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TWO WAY SUBTABLE SUM PROBLEMS AND QUADRATIC GRÖBNER BASES

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Abstract. Hara, Takemura and Yoshida discussed toric ideals arising from two way subtable sum problems and showed that these toric ideals are generated by quadratic binomials if and only if the subtables are either diagonal or triangular. In the present paper, we show that if the subtables are either diagonal or triangular, then their toric ideals possess quadratic Gröbner bases.

Fix positive integers m and n and let $T = \{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Let K be a field and $K[\{u_i\}_{1\leq i\leq m} \cup \{v_j\}_{1\leq j\leq n} \cup \{w,t\}]$ be the polynomial ring in $m+n+2$ variables over K. Given a subset S of T, let R_S denote the semigroup ring generated by those monomials $u_i v_j w$ with $(i,j) \in S$ and those monomials $u_i v_j t$ with $(i,j) \notin S$. Let $K[X] = K[\{x_{i,j}\}_{(i,j)\in T}]$ denote the polynomial ring in mn variables over K. Define the surjective map $\pi : K[X] \to R_S$ by $\pi(x_{i,j}) = u_i v_j w$ if $(i,j) \in S$ and $\pi(x_{i,j}) = u_i v_j t$ if $(i,j) \notin S$. We call the kernel of π *the toric ideal of two way subtable sum problems associated with* S and denote it by I_S. We refer the reader to [1] and [5] for fundamental facts on Gröbner bases and toric ideals.

A subset $S \subset T$ is called 2×2 *block diagonal* if there exist integers r, c such that

$$S = \{(i,j) \mid 1 \leq i \leq r, 1 \leq j \leq c\} \cup \{(i,j) \mid r < i \leq m, c < j \leq n\}$$

after an appropriate interchange of rows and columns. A subset $S \subset T$ is called *triangular* if S satisfies the condition

$$(*) \qquad (i,j) \in S \Rightarrow (i',j') \in S \text{ for all } 1 \leq i' \leq i \text{ and } 1 \leq j' \leq j$$

after an appropriate interchange of rows and columns. Hara, Takemura and Yoshida [2] showed that I_S is generated by quadratic binomials if and only if S is either 2×2 block diagonal or triangular. Our main theorem is as follows:

Theorem. *With the same notation as above, the following conditions are equivalent:*

(i) *the toric ideal I_S is generated by quadratic binomials,*
(ii) *the toric ideal I_S possesses a squarefree initial ideal,*
(iii) *the toric ideal I_S possesses a quadratic Gröbner basis,*
(iv) *the semigroup ring R_S is normal,*

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(v) *the semigroup ring* R_S *is Koszul,*
(vi) *the subset* S *is either* 2×2 *block diagonal or triangular.*

Proof. First of all, (i) $\Leftrightarrow$ (vi) is proved in [2]. Moreover, in [2], it is proved that if S is neither diagonal nor triangular, then there exists an indispensable binomial $f^{(+)} - f^{(-)} \in I_S$ such that neither $f^{(+)}$ nor $f^{(-)}$ is squarefree. Hence, by [4, Corollary 6.3], it follows that (iv) $\Rightarrow$ (vi). On the other hand, in general, it is known that (ii) $\Rightarrow$ (iv) and that (iii) $\Rightarrow$ (v) $\Rightarrow$ (i). Since R_S is generated by monomials of the same degree, it follows from the proof of [3, Proposition 1.6] that (iii) $\Rightarrow$ (ii). We now prove that (vi) $\Rightarrow$ (iii). Suppose (vi) holds.

Suppose that $S \subset T$ is triangular and satisfies $(*)$. Then I_S is generated by

$$\mathcal{G} = \{x_{i,\ell}x_{j,k} - x_{i,k}x_{j,\ell} \mid 1 \le i < j \le m, 1 \le k < \ell \le n, \pi(x_{i,\ell}x_{j,k}) = \pi(x_{i,k}x_{j,\ell})\}.$$

(Note that all monomials appearing in $g \in \mathcal{G}$ are squarefree.) Throughout the proof, we will perform computations using Buchberger's criterion for $\mathcal{G}$. These computations, while routine, are often very long; therefore, we will omit the details and state only the results. Since S is triangular, the following holds:

($\sharp$) $x_{i,\ell}x_{j,k} - x_{i,k}x_{j,\ell}$ $(1 \le i < j \le m,\ 1 \le k < \ell \le n)$ does not belong to $\mathcal{G}$ if and only if $S \cap \{(i,\ell),(j,k),(i,k),(j,\ell)\}$ is either $\{(i,k)\}$ or $\{(i,k),(i,\ell),(j,k)\}$.

Let $<$ be the lexicographic order on $K[X]$ induced by $x_{m,1} > x_{m,2} > \cdots > x_{m,n} > x_{m-1,1} > \cdots > x_{m-1,n} > \cdots > x_{1,1} > \cdots > x_{1,n}$. Then the initial monomial of $x_{i,\ell}x_{j,k} - x_{i,k}x_{j,\ell}$ $(1 \le i < j \le m,\ 1 \le k < \ell \le n)$ is $x_{i,\ell}x_{j,k}$.

