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A REMARK ON THE REGULARITY OF THE DIV-CURL SYSTEM

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ABSTRACT. As a limiting case of the classical Calderón-Zygmund theory, in this note we study the Besov regularity of differential forms u for which du and δu have absolutely integrable coefficients in $\mathbb{R}^n$.

By further refining the work in [3], the following estimates were recently proved in [2]: There exists a constant C such that for any smooth ℓ-form u with compact support in $\mathbb{R}^n$, $n \geq 2$, one has

$$\|u\|_{L^{n/(n-1)}} \leq C[\|du\|_{L^1} + \|\delta u\|_{L^1}] \quad \text{if} \quad \ell \neq 1, n-1, \tag{1}$$

$$\|u\|_{L^{n/(n-1)}} \leq C[\|du\|_{L^1} + \|\delta u\|_{H^1}] \quad \text{if} \quad \ell = 1, \tag{2}$$

$$\|u\|_{L^{n/(n-1)}} \leq C[\|du\|_{H^1} + \|\delta u\|_{L^1}] \quad \text{if} \quad \ell = n-1. \tag{3}$$

Here H^1 denotes the standard Hardy space. The cases $\ell = 0$ and $\ell = 1$ correspond, respectively, to the classical Gagliardo-Nirenberg inequality and to an estimate obtained by J. Bourgain and H. R. Brezis in [1].

In light of (1)–(3), it is of interest to establish similar estimates which emphasize the size of u in Besov spaces $B_s^{p,q}$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$ (see, e.g., [4] for definitions and basic properties). In this regard, we shall prove the following result.

Proposition 1. *Assume that $n > 2$. Then for each $0 < \theta < 1$ there exists $C > 0$ such that*

$$\|u\|_{B_{1-\theta}^{\frac{n}{n-\theta},\frac{2}{\theta}}(\mathbb{R}^n,\Lambda^\ell)} \leq C[\|du\|_{L^1(\mathbb{R}^n,\Lambda^{\ell+1})} + \|\delta u\|_{L^1(\mathbb{R}^n,\Lambda^{\ell-1})}] \quad \text{if} \quad \ell \neq 1, n-1, \tag{4}$$

$$\|u\|_{B_{1-\theta}^{\frac{n}{n-\theta},\frac{2}{\theta}}(\mathbb{R}^n,\Lambda^\ell)} \leq C[\|du\|_{L^1(\mathbb{R}^n,\Lambda^{\ell+1})} + \|\delta u\|_{H^1(\mathbb{R}^n,\Lambda^{\ell-1})}] \quad \text{if} \quad \ell = 1, \tag{5}$$

$$\|u\|_{B_{1-\theta}^{\frac{n}{n-\theta},\frac{2}{\theta}}(\mathbb{R}^n,\Lambda^\ell)} \leq C[\|du\|_{H^1(\mathbb{R}^n,\Lambda^{\ell+1})} + \|\delta u\|_{L^1(\mathbb{R}^n,\Lambda^{\ell-1})}] \quad \text{if} \quad \ell = n-1 \tag{6}$$

for any smooth ℓ-form u with support in a fixed compact subset of $\mathbb{R}^n$.

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In the above, Λ^ℓ stands for the ℓ-th exterior power of $\mathbb{R}^n$ (i.e., ℓ-differential forms) and, given a space of distributions X in $\mathbb{R}^n$, we set $X(\mathbb{R}^n, \Lambda^\ell) := X \otimes \Lambda^\ell$.

The proof of (4)–(6) employs the potential theoretic integral representation formula

$$u = d\Pi(\delta u) + \delta\Pi(du), \tag{7}$$

where Π is the Newtonian potential in $\mathbb{R}^n$ (i.e., the operator of formal convolution with the standard radial fundamental solution for the Laplacian in $\mathbb{R}^n$). Given that $d\Pi$ and $\delta\Pi$ are smoothing operators of order one on the scale of both Besov and Triebel-Lizorkin spaces (the latter containing H^1), the key issue which arises in this scenario is determining the amount of regularity exhibited on this scale by a closed (respectively, co-closed) differential form with absolutely integrable coefficients.

Proposition 2. *If $\ell < n$ and $F = \sum_I F_I dx^I \in L^1(\mathbb{R}^n, \Lambda^\ell)$, $n \geq 2$, is a differential form such that $dF = 0$, then*

$$F \in B_{-\theta}^{\frac{n}{n-\theta}, \frac{2}{\theta}}(\mathbb{R}^n, \Lambda^\ell) \tag{8}$$

for each $\theta \in (0,1)$. Furthermore, if $\ell > 0$ and $G \in L^1(\mathbb{R}^n, \Lambda^\ell)$ satisfies $\delta G = 0$, then

$$G \in B_{-\theta}^{\frac{n}{n-\theta}, \frac{2}{\theta}}(\mathbb{R}^n, \Lambda^\ell) \tag{9}$$

whenever $0 < \theta < 1$.

