PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 137, Number 5, May 2009, Pages 1569–1577
S 0002-9939(08)09681-0
Article electronically published on December 11, 2008

RATIONALITY OF THE FOLSOM-ONO GRID

P. GUERZHOY

(Communicated by Ken Ono)

Abstract. In a recent paper Folsom and Ono constructed a grid of Poincaré series of weights 3/2 and 1/2. They conjectured that the coefficients of the holomorphic parts of these series are rational integers. We prove that these coefficients are indeed rational numbers with bounded denominators.

1. Introduction

In a recent paper [9] Folsom and Ono constructed two collections of Poincaré series of weights 1/2 and 3/2, enumerated respectively by non-positive integers and negative integers. These series are weak harmonic Maass forms and weakly holomorphic modular forms, respectively, and the q-expansion coefficients of their holomorphic parts satisfy a surprising duality identity. This fact constitutes the principal result of [9] and confirms that the families of Poincaré series constructed by Folsom and Ono are very natural and interesting. This duality result is not isolated: similar grids were recently constructed in various settings (e.g. [15, 3, 8, 11]). The setting considered by Folsom and Ono in [9] is of special interest, because the holomorphic part of their weight 1/2 Poincaré series which corresponds to $m = 0$ coincides essentially with the Ramanujan mock theta function

$$f(q) := 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}. \tag{1}$$

This function obviously has integer q-expansion coefficients; moreover, computer calculations suggest that the phenomenon holds for other integers m. Based on this observation, Folsom and Ono conjectured in [9] that all the q-expansion coefficients involved are rational integers. As a clue to how a somewhat weaker statement may be proved, they suggested (see Remark 3 to Theorem 1.1 in [9]) employing the Galois action on $M^!_{3/2}\left(144, \left(\frac{12}{\cdot}\right)\right)$ and the fact that $S_{3/2}\left(144, \left(\frac{12}{\cdot}\right)\right) = 0$. Unfortunately this hint contains a typo (in fact, $\dim S_{3/2}\left(144, \left(\frac{12}{\cdot}\right)\right) = 2$; see [7]) which seemingly completely compromises it. Nonetheless, the general idea is correct and powerful enough. In this paper we develop this idea and make use of the emptiness of the space $S_{1/2}\left((144, \left(\frac{12}{\cdot}\right)\right)$ along with the one-dimensionality of the space $S_{3/2}\left(36, \left(\frac{12}{\cdot}\right)\right)$ in order to prove the following result.

Received by the editors June 23, 2008, and, in revised form, June 28, 2008.
2000 *Mathematics Subject Classification.* Primary 11F37.
The author was supported by NSF grant DMS-0700933.

Theorem 1. *For an integer $m \leq 0$, let*

$$H^+_{0,m} = q^{-1/24}\left(q^m + \sum_{n\geq 0} a_m(n)q^n\right)$$

be the q-expansion of the holomorphic part of the weight $1/2$ Poincaré series constructed by Folsom and Ono. Then all numbers $a_m(n)$ are rational.

Remark 1. The question concerning rationality (algebraicity) of the Fourier coefficients of holomorphic parts of weak harmonic Maass forms may be deep and subtle. In particular, the proof of the rationality in a certain integral weight situation [6] is quite non-trivial. On the other hand, conjectural irrationality in another integral weight setting would imply the opposite side famous Lehmer's conjecture [13]. Most interestingly, in [5] the rationality of particular coefficients in a half-integral weight setting allowed Bruinier and Ono to conclude that a certain elliptic curve has infinite rank and to interpret the rational coefficients in terms of the central derivatives of the associated L-function.

Recall that Folsom and Ono conjectured in [9] that the numbers $a_m(n)$ are in fact rational *integers*. In this direction, we prove the following statement.