Suppose that the S-polynomial f of $g_1, g_2 \in \mathcal{G}$ is not reduced to 0 by $\mathcal{G}$. By [1, Ch.2 §9 Proposition 4], the initial monomials of g_1 and g_2 are not relatively prime. Hence f is a cubic binomial. If the monomials of f have a common variable, then f is reduced to 0 by $\mathcal{G}$. Let $f = x_{i_1,\ell_1}x_{i_2,\ell_2}x_{i_3,\ell_3} - x_{i'_1,\ell'_1}x_{i'_2,\ell'_2}x_{i'_3,\ell'_3}$ with the initial monomial $x_{i_1,\ell_1}x_{i_2,\ell_2}x_{i_3,\ell_3}$. Since $\pi(x_{i_1,\ell_1}x_{i_2,\ell_2}x_{i_3,\ell_3}) = \pi(x_{i'_1,\ell'_1}x_{i'_2,\ell'_2}x_{i'_3,\ell'_3})$, we have $\{i_1,i_2,i_3\} = \{i'_1,i'_2,i'_3\}$ and $\{\ell_1,\ell_2,\ell_3\} = \{\ell'_1,\ell'_2,\ell'_3\}$. Moreover, since the monomials of f have no common variable, $|\{i_1,i_2,i_3\}| = |\{\ell_1,\ell_2,\ell_3\}| = 3$. We may assume $i_1 = i'_1 < i_2 = i'_2 < i_3 = i'_3$. By the definition of $<$, we have $\ell_3 < \ell'_3 \in \{\ell_1,\ell_2\}$. For example, if $\ell_1 < \ell_3 < \ell_2$, then $\ell'_3 = \ell_2$ and hence, by $\ell_1 \ne \ell'_1$, we have $(\ell'_1,\ell'_2) = (\ell_3,\ell_1)$. (This is the case (6) below.) By such an argument, f is one of the following:

(1) $x_{i_1,j_2}x_{i_2,j_3}x_{i_3,j_1} - x_{i_1,j_3}x_{i_2,j_1}x_{i_3,j_2}$
(2) $x_{i_1,j_3}x_{i_2,j_2}x_{i_3,j_1} - x_{i_1,j_2}x_{i_2,j_1}x_{i_3,j_3}$
(3) $x_{i_1,j_3}x_{i_2,j_2}x_{i_3,j_1} - x_{i_1,j_1}x_{i_2,j_3}x_{i_3,j_2}$
(4) $x_{i_1,j_3}x_{i_2,j_1}x_{i_3,j_2} - x_{i_1,j_1}x_{i_2,j_2}x_{i_3,j_3}$
(5) $x_{i_1,j_2}x_{i_2,j_3}x_{i_3,j_1} - x_{i_1,j_1}x_{i_2,j_2}x_{i_3,j_3}$
(6) $x_{i_1,j_1}x_{i_2,j_3}x_{i_3,j_2} - x_{i_1,j_2}x_{i_2,j_1}x_{i_3,j_3}$

where $1 \le i_1 < i_2 < i_3 \le m$ and $1 \le j_1 < j_2 < j_3 \le n$.

In each case we will show that f is reduced by $\mathcal{G}$ to a cubic binomial h where the monomials of h have a common variable. Since I_S is prime, it will follow that the quadratic factor of h belongs to $\mathcal{G}$ and thus that f is reduced to 0 by $\mathcal{G}$.

(1) Suppose that none of $g_1 = x_{i_1,j_3}x_{i_3,j_2} - x_{i_1,j_2}x_{i_3,j_3}$, $g_2 = x_{i_1,j_2}x_{i_3,j_1} - x_{i_1,j_1}x_{i_3,j_2}$ and $g_3 = x_{i_2,j_3}x_{i_3,j_1} - x_{i_2,j_1}x_{i_3,j_3}$ belong to $\mathcal{G}$. Thanks to (♯) together with $g_1 \notin \mathcal{G}$, we have $(i_1, j_2) \in S$. Thanks to (♯) together with $g_2 \notin \mathcal{G}$, we have $(i_1, j_1), (i_3, j_1) \in S$ and $(i_3, j_2) \notin S$. Similarly, $g_3 \notin \mathcal{G}$ implies $(i_2, j_1), (i_2, j_3) \in S$. Thus, $(i_1, j_2), (i_2, j_3), (i_3, j_1) \in S$ and $(i_3, j_2) \notin S$. This contradicts $f \in I_S$.

