DISTINGUISHING PROPERTIES OF ARENS IRREGULARITY

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ABSTRACT. In this paper, we present a number of examples of commutative Banach algebras with various Arens irregularity properties. These examples illustrate in particular that strong Arens irregularity and extreme non-Arens regularity, the two natural concepts of "maximal" Arens irregularity for general Banach algebras as introduced by Dales-Lau and Granirer, respectively, are indeed distinct. Thereby, an open question raised by several authors is answered. We also link these two properties to another natural Arens irregularity property.

1. INTRODUCTION

Let A be a Banach algebra. As is well-known, on the bidual A^{**} of A, there are two Banach algebra multiplications, called the left and right Arens products, each extending the multiplication on A (see [1]). By definition, the left Arens product $\square$ on A^{**} is induced by the left A-module structure on A. That is, for $m, n \in A^{**}$, $f \in A^*$, and $a, b \in A$, we have

$$\langle m\square n, f \rangle = \langle m, n\square f \rangle, \ \langle n\square f, a \rangle = \langle n, f \cdot a \rangle, \ \text{and} \ \langle f \cdot a, b \rangle = \langle f, ab \rangle.$$

Similarly, the right Arens product $\diamondsuit$ on A^{**} is defined by considering A as a right A-module. A is said to be Arens regular if $\square$ and $\diamondsuit$ coincide on A^{**}. Every C^*-algebra is Arens regular. On the other hand, Banach algebras studied in abstract harmonic analysis are typically far from being Arens regular.

By definition, for any fixed $m \in A^{**}$, the maps $n \longmapsto n\square m$ and $n \longmapsto m\diamondsuit n$ are weak*-weak* continuous on A^{**}. The left and the right topological centres of A^{**} are defined, respectively, as

$$Z_t^{(l)}(A^{**}) = \{m \in A^{**} : \text{ the map } n \longmapsto m\square n \text{ is weak}^*\text{-weak}^* \text{ continuous on } A^{**}\},$$

$$Z_t^{(r)}(A^{**}) = \{m \in A^{**} : \text{ the map } n \longmapsto n\diamondsuit m \text{ is weak}^*\text{-weak}^* \text{ continuous on } A^{**}\}.$$

It can be seen that A is Arens regular if and only if $Z_t^{(l)}(A^{**}) = Z_t^{(r)}(A^{**}) = A^{**}$. The reader is referred to Dales [4], Dales-Lau [5], Lau-Ülger [22], and Palmer [26] for more information on Arens products and topological centres.

Clearly, $A \subseteq Z_t^{(l)}(A^{**}) \cap Z_t^{(r)}(A^{**})$ under the canonical embedding $A \hookrightarrow A^{**}$. The following concepts of Arens irregularity were introduced by Dales and Lau [5]: A is called left strongly Arens irregular if $Z_t^{(l)}(A^{**}) = A$, right strongly Arens irregular if $Z_t^{(r)}(A^{**}) = A$, and strongly Arens irregular (SAI for short) if A is both left and right strongly Arens irregular.

Let G be a locally compact group. It was proved by Lau and Losert [19] that the group algebra $L_1(G)$ is always SAI. For the Fourier algebra $A(G)$, Lau and Losert [20, Theorem 6.5] showed that $A(G)$ is SAI if either G is discrete and amenable, or G is second countable and amenable with $\overline{[G, G]}$ non-open in G. Moreover, Lau and Losert [21] also proved that $A(G)$ is SAI when G is a countably infinite product of second countable locally compact groups $\{G_i\}_{i=0}^{\infty}$ with G_i $(i \geq 1)$ compact and non-trivial. Some slight extensions of [20, Theorem 6.5(iii)] and [21, Theorem 4.2] were obtained by the first author in [15]. However, as shown recently by Losert in [24] and [25], respectively, $A(F_2)$ and $A(SU(3))$ are not SAI.