Corollary 1. *For every $m \leq 0$ the denominators of $a_m(n)$ are bounded independently of n.*

Remark 2. As was recently communicated to the author, Sander Zwegers has succeeded in using the results of the present paper to prove that all coefficients $a_m(n)$ are indeed rational integers. He does so by means of an explicit construction, which, in particular, provides an alternative and independent way of calculating these coefficients. The coefficients calculated by Zwegers agree totally with those calculated by Folsom and Ono. The paper [9] contains a typo. The author thanks Ken Ono for communicating a correct version of the numerics, which we display below (in our notation):

$$\begin{aligned}
H^+_{0,0} &= q^{-1/24}\left(1 + q - 2q^2 + 3q^3 - 3q^4 + \ldots + 486q^{47} + \ldots\right)\\
H^+_{0,-1} &= q^{-1/24}\left(q^{-1} - 263q + 2781q^2 - 17960q^3 + \ldots\right)\\
H^+_{0,-2} &= q^{-1/24}\left(q^{-2} + 3400q - 102060q^2 + \ldots\right)\\
H^+_{0,-3} &= q^{-1/24}\left(q^{-3} - 23374q + \ldots\right).
\end{aligned}$$

We review some facts on vector-valued weak harmonic Maass forms in Section 2, and the construction of vector-valued Maass-Poincaré series in Section 3. Section 4 is devoted to the proofs of Theorem 1 and Corollary 1.

2. Preliminaries on vector-valued weak harmonic Maass forms

In this section we introduce some notation and recall some basic facts on vector-valued weak harmonic Maass forms. We closely follow the exposition in [4] and refer to that paper for a detailed discussion. The only discrepancy between our notation and that of [4] is that we allow an arbitrary unitary representation ρ, not necessarily the Weil representation, as the Nebentypus.

Let $Mp_2(\mathbb{R})$ be the two-fold cover of $SL_2(\mathbb{R})$, realized by the two choices of the holomorphic square roots of $j(\gamma,\tau) = c\tau + d$, where $\Im\tau > 0$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. Recall that the elements of $Mp_2(\mathbb{R})$ are of the form $(\gamma, \phi(\tau))$ with $\gamma \in$

$SL_2(\mathbb{R})$ and $\phi(\tau)$ a holomorphic function such that $\phi(\tau)^2 = j(\gamma,\tau)$. For a complex number $z = |z|\exp(i\theta)$ with $\theta \in (-\pi,\pi]$ we assume $\sqrt{z} = z^{1/2} = |z|^{1/2}\exp(i\theta/2)$.

We denote by Γ the inverse image of $SL_2(\mathbb{Z})$ in $Mp_2(\mathbb{R})$ and denote by

$$\tilde{S} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau} \right) \quad \text{and} \quad \tilde{T} = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right)$$

the standard generators of Γ. Let ρ be an r-dimensional unitary representation of Γ. Let $k \in \frac{1}{2}\mathbb{Z}$. We define a weak harmonic Maass form of weight k and Nebentypus ρ as a real-analytic function from the complex upper half-plane to $\mathbb{C}^r$ which satisfies the following conditions:

i) $f(\gamma\tau) = \phi(\tau)^{2k}\rho(\gamma,\phi)f(\tau)$ for all $(\gamma,\phi) \in \Gamma$.

ii) There is a $C > 0$ such that for any cusp $s \in \mathbb{Q} \cup \{\infty\}$ of $SL_2(\mathbb{Z})$ and $(\gamma,\phi) \in Mp_2(\mathbb{R})$ with $\gamma \in SL_2(\mathbb{Z})$ so that $\gamma\infty = s$, the function $f_s(\tau) = \phi^{-2k}\rho^{-1}(\gamma,\phi)f(\gamma,\tau)$ satisfies $f_s(\tau) = \mathcal{O}(\exp(Cv))$ as $v \to \infty$ (uniformly in u, where $\tau = u + iv$).

iii) $\Delta_k f(\tau) = 0$, where

$$\Delta = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

We denote the space of harmonic weak Maass forms of weight k and Nebentypus ρ by $H_{k,\rho}$. Many properties of our weak harmonic Maass form coincide with those described in [4, Section 3] with literally the same proofs since these proofs do not use any specifics about the Weil representation. In particular, any weak harmonic Maass form f of weight k (we think of f as a column vector of r functions) has a unique decomposition $f = f^+ + f^-$ into its holomorphic and non-holomorphic parts, which have Fourier expansions

$$f^+(\tau) = \sum_{n \in \mathbb{Q}} a^+(n)\exp(2\pi inu) \tag{2}$$

and

$$f^-(\tau) = a^-(0)v^{1-k} + \sum_{n \in \mathbb{Q},\ n \neq 0} a^-(n)H(2\pi nv)\exp(2\pi inu), \tag{3}$$

