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POLYHEDRAL EMBEDDINGS OF SNARKS IN ORIENTABLE SURFACES

MARTIN KOCHOL

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Abstract. An embedding of a 3-regular graph in a surface is called *polyhedral* if its dual is a simple graph. An old graph-coloring conjecture is that every 3-regular graph with a polyhedral embedding in an orientable surface has a 3-edge-coloring. An affirmative solution of this problem would generalize the dual form of the Four Color Theorem to every orientable surface. In this paper we present a negative solution to the conjecture, showing that for each orientable surface of genus at least 5, there exist infinitely many 3-regular non-3-edge-colorable graphs with a polyhedral embedding in the surface.

1. Introduction

By Tait [10], a cubic (3-regular) planar graph is 3-edge-colorable if and only if its geometric dual is 4-colorable. Thus the dual form of the Four Color Theorem (see Appel and Haken [2]) is that every 2-edge-connected planar cubic graph has a 3-edge-coloring.

An embedding of a cubic graph in a surface is called *polyhedral* if its dual is a simple graph, i.e., has no multiple edges and loops. Denote by S_n the orientable surface of genus n (see, e.g., [5, 6] for formal definitions). One of the classical conjectures about graphs is that each 3-regular graph with a polyhedral embedding in an orientable surface has a 3-edge-coloring. This problem was introduced by Grünbaum [7] during a conference in 1968. A positive solution of this conjecture would generalize the dual form of the Four Color Theorem to every S_n, $n \geq 0$. The conjecture holds true for S_0, as follows from results of Tait [10] and Appel and Haken [2]. For some special classes of graphs it was verified in [1, 3, 4]. Note that an analogous property does not hold for nonorientable surfaces, because the Petersen graph has a polyhedral embedding in the projective plane.

In this paper we construct counterexamples to the conjecture by using superposition techniques we introduced in [8, 9]. In particular, for any S_n, $n \geq 5$, we construct infinitely many non-3-edge-colorable cubic graphs with a polyhedral embedding in S_n.

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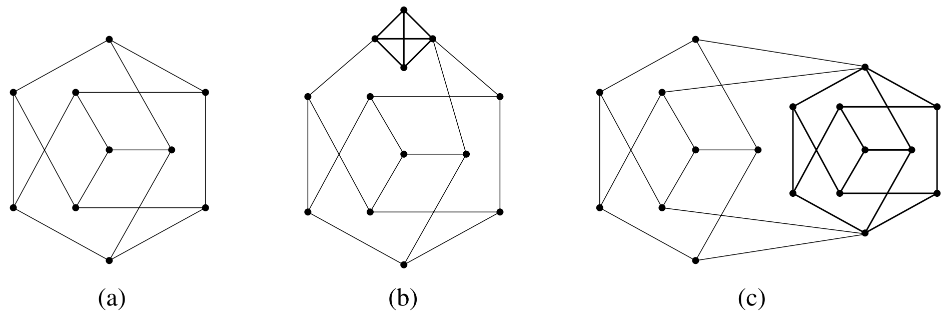

FIGURE 1

2. Snarks and superposition

Nontrivial cubic graphs without a 3-edge-coloring are called *snarks*. By nontrivial we mean cyclically 4-edge-connected and with girth (length of the shortest circuit) at least 5. Note that a graph is *cyclically k-edge-connected* if deleting fewer than k edges does not disconnect the graph into components so that at least two of these components have circuits. The best known snark is the Petersen graph (shown in part (a) of Figure 1). Snarks represent an important class of graphs, because the smallest counterexamples to many conjectures about graphs must be snarks (see, e.g., [8, 9]).

If G is a graph, then $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. If v is a vertex of G, then $\omega_G(v)$ denotes the set of edges having one end v and the other end from $V(G) \setminus \{v\}$. By a *nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow* φ in G we mean a mapping $\varphi : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\varphi(e) \neq 0$ for each edge e of G and $\partial\varphi(v) = \sum_{e \in \omega_G(v)} \varphi(e) = 0$ for each vertex v of G.

By a 4-*snark* we mean a graph without a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. It is well known that nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows in a cubic graph G correspond to 3-edge-colorings of G by nonzero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see, e.g., [9]). Thus snarks form a proper subclass of 4-snarks. In [9] we introduced a general method for constructing 4-snarks. It is based on the following two steps.

Suppose v is a vertex of a graph G and G' arises from G by the following process. Replace v by a graph H_v so that each edge e of G having one end v now has one end from H_v. If e is a loop and has both ends equal to v, then both ends of e will now be from H_v. Thus constructed, G' is called a *vertex superposition* of G.