There is another notion of Arens irregularity for a general Banach algebra A in the literature. Let $WAP(A)$ be the space of weakly almost periodic functionals on A. It was shown by Pym [27] that A is Arens regular if and only if $WAP(A) = A^*$. Hence, the size of the quotient space $A^*/WAP(A)$ measures the Arens irregularity of A in a certain sense. In [11], Granirer introduced the concept of "extreme non-Arens regularity": A is called extremely non-Arens regular (ENAR for short) if $A^*/WAP(A)$ contains a closed linear subspace which has A^* as a continuous linear image.

Corollary 7 in Granirer [11] shows that $A(G)$ is ENAR if G is non-discrete and second countable. The first author extended Granirer's result by proving that $A(G)$ is ENAR if G is infinite and satisfies $\kappa(G) \leq \chi(G)$, where $\kappa(G)$ is the compact covering number of G and $\chi(G)$ is the local weight of G (see [12, Corollary 4.2]). Versions for $L_1(G)$ of some results on $A(G)$ by Granirer [11] and the first author [12, 13] were obtained by Fong and the second author in [6], and also recently by Bouziad and Filali in [3]. In particular, it was shown that $L_1(G)$ is ENAR for all σ-compact non-compact locally compact groups G (cf. [6, Theorem 4.7] and [3, Theorem 4.4]).

The counterexample $A(SU(3))$ to the strong Arens irregularity problem of Fourier algebras, presented by Losert, gives a Banach algebra which is ENAR but not SAI. The question of whether there exists an SAI and Arens irregular Banach algebra which fails to be ENAR was raised by the authors in [16] (see also Bouziad-Filali [3]).

In this note, we present a number of examples of Arens irregular commutative Banach algebras. These examples illustrate in particular that these two natural concepts of "maximal" Arens irregularity for general Banach algebras are indeed distinct (cf. Theorems 4 and 5). Thereby, the open question on the relation between strong Arens irregularity and extreme non-Arens regularity is answered, as conjectured by the authors in [16]. We also link these two properties to yet another Arens irregularity property, which will be introduced in the next section.

2. Banach algebras with weak factorization

Let A be a Banach algebra. Let $\langle A^*A \rangle$ and $\langle A^*A^{**} \rangle$ denote the closed linear spans of the module products $A^* \cdot A$ and $A^* \diamondsuit A^{**}$, respectively. Then $\langle A^*A \rangle \subseteq \langle A^*A^{**} \rangle \subseteq A^*$. By Cohen's factorization theorem, $\langle A^*A \rangle = A^* \cdot A$ if A has a

bounded approximate identity (BAI). In this situation, A^* is said to factor on the left if $A^* \cdot A = A^*$ (see Lau-Ülger [22]). In [22, Theorem 2.6] Lau and Ülger proved that if A is weakly sequentially complete with a sequential BAI, then A^* factors on the left if and only if A is unital. This result of Lau and Ülger has been extended recently by the authors and Ruan to a larger class of Banach algebras, which includes all convolution Beurling algebras $L_1(G, \omega)$ with $\omega \geq 1$ and Fourier algebras $A(G)$ with G amenable (see [17, 18]).

Definition 1. Let A be a Banach algebra and let X be a right Banach A-module. X is said to factor over A on the left if $\langle X \cdot A \rangle = X$. When $X = A^*$ has its canonical A-module structure, the term "over A" is omitted. When X is a symmetric Banach A-module, the term "on the left" is omitted.

If A is commutative, $A^{**} \square A^* = A^* \diamondsuit A^{**}$. In this case, we say that A has WF (weak factorization) if A^* factors over A^{**} (i.e., $\langle A^* A^{**} \rangle = A^*$), and A has PWF (proper weak factorization) if A^* factors over A^{**} but not over A (i.e., $\langle A^* A \rangle \subsetneq \langle A^* A^{**} \rangle = A^*$).

Let G be a locally compact group. If $A = L_1(G)$, then $\langle A^* A \rangle$ is the C^*-algebra $LUC(G)$ of left uniformly continuous functions on G; hence $L_1(G)^*$ factors on the left if and only if G is discrete. If $A = A(G)$, then $\langle A^* A \rangle$ is the C^*-algebra $UCB(\widehat{G})$ of uniformly continuous linear functionals on $A(G)$ (see Granirer [9]). By [10, Theorem 3], $A(G)^*$ factors if and only if G is compact.