where

$$H(w) = e^{-w}\int_{-2w}^{\infty} e^{-t}t^{-k}dt.$$

All but finitely many coefficients $a^+(n)$ with $n < 0$ (resp. $a^-(n)$ for $n > 0$) in (2) (resp. (3)) vanish. If $f^- = 0$, then f is a vector-valued weakly holomorphic modular form of weight k and Nebentypus ρ. (Note that f may have fractional powers of q in its q-expansion.) We denote the space of these forms by $M^!_{k,\rho}$. We denote by $M_{k,\rho}$ and $S_{k,\rho}$ the subspaces of modular (bounded at cusps) and cusp (vanishing at cusps) forms.

The operator $\xi : H_{k,\rho} \to M^!_{2-k,\bar{\rho}}$ is defined by

$$\xi := -2iv^k\overline{\frac{\partial}{\partial\bar{\tau}}}.$$

Its action on $f \in H_{k,\rho}$ is given by

$$\xi(f) = \xi(f^-) = -2v^k\overline{(k-1)a^-(0) + \sum_{n \in \mathbb{Q},\ n \neq 0} a^-(n)(-4\pi n)^{1-k}\exp(2\pi n\bar{\tau})}. \tag{4}$$

The inverse image of $S_{2-k,\rho}$ under ξ is denoted by $H^+_{k,\rho}$.

The bilinear pairing between the spaces $S_{2-k,\bar{\rho}}$ and $H^+_{k,\rho}$ is defined by

$$\{g, f\} = \text{constant term of the } q\text{-series} \quad \langle g, f^+\rangle.$$

The argument in the proof of [4, Proposition 3.5] remains unaltered and proves

$$\{g, f\} = (g, \xi(f)) \tag{5}$$

with the Petersson scalar product in the right-hand side.

3. Vector-valued Maass-Poincaré series

In this section we present a construction of a collection of vector-valued weak harmonic Maass forms that makes use of Poincaré series. For an integer l, we use the notation $\zeta_l := \exp(2\pi i/l)$. In order to introduce the Poincaré series constructed by Folsom and Ono [9], we need some additional notation.

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2)$ with $c > 0$, put

$$\chi(\gamma) := \begin{cases} \zeta_{24}^{-b} & c = 0, \\ i^{-1/2}(-1)^{\frac{1}{2}(c+ad+1)} \exp(2\pi i(-\frac{a+d}{24c} - \frac{a}{4} + \frac{3dc}{8}))\omega^{-1}_{-d,c} & c > 0, \end{cases}$$

where

$$\omega_{d,k} := \exp(\pi i s(d,k)).$$

The Dedekind sum $s(d, k)$ is defined by

$$s(d,k) := \sum_{j \mod k} \left(\left(\frac{j}{k}\right)\right)\left(\left(\frac{dj}{k}\right)\right),$$

with

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & x \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & x \in \mathbb{Z}. \end{cases}$$

Following [9] we now define the weight 1/2 Poincaré series for $m \leq 0$ as

$$H_{0,m}(\tau) := \frac{2}{\sqrt{\pi}} \sum_{\gamma\in\Gamma_0(2)/\pm\Gamma_\infty} \chi(\gamma)^{-1}(c\tau + d)^{-1/2}\phi_{1/2,3/4}(m - 1/24, \gamma\tau),$$

where γ is represented by $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here the function $\phi_{\kappa,s}(r, \tau)$ is defined by the M-Whittaker function $M_{\nu,\mu}$ for $v \geq 0$:

$$\phi_{\kappa,s}(r, z) := (4\pi|r|v)^{-\kappa/2} \exp(2\pi iru)M_{-\kappa/2,s-1/2}(4\pi|r|v).$$

Along with $H_{0,m}$ we define the following two companion series (cf. [10]):

$$H_{1,m}(\tau) := \frac{2\sqrt{-i}}{\sqrt{\pi}} \sum_{\gamma\in\Gamma_0(2)/\pm\Gamma_\infty} \chi(\gamma)^{-1}(d\tau - c)^{-1/2}\phi_{1/2,3/4}(m - 1/24, \gamma S\tau)$$

and

$$H_{2,m}(\tau) := \frac{2\sqrt{-i}\zeta_3}{\sqrt{\pi}} \sum_{\gamma\in\Gamma_0(2)/\pm\Gamma_\infty} \chi(\gamma)^{-1}(d\tau - (d-c))^{-1/2}\phi_{1/2,3/4}(m-1/24, \gamma ST\tau),$$

where

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let ρ be the three-dimensional representation of Γ, defined on the generators by

$$\rho(\tilde{S}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \rho(\tilde{T}) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix}.$$

We collect all that we need about these Poincaré series for further applications in the following proposition.

