_1.
\]

From this and (21) we conclude that

\[
|J^2(t)| \leq CA^{-1+1/n} \|F\|_{L^1(\mathbb{R}^n)} \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.
\]

Together, (22) and (27) yield

\[
|J(t)| \leq C \left(\lambda^{-1+1/n} \|F\|_{L^1(\mathbb{R}^n)} + \lambda^{1/n} \|F(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})}\right) \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.
\]

To minimize the right-hand side of the above inequality in the parameter $\lambda > 0$, for each $t \in \mathbb{R}$ choose $\lambda = \lambda(t) := (n-1) \|F\|_{L^1(\mathbb{R}^n)} \|F(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})}^{-1}$. This leads to

\[
|J(t)| \leq C \|F\|_{L^1(\mathbb{R}^n)}^{1/n} \|F(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})}^{1-1/n} \|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.
\]

Integrating in $t \in \mathbb{R}$ and using Hölder’s inequality with exponents $n/(n-1)$ and $n$ readily give (5). \hfill \Box

Remark. Note that (1)–(3) are simple consequences of the integral representation formula (7), the regularity result (10) along with its Hodge dual, and the mapping properties of the Newtonian potential.

A natural conjecture is that the estimates (4)–(6) hold with the stronger norm $\|u\|_{W^{1-\theta, n/(n-1)}(\mathbb{R}^n, \mathcal{A}^t)}$ in place of $\|u\|_{B^{1-\theta, n/(n-1)}_{1, 1}(\mathbb{R}^n, \mathcal{A}^t)}$. Then the estimates (1)–(3) would follow directly from this strengthened version of (4)–(6) and the embedding

\[
W^{1-\theta, n/(n-1)}(\mathbb{R}^n) \hookrightarrow L^{n/(n-1)}(\mathbb{R}^n).
\]

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References


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