In the rest of this section, A will denote a *commutative* Banach algebra with a faithful multiplication. In this situation, $Z_t^{(l)}(A^{**}) = Z_t^{(r)}(A^{**})$ is just the algebraic centre $Z(A^{**})$ of A^{**}, and A^* is both an A-module and an A^{**}-module, where A^{**} is equipped with either of the Arens products.

Obviously, if A has a BAI, then $\langle A^* A^{**} \rangle = A^*$. Hence, when considering Banach algebras with WF or PWF, we are most interested in Banach algebras which do not have any BAI.

It turns out that the two extreme cases in the inclusions $\langle A^* A \rangle \subseteq \langle A^* A^{**} \rangle \subseteq A^*$ are related to Arens (ir)regularity of A. In fact, it was shown by the first author [15, Lemma 2.2(iii)] that for all commutative Banach algebras A, $Z(A^{**}) \cap \langle A^* A \rangle^\perp = \langle A^* A^{**} \rangle^\perp$. Thus, we have

(1) $\qquad\qquad A$ is Arens regular $\implies \langle A^* A \rangle = \langle A^* A^{**} \rangle$

and

(2) $\qquad\qquad A$ is SAI $\implies \langle A^* A^{**} \rangle = A^*$.

It follows from (1) that

(3) $\qquad\qquad A$ has PWF $\implies A$ is Arens irregular and non-unital.

Therefore, having PWF indeed describes an Arens irregularity property for non-unital commutative Banach algebras. It will be seen in Section 3 that this property is also distinct from strong Arens irregularity and extreme non-Arens regularity.

In general, the converses of (1) and (2) are not true, since if A is unital, then $\langle A^* A \rangle = \langle A^* A^{**} \rangle = A^*$, but A may be neither Arens regular nor SAI. However, if A is semisimple, weakly sequentially complete, and completely continuous with A^* a von Neumann algebra, Ülger [28, Theorem 3.3(e)] shows that the converse of (1) holds. Therefore, for all discrete groups G,

(4) $\qquad\qquad A(G)$ is Arens regular $\iff \langle A(G)^* A(G)^{**} \rangle = UCB(\widehat{G})$.

In Section 3, we will construct a "non-trivial" counterexample to the converse of (2), i.e., a Banach algebra which does not have a BAI and for which the converse of (2) fails (Theorem 5).

On the other hand, the first author proved that for all locally compact groups G, $A(G)$ is SAI if and only if $A(G)$ has WF and $A(H)$ is SAI for every σ-compact open subgroup H of G (see [15, Theorem 3.9]). For discrete groups G containing the free group F_r on r generators with $2 \leq r < \infty$, beyond showing that $A(G)$ is not SAI, Losert [24, Corollary 1] proved that $A(G)$ does not even have WF (cf. (2)). Note that $A(F_2)$ is non-unital and Arens irregular (see e.g. Forrest [7, Corollary 3.8]). Therefore, the converse of (3) does not hold in general.

We close this section by recalling that $UCB(\widehat{G}) \subsetneqq A(G)^*$ if G is not compact (see Granirer [10, Theorem 3]). Therefore, for all non-compact locally compact groups G, we have

$$(5) \qquad\qquad A(G) \text{ is SAI} \Longrightarrow A(G) \text{ has PWF}.$$

In particular, we have

$$(6) \qquad L_1(G) \text{ has PWF for all non-discrete locally compact abelian groups } G,$$

since $L_1(G)$ is always SAI.

3. Examples of Arens irregular Banach algebras

In this section, we construct a number of commutative Banach algebras which have various Arens irregularity properties. Most of these Banach algebras are obtained from some Fourier algebras whose Arens irregularity properties are known.