Proof of Proposition 1. The estimate (4) follows from (7), given the regularity exhibited by $F := du$ and $G := \delta u$, respectively. The estimates (5) and (6) follow as well, using the mapping properties of the Newtonian potential on Triebel-Lizorkin spaces. □

Proof of Proposition 2. Of course, it suffices to establish (8), as (9) follows from this and Hodge duality. With this goal in mind, let $W^{s,p}(\mathbb{R}^n)$, $1 < p < \infty$, $s \in \mathbb{R}$, stand for the L^p-based Sobolev space of order s. We shall first show that

$$F \in W^{-1, \frac{n}{n-1}}(\mathbb{R}^n, \Lambda^\ell). \tag{10}$$

Then (10) and complex interpolation give

$$F \in \Big[W^{-1,\frac{n}{n-1}}(\mathbb{R}^n, \Lambda^\ell)\,,\, L^1(\mathbb{R}^n, \Lambda^\ell)\Big]_\theta = \Big[W^{1,n}(\mathbb{R}^n, \Lambda^\ell)\,,\, L^\infty(\mathbb{R}^n, \Lambda^\ell)\Big]'_\theta \tag{11}$$

for any $0 \leq \theta \leq 1$. The last equality is a consequence of Calderón's duality theorem for the complex interpolation method (see, e.g., p. 72 in [4]). Now, since

$$B_0^{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \quad \text{and} \quad B_1^{n,2}(\mathbb{R}^n) \hookrightarrow W^{1,n}(\mathbb{R}^n) \tag{12}$$

(here $n \geq 2$ is used), it follows that

$$B_\theta^{\frac{n}{\theta}, \frac{2}{2-\theta}}(\mathbb{R}^n) = \Big[B_0^{\infty,1}(\mathbb{R}^n), B_1^{n,2}(\mathbb{R}^n)\Big]_\theta \hookrightarrow \Big[L^\infty(\mathbb{R}^n), W^{1,n}(\mathbb{R}^n)\Big]_\theta \tag{13}$$

for every $\theta \in [0,1]$. In particular,

$$\Big[W^{1,n}(\mathbb{R}^n)\,,\, L^\infty(\mathbb{R}^n)\Big]'_\theta \hookrightarrow \Big(B_\theta^{\frac{n}{\theta}, \frac{2}{2-\theta}}(\mathbb{R}^n)\Big)' = B_{-\theta}^{\frac{n}{n-\theta}, \frac{2}{\theta}}(\mathbb{R}^n), \tag{14}$$

and (8) follows readily from this.

The proof of (10) follows closely the presentation in [2]. The departure point is the observation that, given $N \in \mathbb{N}$, $p > N$ and $\lambda > 0$, there exists $C > 0$ such that any function $\Phi \in C_0^\infty(\mathbb{R}^N)$ can be written in the form

(15)
$$\Phi = \Phi^1 + \Phi^2 \text{ with } \lambda^{-1+N/p}\|\Phi^1\|_{L^\infty(\mathbb{R}^N)} + \lambda^{N/p}\|\nabla\Phi^2\|_{L^\infty(\mathbb{R}^N)} \leq C\|\nabla\Phi\|_{L^p(\mathbb{R}^N)}.$$

See Lemma 2 in [2], where a slightly more general result is proved.

Note that (10) follows as soon as we show that

$$\Big|\int_{\mathbb{R}^n}\langle F, G\rangle\,dx\Big| \leq C\|F\|_{L^1}\|\nabla G\|_{L^n} \tag{16}$$

for any differential form $G = \sum_I G_I dx^I \in C_0^\infty(\mathbb{R}^n, \Lambda^\ell)$. In turn, since $\ell < n$ and

$$\int_{\mathbb{R}^n}\langle F, G\rangle\,dx = \sum_I \int_{\mathbb{R}^n} F_I G_I\,dx = (n-\ell)\sum_{j=1}^n \sum_{I \not\ni j}\int_{\mathbb{R}^n} F_I G_I\,dx, \tag{17}$$

it suffices to show that there exists $C > 0$ such that

$$\Big|\sum_{I\not\ni j}\int_{\mathbb{R}^n} F_I G_I\,dx\Big| \leq C\|F\|_{L^1}\|\nabla G\|_{L^n}, \qquad \forall\, j \in \{1, ..., n\}. \tag{18}$$

In the proof of (18) there is no loss of generality in assuming that $j = 1$. Next, set

$$J(t) := \sum_{I\not\ni 1}\int_{\mathbb{R}^{n-1}} F_I(t, x')G_I(t, x')\,dx', \qquad t \in \mathbb{R}. \tag{19}$$

For each $t \in \mathbb{R}$ and $\lambda > 0$ to be specified later, use the decomposition (15) with $p = n$ and $N = n - 1$ in order to write $G(t, \cdot) = G^1(t, \cdot) + G^2(t, \cdot)$ where