(2) Suppose that none of $g_1 = x_{i_1,j_2}x_{i_2,j_1} - x_{i_1,j_1}x_{i_2,j_2}$, $g_2 = x_{i_2,j_2}x_{i_3,j_1} - x_{i_2,j_1}x_{i_3,j_2}$ and $g_3 = x_{i_1,j_3}x_{i_2,j_2} - x_{i_1,j_2}x_{i_2,j_3}$ belong to $\mathcal{G}$. Thanks to (♯) together with $g_1 \notin \mathcal{G}$, we have $(i_2, j_2) \notin S$. Thanks to (♯) together with $g_2 \notin \mathcal{G}$, we have $(i_2, j_1) \in S$ and $(i_3, j_1) \notin S$. Similarly, $g_3 \notin \mathcal{G}$ implies $(i_1, j_3) \notin S$. Thus, $(i_1, j_3), (i_2, j_2), (i_3, j_1) \notin S$ and $(i_2, j_1) \in S$. This contradicts $f \in I_S$.

(3) Suppose that none of $g_1 = x_{i_2,j_3}x_{i_3,j_2} - x_{i_2,j_2}x_{i_3,j_3}$, $g_2 = x_{i_1,j_3}x_{i_2,j_2} - x_{i_1,j_2}x_{i_2,j_3}$ and $g_3 = x_{i_2,j_2}x_{i_3,j_1} - x_{i_2,j_1}x_{i_3,j_2}$ belong to $\mathcal{G}$. Thanks to (♯) together with $g_1 \notin \mathcal{G}$, we have $(i_2, j_2) \in S$. Thanks to (♯) together with $g_2 \notin \mathcal{G}$, we have $(i_1, j_3) \in S$ and $(i_2, j_3) \notin S$. Similarly, $g_3 \notin \mathcal{G}$ implies $(i_3, j_1) \in S$. Thus, $(i_1, j_3), (i_2, j_2), (i_3, j_1) \in S$ and $(i_2, j_3) \notin S$. This contradicts $f \in I_S$.

(4) Suppose that none of $g_1 = x_{i_1,j_3}x_{i_2,j_1} - x_{i_1,j_1}x_{i_2,j_3}$ and $g_2 = x_{i_1,j_3}x_{i_3,j_2} - x_{i_1,j_2}x_{i_3,j_3}$ belong to $\mathcal{G}$. Thanks to (♯) together with $g_1 \notin \mathcal{G}$, we have $(i_1, j_1) \in S$. Since $f \in I_S$, $\{(i_1, j_3), (i_2, j_1), (i_3, j_2)\} \cap S \neq \emptyset$. Again, thanks to (♯) together with $g_1, g_2 \notin \mathcal{G}$, we have $\{(i_1, j_3), (i_2, j_1), (i_3, j_2)\} \subset S$. Since $f \in I_S$, we have $(i_3, j_3) \in S$. This contradicts $g_2 \notin \mathcal{G}$.

(5) Suppose that none of $g_1 = x_{i_1,j_2}x_{i_3,j_1} - x_{i_1,j_1}x_{i_3,j_2}$ and $g_2 = x_{i_2,j_3}x_{i_3,j_1} - x_{i_2,j_1}x_{i_3,j_3}$ belong to $\mathcal{G}$. Thanks to (♯) together with $g_1 \notin \mathcal{G}$, we have $(i_1, j_1) \in S$. Since $f \in I_S$, $\{(i_1, j_2), (i_2, j_3), (i_3, j_1)\} \cap S \neq \emptyset$. Again, thanks to (♯) together with $g_1, g_2 \notin \mathcal{G}$, we have $\{(i_1, j_2), (i_2, j_3), (i_3, j_1)\} \subset S$. Since $f \in I_S$, we have $(i_3, j_3) \in S$. This contradicts $g_2 \notin \mathcal{G}$.

(6) Suppose that none of $g_1 = x_{i_1,j_2}x_{i_2,j_1} - x_{i_1,j_1}x_{i_2,j_2}$ and $g_2 = x_{i_2,j_3}x_{i_3,j_2} - x_{i_2,j_2}x_{i_3,j_3}$ belong to $\mathcal{G}$. Thanks to (♯) together with $g_2 \notin \mathcal{G}$, we have $(i_2, j_2) \in S$. This contradicts $g_1 \notin \mathcal{G}$.

Thus, in all cases, $g_i \in \mathcal{G}$ for some i. Then f is reduced to the cubic binomial h by g_i, where monomials of h have a common variable. Hence f is reduced to 0 by $\mathcal{G}$.

On the other hand, suppose that S is 2×2 block diagonal; that is, there exist integers r, c such that $S = \{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq c\} \cup \{(i, j) \mid r < i \leq m, c < j \leq n\}$. Let $S' = \{(i, j) \mid 1 \leq i \leq r, 1 \leq j \leq c\}$. Then S' is triangular. Let $f = x_{i,\ell}x_{j,k} - x_{i,k}x_{j,\ell}$ with $1 \leq i < j \leq m, 1 \leq k < \ell \leq n$. Then $f \notin I_{S'}$ if and only if $i \leq r < j$ and $k \leq c < \ell$ if and only if $f \notin I_S$. Hence we have $I_S = I_{S'}$. □

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