Let A and B be commutative Banach algebras and let $C = A \oplus_1 B$. It can be seen that

$$C^* = A^* \oplus_\infty B^*, \quad WAP(C) = WAP(A) \oplus_\infty WAP(B),$$
$$\langle C^*C \rangle = \langle A^*A \rangle \oplus_\infty \langle B^*B \rangle, \quad \langle C^*C^{**} \rangle = \langle A^*A^{**} \rangle \oplus_\infty \langle B^*B^{**} \rangle,$$
$$C^{**} = A^{**} \oplus_1 B^{**}, \text{ and } Z(C^{**}) = Z(A^{**}) \oplus_1 Z(B^{**}).$$

It can also be seen that C is Arens regular (respectively, SAI) if and only if both A and B are Arens regular (respectively, SAI), and C has PWF if and only if both A and B have WF and at least one of A and B has PWF. These facts will be frequently used in what follows.

For convenience, we record below some known results which will be used throughout this section.

Theorem 2. *Let G be a locally compact group.*
  (I) *$L_1(G)$ is SAI.*
  (II) *If $A(G)$ is Arens regular, then G is discrete.*
  (III) *If G is σ-compact and non-discrete, then $A(G)$ is ENAR.*
  (IV) *$A(SU(3))$ is not SAI.*

The reader is referred to Lau-Losert [19], Forrest [7], Granirer [11] and Hu [12], and Losert [25], respectively, for Theorem 2(I) - (IV) and related results.

The Banach algebra given below will play an important role in the constructions of examples. We define it as follows:

Definition 3. Let $l_2(2^{\mathbb{R}})$ be the usual Banach space with pointwise multiplication. Let $\mathcal{Y}$ be the unitization of $l_2(2^{\mathbb{R}})$. Then $\mathcal{Y}$ is a unital commutative Banach algebra and hence $\mathcal{Y}$ has WF but not PWF. Since $\mathcal{Y}$ is reflexive, $\mathcal{Y}$ is both Arens regular and SAI. In particular, $WAP(\mathcal{Y}) = \mathcal{Y}^*$. Also, we have $|\mathcal{Y}^*| \geq |2^{\mathbb{R}}| > 2^{\aleph_0}$.

We start with some examples of unital commutative Banach algebras showing all possible combinations of Arens irregularities.

Theorem 4. *There exist unital Arens irregular commutative Banach algebras A_1, B_1, C_1, and D_1 such that*
    (i) *A_1 is SAI but not ENAR,*
    (ii) *B_1 is ENAR but not SAI,*
    (iii) *C_1 is neither SAI nor ENAR,*
    (iv) *D_1 is both SAI and ENAR.*

Proof. (i) Let $A_1 = A(\mathbb{T}) \oplus_1 \mathcal{Y}$, where $\mathcal{Y}$ is the Banach algebra given in Definition 3. Then A_1 is a unital commutative Banach algebra. Since both $A(\mathbb{T})$ and $\mathcal{Y}$ are SAI and $A(\mathbb{T})$ is Arens irregular, A_1 is SAI but not Arens regular.
    Note that $\mathcal{Y}^* = WAP(\mathcal{Y}) \subseteq WAP(A_1)$. We have

$$|A_1^*/WAP(A_1)| \leq |A_1^*/\mathcal{Y}^*| = |A(\mathbb{T})^*| = |l_\infty(\mathbb{Z})| = 2^{\aleph_0} < |\mathcal{Y}^*| \leq |A_1^*|,$$