(20)
$$\lambda^{-1/n}\|G^1(t, \cdot)\|_{L^\infty(\mathbb{R}^{n-1})} + \lambda^{1-1/n}\|\nabla G^2(t, \cdot)\|_{L^\infty(\mathbb{R}^{n-1})} \leq C\|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}.$$

Accordingly, write $J(t) = J^1(t) + J^2(t)$ where

$$J^j(t) := \sum_{I\not\ni 1}\int_{\mathbb{R}^{n-1}} F_I(t, x')G_I^j(t, x')\,dx', \qquad j = 1, 2, \tag{21}$$

and note that

$$|J^1(t)| \leq C\lambda^{1/n}\|F(t, \cdot)\|_{L^1(\mathbb{R}^{n-1})}\|\nabla G(t, \cdot)\|_{L^n(\mathbb{R}^{n-1})}. \tag{22}$$

To estimate $J^2(t)$ we first note that

$$0 = dF = \sum_{j=1}^n dx_j \wedge \partial_{x_j} F; \tag{23}$$

hence, after taking the interior product with dx_1,

$$\begin{aligned} 0 &= dx_1 \vee \Big(\sum_{j=2}^n dx_j \wedge \partial_{x_j} F\Big) + dx_1 \vee (dx_1 \wedge \partial_{x_1} F) \\ &= dx_1 \vee \Big(\sum_{j=2}^n dx_j \wedge \partial_{x_j} F\Big) + \sum_{I\not\ni 1} \partial_{x_1} F_I dx^I. \end{aligned} \tag{24}$$

In particular,

$$\sum_{I\not\ni 1}\partial_{x_1}F_I dx^I = -\sum_{j=2}^{n} dx_1 \vee \Big(dx_j \wedge \partial_{x_j}F\Big). \tag{25}$$

As a consequence of the Fundamental Theorem of Calculus, the identity (25) and integration by parts, we may then write

$$\begin{aligned} J^2(t) &= \sum_{I\not\ni 1}\int_{-\infty}^{t}\int_{\mathbb{R}^{n-1}}(\partial_{x_1}F_I)(x_1,x')G^2_I(t,x')\,dx'dx_1 \\ &= -\sum_{j=2}^{n}\int_{-\infty}^{t}\int_{\mathbb{R}^{n-1}}\langle dx_1\vee(dx_j\wedge\partial_{x_j}F(x_1,x')),G^2(t,x')\rangle\,dx'dx_1 \\ &= \sum_{j=2}^{n}\int_{-\infty}^{t}\int_{\mathbb{R}^{n-1}}\langle dx_1\vee(dx_j\wedge F(x_1,x')),\partial_{x_j}G^2(t,x')\rangle\,dx'dx_1. \end{aligned} \tag{26}$$

From this and (20) we conclude that

$$|J^2(t)| \leq C\lambda^{-1+1/n}\|F\|_{L^1(\mathbb{R}^n)}\|\nabla G(t,\cdot)\|_{L^n(\mathbb{R}^{n-1})}. \tag{27}$$

Together, (22) and (27) yield

$$|J(t)| \leq C\Big(\lambda^{-1+1/n}\|F\|_{L^1(\mathbb{R}^n)} + \lambda^{1/n}\|F(t,\cdot)\|_{L^1(\mathbb{R}^{n-1})}\Big)\|\nabla G(t,\cdot)\|_{L^n(\mathbb{R}^{n-1})}. \tag{28}$$

To minimize the right-hand side of the above inequality in the parameter $\lambda > 0$, for each $t\in\mathbb{R}$ choose $\lambda = \lambda(t) := (n-1)\|F\|_{L^1(\mathbb{R}^n)}\|F(t,\cdot)\|^{-1}_{L^1(\mathbb{R}^{n-1})}$. This leads to

$$|J(t)| \leq C\|F\|^{1/n}_{L^1(\mathbb{R}^n)}\|F(t,\cdot)\|^{1-1/n}_{L^1(\mathbb{R}^{n-1})}\|\nabla G(t,\cdot)\|_{L^n(\mathbb{R}^{n-1})}. \tag{29}$$

Integrating in $t\in\mathbb{R}$ and using Hölder's inequality with exponents $n/(n-1)$ and n readily give (8). □

Remark. Note that (1)–(3) are simple consequences of the integral representation formula (7), the regularity result (10) along with its Hodge dual, and the mapping properties of the Newtonian potential.

A natural conjecture is that the estimates (4)–(6) hold with the stronger norm $\|u\|_{W^{1-\theta,\frac{n}{n-\theta}}(\mathbb{R}^n,\Lambda^\ell)}$ in place of $\|u\|_{B^{\frac{n}{n-\theta},\frac{2}{\theta}}_{1-\theta}(\mathbb{R}^n,\Lambda^\ell)}$. Then the estimates (1)–(3) would follow directly from this strengthened version of (4)–(6) and the embedding

$$W^{1-\theta,\frac{n}{n-\theta}}(\mathbb{R}^n) \hookrightarrow L^{n/(n-1)}(\mathbb{R}^n). \tag{30}$$

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