Proposition 1. (i) *For every* $m \leq 0$,

$$H_m := (H_{0,m}, H_{1,m}, H_{2,m})^T \in H_{1/2,\rho}.$$

(ii) *We have the Fourier expansions*

$$\begin{aligned} H_{0,m}^+ &= q^{m-1/24} + \mathcal{O}(q^{23/24}) \\ H_{1,m}^+ &= \mathcal{O}(q^{1/3}) \\ H_{2,m}^+ &= \mathcal{O}(q^{1/3}). \end{aligned}$$

Sketch of the proof. The series $H_{1,m}$ and $H_{2,m}$ are constructed in such a way that condition i) of the definition of weak harmonic Maass forms of weight 1/2 with representation ρ is satisfied automatically (see [10] for a similar construction). After that, one computes the Fourier coefficients of all three series using Poisson summation (cf. [9, Lemma 3.3]) and then justifies the convergence of the coefficients of the expressions obtained for the Fourier coefficients. This argument is provided in full detail in [9] (Theorems 3.1 and 3.3) for the series $H_{0,m}$; the arguments for the series $H_{1,m}$ and $H_{2,m}$ are similar. As soon as the Fourier expansions are calculated, (ii) becomes clear, and it is easy to check that $H_m \in H_{1/2,\rho}$. We have to show further that the absolute values of the Fourier coefficients of $\xi(H_m)$ grow polynomially with n. In view of (4), this follows from a similar estimate for the Fourier coefficients of the non-holomorphic part of H_m. In the special case of $H_{0,0}$, this was done by Bringmann and Ono in [2, Corollary 4.2]. The general case is similar. The polynomial growth of these Fourier coefficients implies that $\xi(H_m) \in M_{3/2,\bar{\rho}}$. However, the transformation law with respect to $\tilde{T}$ shows that none of the three series can have a non-zero constant term. This observation implies that $\xi(H_m) \in S_{3/2,\bar{\rho}}$, completing the proof of the proposition. □

4. Proofs of Theorem 1 and Corollary 1

Proposition 2. *The space* $S_{3/2,\bar{\rho}}$ *is one-dimensional.*

Proof. Following Zwegers [16] we consider the theta functions

$$\begin{aligned} g_0(\tau) &= \sum_{n\in\mathbb{Z}} (-1)^n (n+1/3) \exp(3\pi i (n+1/3)^2 \tau) \\ g_1(\tau) &= -\sum_{n\in\mathbb{Z}} (n+1/6) \exp(3\pi i (n+1/6)^2 \tau) \\ g_2(\tau) &= \sum_{n\in\mathbb{Z}} (n+1/3) \exp(3\pi i (n+1/3)^2 \tau). \end{aligned}$$

The transformation law for these theta-functions, proved by Zwegers in [16], tells us that $\Phi := (g_1, g_0, -g_2)^T \in S_{3/2,\bar{\rho}}$. Note that

$$3g_2(6\tau) = \frac{1}{2} \sum_{n\in\mathbb{Z}} \left(\frac{-3}{n}\right) n q^{n^2} \in S_{3/2}\left(\Gamma_0(36), \left(\frac{12}{\cdot}\right)\right).$$

It is easy to check (see e.g. [7]) that $\dim S_{3/2}\left(\Gamma_0(36), \left(\frac{12}{\cdot}\right)\right) = 1$. At the same time, if $(h_1, h_0, -h_2) \in S_{3/2,\bar{\rho}}$, then, since the transformation law of all three elements is determined completely, we must have $h_2 = cg_2$ with a constant c. This immediately implies $h_0 = cg_0$ and $h_1 = cg_1$, which proves the proposition. □