i.e., $|A_1^*/WAP(A_1)| < |A_1^*|$. By the definition of extreme non-Arens regularity, A_1 cannot be ENAR.
    (ii) Let $B_1 = A(SU(3))$. Then B_1 is a unital Arens irregular commutative Banach algebra which is ENAR but not SAI (by Theorem 2(II) - (IV)).
    (iii) Let $C_1 = A(SU(3)) \oplus_1 \mathcal{Y}$, where $\mathcal{Y}$ is the same as above. Then C_1 is a unital commutative Banach algebra which is neither Arens regular nor SAI, since $A(SU(3))$ is Arens irregular and not SAI.
    Since $A(SU(3))$ is separable, $|A(SU(3))^*| \leq 2^{\aleph_0}$. By similar arguments as used in the proof of (i), we have $|C_1^*/WAP(C_1)| \leq |A(SU(3))^*| \leq 2^{\aleph_0} < |\mathcal{Y}^*| \leq |C_1^*|$. Therefore, C_1 is not ENAR.
    (iv) Let $D_1 = A(\mathbb{T})$. Then D_2 is a unital Arens irregular commutative Banach algebra which is both SAI and ENAR (by Theorem 2(I) - (III)). $\qquad\square$

For any locally compact group G, $WAP(\widehat{G})$ will denote $WAP(A(G))$. When G is abelian with dual group $\widehat{G}$, $WAP(A(G))$ is precisely the space of weakly almost periodic functions on $\widehat{G}$.
    By (6), $L_1(\mathbb{R})$ has PWF. Like $A(\mathbb{T})$, $L_1(\mathbb{R})$ ($\cong A(\mathbb{R})$) is both SAI and ENAR (by Theorem 2(I) and (III)). Note that $A(\mathbb{T})$ is unital and $L_1(\mathbb{R})$ has a BAI. In the following, we give more examples of commutative Banach algebras with different combinations of Arens irregularities such that each of them has PWF and can be constructed without possessing any BAI.

Theorem 5. *There exist commutative Banach algebras A_2, B_2, C_2, and D_2 which have PWF (and hence are Arens irregular) and are such that*
    (i) *A_2 is SAI but not ENAR,*
    (ii) *B_2 is ENAR but not SAI,*
    (iii) *C_2 is neither SAI nor ENAR,*
    (iv) *D_2 is both SAI and ENAR.*
Moreover, such Banach algebras A_2, B_2, C_2, and D_2 can be constructed with or without a BAI.

Proof. (i) Let $A_2 = L_1(\mathbb{T}) \oplus_1 \mathcal{Y}$, where $\mathcal{Y}$ is the Banach algebra defined in Definition 3. A similar argument as used in the proof of Theorem 4(i) shows that A_2 is SAI but not ENAR. Clearly, both $L_1(\mathbb{T})$ and $\mathcal{Y}$ have WF. By (6), $L_1(\mathbb{T})$ has PWF. Therefore, $A_2 = L_1(\mathbb{T}) \oplus_1 \mathcal{Y}$ has PWF.

(ii) Let G be a locally compact abelian group satisfying $\chi(G) > \kappa(G) = \aleph_0$ (e.g., take G to be the product of $\mathbb{Z}$ and a non-metrizable compact abelian group). Let $B_2 = A(SU(3)) \oplus_1 A(G)$. Then B_2 is not SAI because $A(SU(3))$ is not SAI.

However, since $A(SU(3))$ is unital and $A(G)$ is SAI, they both have WF. By (5), $A(G)$ has PWF. Therefore, $B_2 = A(SU(3)) \oplus_1 A(G)$ has PWF.

We show below that B_2 is ENAR. For a Banach space M, let $\mathcal{D}(M)$ denote the density character of M. By [14, Theorem A3], we have

$$\mathcal{D}(B_2) = \max\{\mathcal{D}(A(SU(3))), \mathcal{D}(A(G))\} = \max\{\aleph_0, \chi(G)\} = \chi(G).$$

Note that $B_2^* = A(SU(3))^* \oplus_\infty A(G)^*$ and $WAP(B_2) = WAP(\widehat{SU(3)}) \oplus_\infty WAP(\widehat{G})$. It can be seen that $A(G)^*/WAP(\widehat{G})$ is the image of $B_2^*/WAP(B_2)$ under the linear contraction $B_2^*/WAP(B_2) \longrightarrow A(G)^*/WAP(\widehat{G}), (S,T)+WAP(B_2) \longmapsto T+WAP(\widehat{G})$. By [12, Theorem 4.1] (the case $J = \{0\}$) and its proof, we see that there exists a bounded linear map from $A(G)^*/WAP(\widehat{G})$ onto $l_\infty(\chi(G))$. Thus, there exists a bounded linear map from $B_2^*/WAP(B_2)$ onto $l_\infty(\mathcal{D}(B_2))$ $(= l_\infty(\chi(G)))$. It follows from [12, Lemma 2.1] that B_2 is ENAR.