Suppose e is an edge of G with ends u and v and G' arises from G by the following process. Replace e by a graph H_e having at least two vertices, i.e., we delete e; pick two distinct vertices u', v' of H_e; and identify u' with u and v' with v. Then G' is called an *edge superposition* of G. Furthermore, if H_e is a 4-snark, then G' is called a 4-*strong* edge superposition of G.

We say that a graph G' is a (4-*strong*) *superposition* of G if G' arises from G after finitely many vertex and (4-strong) edge superpositions. The following statement was proved in [9, Lemma 4.4].

Lemma 2.1. *Let G be a 4-strong superposition of a 4-snark. Then G is a 4-snark.*

In other words, if we take a 4-snark and replace some of its vertices by arbitrary graphs and some of its edges by 4-snarks, we get a new 4-snark. For example, in part (a) of Figure 1 we show the Petersen graph. In parts (b) and (c) of the figure we depict a vertex and an edge superposition of the Petersen graph, respectively. Since the edge superposition is 4-strong (an edge is replaced by another copy of the Petersen graph) and the Petersen graph is a snark, all graphs from Figure 1 are 4-snarks by Lemma 2.1.

For more details about superposition techniques we refer to [8, 9].

3. Constructions

In part (a) of Figure 2, we show an embedding of the Petersen graph in the torus. In order to describe our constructions, it is convenient to present this embedding in the form indicated in part (b). If we identify the opposite segments of the square in part (a), we get a torus. If we identify these segments in part (b), we get a handle on the plane. The two embeddings correspond to each other, the only difference being that face f_0 is infinite in part (b) and finite in part (a). The following properties hold true.

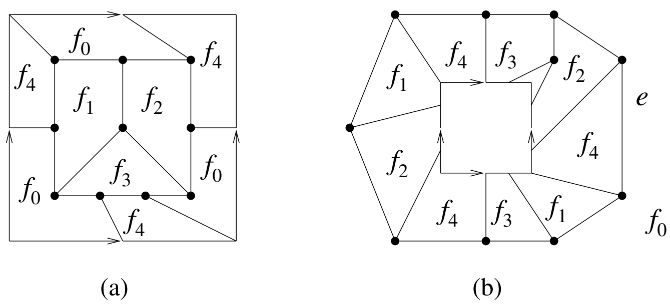

Figure 2

(1) Any two faces f_i and f_j, $i, j \in \{1, \ldots, 4\}$, share exactly one edge.
(2) Face f_0 shares exactly two edges with each f_i, $i \in \{1, 2, 3\}$, and exactly three edges with f_4.

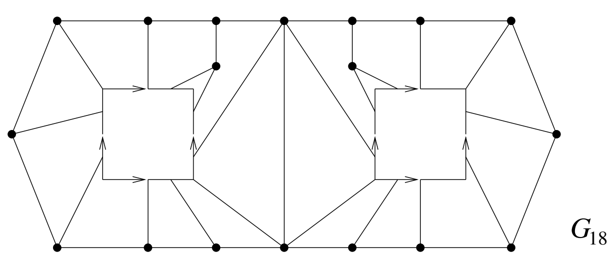

Figure 3

Replacing edge e by another copy of the Petersen graph, we get the graph G_{18} drawn in Figure 3. It is shown as an embedding in the plane with two handles. By Lemma 2.1, G_{18} is a 4-snark, because it is a 4-strong edge superposition of the Petersen graph. Replacing each vertex of degree 5 in G_{18} by a path of length 2, we get the graph G_{22} in Figure 4. By Lemma 2.1, G_{22} is a 4-snark. Since it is cubic, cyclically 4-edge-connected, and has girth 5, we can conclude that

(3) G_{22} is a snark.

The boundary of the infinite face f_0 is a circuit C, which is composed of two paths P_1 and P_2 with ends u and v. The faces f_i and f_{i+4}, $i = 1, \dots, 4$, indicated in Figure 4 are copies of the face f_i from Figure 2. Thus, using (1) and (2), we can prove the following properties.

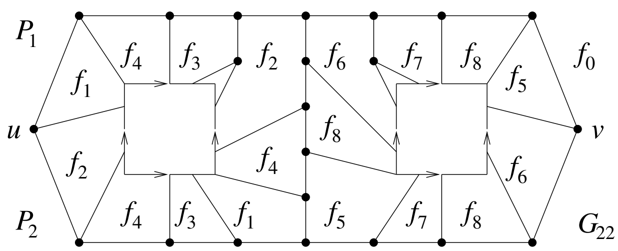

FIGURE 4

(4) Any two faces f_i and f_j, $i, j \in \{1, \dots, 8\}$, share at most one edge.
(5) Face f_0 shares exactly two edges with each f_i, $i \in \{1, \dots, 8\}$, so that P_1 and P_2 each contain exactly one of these edges.