Proposition 3. *For an integer $m \le 0$, let*

$$C_m = \begin{cases} \left(\frac{-6}{n}\right) n & \text{if } 1 - 24m = n^2 \text{ for an integer } n \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

The function $H_m - C_m H_0$ is a vector-valued weakly holomorphic modular form of weight $1/2$:

$$F_m := H_m - C_m H_0 \in M^!_{1/2,\rho}.$$

Proof. By construction (see Proposition 1) $F_m \in H_{1/2,\rho}$. Thus all we have to check is that $F_m^- = 0$, that is, $\xi(F_m) = 0$. We employ (5) in order to calculate $\xi(H_m)$. By Proposition 2 there is a $c_m \in \mathbb{C}$ such that $\xi(H_m) = c_m\Phi$. Note that

$$g_1(\tau) = -\frac{1}{6}\sum_{n\ge 1}\left(\frac{-6}{n}\right) n q^{n^2/24}$$

and that the constant term of the q-expansion of $g_0 H^+_{1,m} - g_2 H^+_{2,m}$ vanishes for every $m \le 0$. It follows that for all $m \le 0$,

$$\begin{aligned} c_m(\Phi,\Phi) = \{\Phi, H_m\} &= \text{constant term of } g_1 H^+_{0,m} \\ &= \begin{cases} -\frac{1}{6}\left(\frac{-6}{n}\right) n & \text{if} \quad m = \frac{1-n^2}{24} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus $c_m = C_m c_0$, and

$$\xi(F_m) = \xi(H_m - C_m H_0) = c_m\Phi - C_m c_0 \Phi = 0$$

as required. □

Since, by Proposition 3, $F_m(\tau) \in M^!_{1/2,\rho}$, we derive by an argument identical to the proof of [2, Corollary 2.3] that

$$F_{0,m}(24\tau) = H_{0,m}(24\tau) - C_m H_{0,0}(24\tau) \in M^!_{1/2}\left(\Gamma_0(144), \left(\frac{12}{\cdot}\right)\right).$$

Note also that $C_m H^+_{0,0}(24\tau)$ has rational integral Fourier coefficients since C_m is an integer by definition, and, by the prominent result of Bringmann and Ono [2], $H^+_{0,0} = q^{-1/24} f(q)$ where $f(q)$ is the Ramanujan mock theta-function (1) which manifestly has rational integral Fourier coefficients. The rationality of the Fourier coefficients of $H^+_{0,m}$ (claimed in Theorem 1) and the boundedness of their denominators are thus equivalent to the same properties of the Fourier coefficients of $F_{0,m}$, which we are going to prove now.

Proposition 1 implies that function $F_{0,m}(\tau)$, which has a pole at infinity, has a pole at every cusp c/d with $g.c.d.(c,d) = 1$ and $2|d$, and has a zero at all other cusps. It follows that the weakly holomorphic modular form $F_{0,m}(24\tau)$ has a pole at infinity and at every cusp c/d with $g.c.d.(c,d) = 1$ and $48|d$ and has a zero at all other cusps. Note that the principal part of the Fourier expansion of $F_{0,m}(24\tau)$ at infinity is a polynomial in q^{-1} with rational coefficients. Let

$$\eta(\tau) = q^{1/24} \prod_{n\geq 1}(1-q^n)$$

be Dedekind's eta-function. We now introduce two auxiliary eta-quotients (cf. [12, Theorem 1.64]):

$$\Phi(\tau) := \frac{\eta(8\tau)\eta(48\tau)^8\eta(72\tau)^3}{\eta(16\tau)^2\eta(24\tau)^4\eta(144\tau)^6} \in M_0\left(\Gamma_0(144), \xi_0\right),$$

where ξ_0 denotes the trivial character modulo 144, and

$$\Psi(\tau) := \frac{\eta(24\tau)\eta(144\tau)^6}{\eta(48\tau)^2\eta(72\tau)^3} \in M_1\left(\Gamma_0(144), \left(\frac{-3}{\cdot}\right)\right).$$