(iii) Let $C_2 = A(SU(3)) \oplus_1 L_1(\mathbb{R}) \oplus_1 \mathcal{Y}$, where $\mathcal{Y}$ is the same as above. Then C_2 is not SAI because $A(SU(3))$ is not SAI. Since both $A(SU(3))$ and $L_1(\mathbb{R})$ are separable, $|A(SU(3))^* \oplus_\infty L_1(\mathbb{R})^*| \leq 2^{\aleph_0}$. Following the proof of Theorem 4(i), we have

$$|C_2^*/WAP(C_2)| \leq |A(SU(3))^* \oplus_\infty L_1(\mathbb{R})^*| \leq 2^{\aleph_0} < |\mathcal{Y}^*| \leq |C_2^*|,$$

and thus C_2 is not ENAR.

Since $A(SU(3))$ is unital and $L_1(\mathbb{R}) \oplus_1 \mathcal{Y}$ is SAI, both $A(SU(3))$ and $L_1(\mathbb{R}) \oplus_1 \mathcal{Y}$ have WF. Note that $L_1(\mathbb{R})$ has PWF due to (6). Therefore, $C_2 = A(SU(3)) \oplus_1 L_1(\mathbb{R}) \oplus_1 \mathcal{Y}$ has PWF.

(iv) Let $D_2 = A(G)$, where G is the group as in (ii) above. Then D_2 is both SAI and ENAR and hence, by (5), D_2 has PWF.

Clearly, each of A_2, B_2, C_2, and D_2 constructed above has a BAI.

The last assertion of the theorem follows if we replace the Banach algebras constructed above by $A_2 \oplus_1 l_2$, $B_2 \oplus_1 l_2$, $C_2 \oplus_1 l_2$, and $D_2 \oplus_1 l_2$, respectively, where the Banach algebra l_2 with pointwise multiplication does not have any BAI. $\square$

Let A be a commutative Banach algebra. Recall assertions (1) and (2) from Section 2. We see that if

$$(7) \qquad\qquad \langle A^*A \rangle \subsetneqq \langle A^*A^{**} \rangle \subsetneqq A^*,$$

then A satisfies

$$(8) \qquad\qquad A \subsetneqq Z(A^{**}) \subsetneqq A^{**},$$

and A does not have PWF.

We shall give two more examples below showing that commutative Banach algebras can be constructed to satisfy (7) (and hence not being SAI or having PWF) with or without being ENAR.

Theorem 6. *There exist commutative Banach algebras A_3 and B_3 which satisfy* (7) *and are such that*

  (i) A_3 *is ENAR;*

  (ii) B_3 *is not ENAR.*

Proof. (i) Let G be a non-metrizable compact group, and let $A_3 = A(F_2) \oplus_1 A(G)$. Then A_3 is a commutative Banach algebra which satisfies (7), since so does $A(F_2)$. Following the argument of the last paragraph in the proof of Theorem 5(ii), one sees that A_3 is ENAR.

  (ii) Let $B_3 = A(F_2) \oplus_1 \mathcal{Y}$, where $\mathcal{Y}$ is the Banach algebra given in Definition 3. Then B_3 satisfies condition (7). By the arguments used in the proof of Theorem 4(i) again, one can derive that B_3 is not ENAR.   □

We recall that (7) holds for $A(F_2)$ (see Section 2). However, the Banach algebras B_1, C_1, B_2 and C_2 in Theorems 4 and 5 illustrate that, in general, we do not have (8) $\Longrightarrow$ (7). In [5, Chapter 9], Dales and Lau constructed numerous convolution Beurling algebras $l_1(G, \omega)$ over discrete abelian groups satisfying (8). We note that these Beurling algebras are unital, and hence (7) fails automatically. The reader is referred to [5] for some natural examples of non-commutative Banach algebras A whose two topological centres are strictly between A and A^{**}.