By (4) and (5), the embedding of G_{22} into S_2 can be transformed into a polyhedral one if we divide f_0 by a new edge with ends u and v. Therefore, by taking a nonpolyhedral embedding of a snark in an orientable surface and replacing some of its edges by copies of G_{22} so that the ends of replaced edges are identified with copies of u and v, we can get a 4-snark with a polyhedral embedding in an orientable surface. In order to get a snark from this graph, we need to replace vertices of degrees higher than 4 by "suitable" graphs. We will use this idea in the proof of the following statement.

Theorem 3.1. *For every $n \geq 5$, there exist infinitely many snarks with a polyhedral embedding in S_n.*

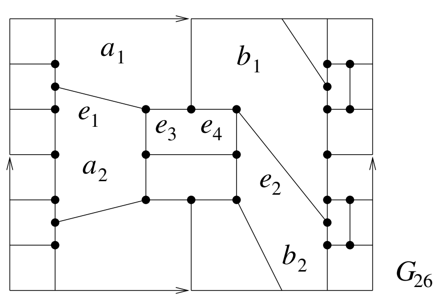

FIGURE 5

Proof. In Figure 5 is a snark G_{26} constructed by dot product (see, e.g., [9, Section 4]) from three copies of the Petersen graph. It was explicitly constructed in Belcastro and Kaminski [4, Figure 8]. The embedding in S_1 shown in the figure is not

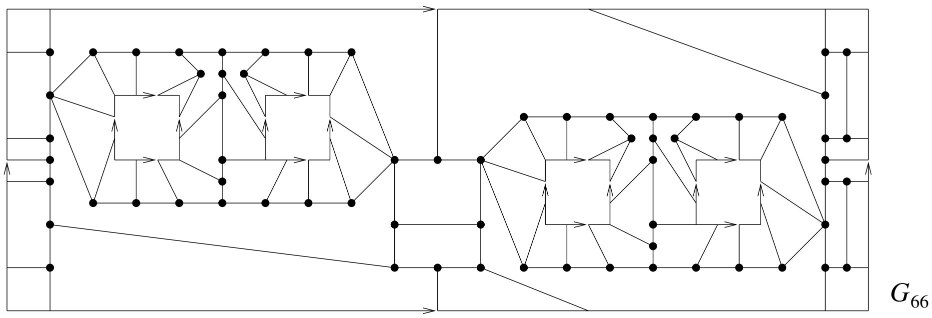

FIGURE 6

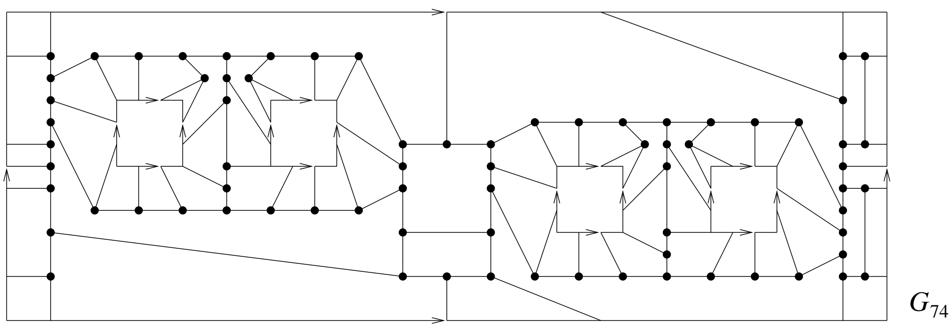

FIGURE 7

polyhedral, because the pairs of faces a_1, a_2 and b_1, b_2 have two edges in common (i.e., the geometric dual has two pairs of parallel edges). We apply 4-strong edge superpositions so that we replace edges e_1 and e_2 by copies of G_{22}, getting graph G_{66}, indicated in Figure 6. It is shown as an embedding in S_5, the surface arising from the torus after adding four handles.

Replacing all vertices of degree 5 in G_{66} by paths of order 3, we get the graph G_{74} in Figure 7. It has order 74 and, by Lemma 2.1, is a snark. Furthermore, by (4) and (5), any two faces of this graph have at most one edge in common (i.e., the pairs of faces a_1, a_2 and b_1, b_2 are "separated" by copies of graph G_{22}). Thus the geometric dual of G_{74} is a simple graph, i.e., G_{74} is a snark with a polyhedral embedding in S_5.