It follows from [12, Theorem 1.65] that the eta-quotient $\Phi(\tau)$ has a pole at infinity, a positive order of vanishing at the cusp $1/48$ (i.e. c/d with $g.c.d.(c,d)=1$ and $g.c.d.(144,d)=48$), and zero order of vanishing at all the other cusps of $\Gamma_0(144)$. Thus for a big enough positive integer A,

$$F_{0,m}(24\tau)\Phi(\tau)^A \in M^!_{1/2}\left(\Gamma_0(144), \left(\frac{12}{\cdot}\right)\right)$$

has no poles other than at infinity and has a positive order of vanishing at all the other cusps of $\Gamma_0(144)$. It follows from [12, Theorem 1.65] that the eta-quotient $\Psi(\tau)$ has a zero at infinity and zero order of vanishing at all the other cusps of $\Gamma_0(144)$. Thus for a big enough positive integer B,

$$f := F_{0,m}(24\tau)\Phi(\tau)^A\Psi(\tau)^{2B-1} \in S_{2B-1/2}\left(\Gamma_0(144), \left(\frac{-12}{\cdot}\right)\right).$$

We now need the following proposition, which is a consequence and half-integral weight analog of a result of Shimura [14, Theorem 3.52].

Proposition 4. *For positive integers B and N with $4|N$ and for a quadratic character ξ, the space $S_{2B-1/2}(\Gamma_0(4N),\xi)$ has a basis consisting of cusp forms for which the Fourier coefficients at infinity are rational integers.*

Remark 3. The author is grateful to Prof. Skoruppa for a hint on the proof of this proposition. As has been communicated to the author, the proposition in fact stays true in a more general setting. In particular, the condition $4|N$ is redundant, and the statement holds for any positive half-integral weight. In this general setting, however, the proof requires a way of constructing the half-integral weight modular form spaces which was recently invented by Skoruppa.

Proof. Using the classical properties of the theta functions

$$\theta_2(\tau) = \sum_{m\geq 0} q^{(2m+1)^2} \quad \text{and} \quad \theta_3(\tau) = 1 + 2\sum_{m>0} q^{m^2},$$

it is easy to show (see [1, Chapter 4.2] for details) that under the conditions of Proposition 4 the map

$$S_{2B-1/2}(\Gamma_0(4N),\xi) \to S_{2B}(\Gamma_0(4N),\xi) \times S_{2B}(\Gamma_0(4N),\xi)$$

taking f to the pair $(f\theta_2, f\theta_3)$ is an isomorphism of the space $S_{2B-1/2}(\Gamma_0(4N),\xi)$ to the subspace $U \subset S_{2B}(\Gamma_0(4N),\xi) \times S_{2B}(\Gamma_0(4N),\xi)$ which consists of pairs (f_2, f_3)

that satisfy the condition $f_2\theta_3 = f_3\theta_2$. It follows from [14, Theorem 3.52] that the space $S_{2B}(\Gamma_0(4N), \xi)$ admits a basis with the required properties. This fact implies the claim of Proposition 4. □

We are now in a position to finish the proof of Theorem 1. Let $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$, and denote by f^σ the series obtained by applying σ to the Fourier coefficients of f at infinity. It follows from Proposition 4 that $f^\sigma \in S_{2B-1/2}\left(\Gamma_0(144), \left(\frac{-12}{\cdot}\right)\right)$. Since the principal part of the Fourier expansion of $F_{0,m}(24\tau)$ (and therefore, by construction, of f) has rational coefficients and $\Psi(\tau)$ does not vanish at cusps other than infinity,

$$\frac{f - f^\sigma}{\Psi^{2B-1}}(\tau) = (F_{0,m}(24\tau) - F^\sigma_{0,m}(24\tau))\Phi(\tau)^A \in S_{1/2}\left(\Gamma_0(144), \left(\frac{12}{\cdot}\right)\right).$$

However, this space is empty: $\dim S_{1/2}\left(\Gamma_0(144), \left(\frac{12}{\cdot}\right)\right) = 0$, and it follows that $F_{0,m}(24\tau) = F^\sigma_{0,m}(24\tau)$ for every $\sigma \in Aut(\mathbb{C}/\mathbb{Q})$, that is, $F_{0,m}(24\tau) \in \mathbb{Q}[[q^{-1}, q]]$. Thus Theorem 1 is proved. □