Due to the fact that $A(F_2)$ satisfies (7), the following question was raised by the first author in [15]:

Problem 1. Does $\langle A(G)^* A(G)^{**} \rangle = A(G)^*$ imply that G is amenable?

By (2), a positive answer to Problem 1 would also give a positive answer to the following

Problem 2. Does strong Arens irregularity of $A(G)$ imply that G is amenable?

It is known that a locally compact group G is amenable if and only if $A(G)$ has a BAI (see Leptin [24]). Our Theorem 5 shows that there *do* exist commutative Banach algebras closely related to Fourier algebras which are SAI and have PWF but do not have any BAI. While we do not know the answers to Problems 1 and 2, this seems to indicate that they are both negative.

In [20, Remark 6.6(a)], Lau and Losert asked whether $\langle A(G)^* B_\rho(G) \rangle = A(G)^*$ is true when G is not amenable, where $B_\rho(G)$ is the reduced Fourier-Stieltjes algebra of G. Note that $\langle A(G)^* B_\rho(G) \rangle \subseteq \langle A(G)^* A(G)^{**} \rangle$ and the equality holds when G is discrete (see Hu [15, Corollary 3.1]). As shown by Losert [24], F_2 gives an example of a non-amenable group G for which $A(G)^*$ does not factor over $B_\rho(G)$. This leads naturally to the question below, which coincides with Problem 1 when G is discrete.

Problem 3. If $A(G)^*$ factors over $B_\rho(G)$, is G amenable?

Note that for all discrete groups G,

$$A(G) \oplus \langle A(G)^* B_\rho(G) \rangle^\perp \subseteq Z(A(G)^{**})$$

by [15, Corollary 3.1]. Comparing with Ülger [28, Theorem 3.3(e)] for the Fourier algebra case (cf. (4)), we may consider the following.

Problem 4. For a (non-amenable) discrete group G, if $A(G)^*$ factors over $B_\rho(G)$, is $A(G)$ SAI?

We note that an affirmative answer to Problem 1 would also yield affirmative answers to Problems 3 and 4. The reader is referred to Lau-Losert [21], Losert [24], and Ülger [28] for more discussions on problems relating to topological centres of Fourier algebras.

We feel it is still a very interesting question to exhibit a general class of Banach algebras (containing such natural Banach algebras as $L_1(\mathbb{R})$) which are both SAI and ENAR.

Let $H^1(\mathbb{T}) = \{f \in L_1(\mathbb{T}) : \widehat{f}(-n) = 0 \text{ for all } n \in \mathbb{N}\}$. Then $H^1(\mathbb{T})$ is a closed ideal in $L_1(\mathbb{T})$. Among other results proved by Ülger [29] on Riesz sets in locally compact abelian groups, [29, Corollary 2.5] shows that $H^1(\mathbb{T})$ is Arens regular. This result was also proved by Graham in [8]. This example tells us in particular that an SAI and Arens irregular Banach algebra can have an infinite dimensional closed ideal which is Arens regular. See also Auger [2, Theorem 4.1.1] for Banach algebras with this property.

We conclude this paper by showing that it is possible for a commutative Banach algebra to contain an infinite dimensional Arens regular closed ideal while possessing all three Arens irregularity properties as considered in the present paper.

Proposition 7. *There exists an ENAR and SAI commutative Banach algebra A with PWF such that A has an infinite dimensional closed ideal which is Arens regular.*

Proof. Let G be a locally compact abelian group satisfying $\chi(G) > \kappa(G) = \aleph_0$. Let $A = A(G) \oplus_1 l_2$. Then A is both ENAR and SAI, and it has PWF (see the proof of Theorem 5). Clearly, l_2 is an infinite dimensional closed ideal in A and is Arens regular. $\qquad\square$

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