In G_{66}, replace vertices of degree 5 by paths or by graphs $H_{i,5}$, $i \geq 1$, such as those in Figure 8. Dashed lines indicate the edges incident in G_{66} with a vertex of degree 5. The resulting graphs are vertex superpositions of G_{66}; hence, by Lemma 2.1, they are snarks. In this way we get snarks of any even order ≥ 74 with a polyhedral embedding in S_5.

In order to get snarks with embeddings in S_n for $n > 5$, we replace some vertices of degree 5 in G_{66} by "suitable" graphs with polyhedral embeddings in S_n, $n \geq 1$. For example, we can use embeddings of the graphs shown in Figure 9. This proves the theorem. □

Theorem 3.2. *For every $n \geq 9$, there exist infinitely many cyclically 5-edge-connected snarks with a polyhedral embedding in S_n.*

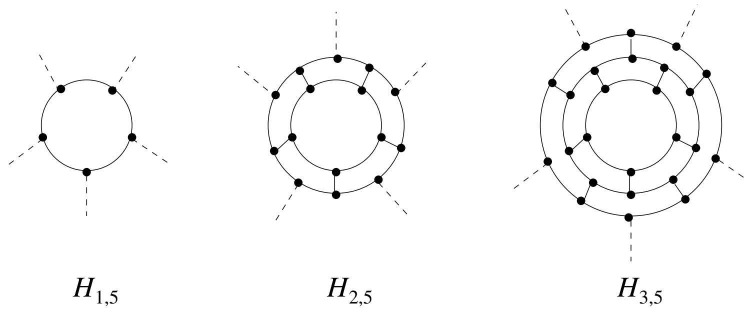

FIGURE 8

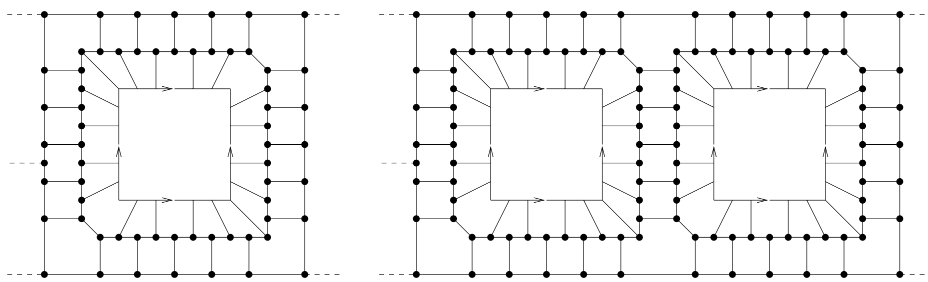

FIGURE 9

Proof. Replace edges $e_1, \dots, e_4$ of G_{26} by copies of G_{22} and then replace vertices of degrees 5 and 7 by paths of length 2 and 4, respectively. We get a cyclically 5-edge-connected snark of order 122 with a polyhedral embedding in S_9, indicated in Figure 10. Similarly as in the proof of Theorem 3.1, we can construct cyclically 5-edge-connected snarks of any even order ≥ 122 with a polyhedral embedding in S_9 and cyclically 5-edge-connected snarks with polyhedral embeddings in S_n for each $n \geq 9$. This proves the theorem. $\square$

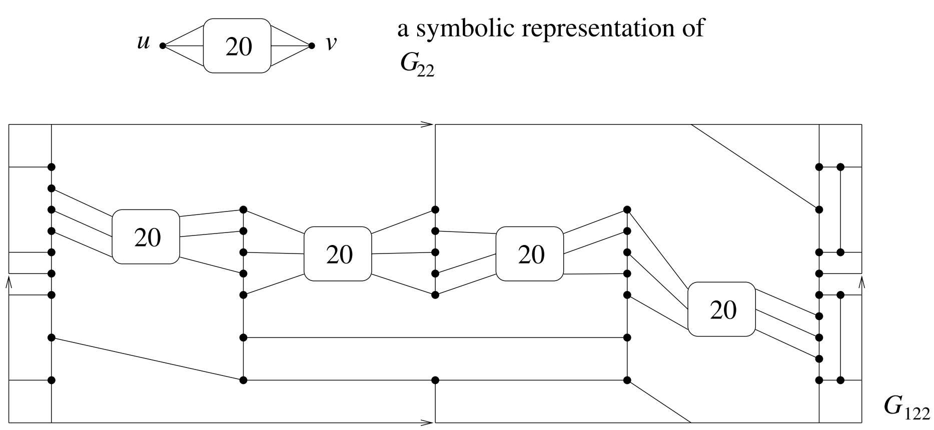

FIGURE 10

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Suché mýto 19, 811 03 Bratislava 1, Slovakia
E-mail address: kochol@savba.sk