Proof of Corollary 1. Since $F_{0,m}(24\tau)$ has rational Fourier coefficients, the same is true of f, and Proposition 4 implies that the denominators of the Fourier coefficients of f are bounded. Since $F_{0,m}(24\tau) = f(\tau)\Phi(\tau)^{-A}\Psi(\tau)^{1-2B}$ and the q-expansion

$$\Phi(\tau)^{-A}\Psi(\tau)^{1-2B} \in q^{16A+24(1-2B)}(1 + \mathbb{Z}[[q]]),$$

the denominators of the Fourier coefficients of $F_{0,m}$ are bounded for every $m \leq 0$. □

Acknowledgements

The author thanks Sander Zwegers for the idea of Proposition 3, which was the starting point of this project. The author thanks Nils-Peter Skoruppa for a hint on the proof of Proposition 4. The author is very grateful to Sander Zwegers and Ken Ono for verifying the numerical data and communicating this data to the author.

References

[1] Basmaji, Jacques, Em Algorithmus zur Berechnung von Hecke-Operatoren Anwendung auf modulare Kurven, Dissertation Essen (1996).

[2] Bringmann, Kathrin; Ono, Ken, The $f(q)$ mock theta function conjecture and partition ranks, Invent. Math. 165 (2006), no. 2, 243–266. MR2231957 (2007e:11127)

[3] Bringmann, Kathrin; Ono, Ken, Arithmetic properties of coefficients of half-integral weight Maass-Poincaré series, Math. Ann. 337 (2007), no. 3, 591–612. MR2274544 (2007m:11064)

[4] Bruinier, Jan Hendrik; Funke, Jens, On two geometric theta lifts, Duke Math. J. 125 (2004), no. 1, 45–90. MR2097357 (2005m:11089)

[5] Bruinier, Jan H.; Ono, Ken, Heegner divisors, L-functions and harmonic weak Maass forms, preprint.

[6] Bruinier, Jan H.; Ono, Ken; Rhoades, Robert C., Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues, Math. Ann. 342 (2008), no. 3, 673–693.

[7] Cohen, H.; Oesterlé, J., Dimensions des espaces de formes modulaires (French). Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), pp. 69–78. Lecture Notes in Math., Vol. 627, Springer, Berlin, 1977. MR0472703 (57:12396)

[8] Duke, W.; Jenkins, Paul, On the zeros and coefficients of certain weakly holomorphic modular forms, Pure Appl. Math. Q. 4 (2008), no. 4, part 1, 1327–1340.

[9] Folsom, Amanda; Ono, Ken, Duality involving the mock theta function $f(q)$, J. Lond. Math. Soc. (2) 77 (2008), no. 2, 320–334. MR2400394

[10] Garthwaite, Sharon Anne, Vector-valued Maass-Poincaré series, Proc. Amer. Math. Soc. 136 (2008), no. 2, 427–436. MR2358480

[11] Guerzhoy, P., On weak harmonic Maass-modular grids of even integral weights, Math. Res. Lett., to appear.

[12] Ono, Ken, The web of modularity: arithmetic of the coefficients of modular forms and q-series, CBMS Regional Conference Series in Mathematics, 102. Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 2004. MR2020489 (2005c:11053)

[13] Ono, Ken, A mock theta function for the Delta-function, Proceedings of the 2007 Integers Conference, accepted for publication.

[14] Shimura, Goro, Introduction to the arithmetic theory of automorphic functions. Reprint of the 1971 original. Publications of the Mathematical Society of Japan, 11. Kanã Memorial Lectures, 1. Princeton University Press, Princeton, NJ, 1994. MR1291394 (95e:11048)

[15] Zagier, Don, Traces of singular moduli, in: Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998), pp. 211–244. Int. Press Lect. Ser., 3, I, Int. Press, Somerville, MA, 2002. MR1977587 (2004h:11037)

[16] Zwegers, S. P., Mock θ-functions and real analytic modular forms (English summary). q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000), pp. 269–277. Contemp. Math., 291, Amer. Math. Soc., Providence, RI, 2001. MR1874536 (2003f:11061)

Department of Mathematics, University of Hawaii, 2565 McCarthy Mall, Honolulu, Hawaii 96822-2273

E-mail address: pavel@math.hawaii.edu