POLYHEDRAL EMBEDDINGS OF SNARKS IN ORIENTABLE SURFACES

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Abstract. An embedding of a 3-regular graph in a surface is called polyhedral if its dual is a simple graph. An old graph-coloring conjecture is that every 3-regular graph with a polyhedral embedding in an orientable surface has a 3-edge-coloring. An affirmative solution of this problem would generalize the dual form of the Four Color Theorem to every orientable surface. In this paper we present a negative solution to the conjecture, showing that for each orientable surface of genus at least 5, there exist infinitely many 3-regular non-3-edge-colorable graphs with a polyhedral embedding in the surface.

1. Introduction

By Tait [10], a cubic (3-regular) planar graph is 3-edge-colorable if and only if its geometric dual is 4-colorable. Thus the dual form of the Four Color Theorem (see Appel and Haken [2]) is that every 2-edge-connected planar cubic graph has a 3-edge-coloring.

An embedding of a cubic graph in a surface is called polyhedral if its dual is a simple graph, i.e., has no multiple edges and loops. Denote by $S_n$ the orientable surface of genus $n$ (see, e.g., [5, 6] for formal definitions). One of the classical conjectures about graphs is that each 3-regular graph with a polyhedral embedding in an orientable surface has a 3-edge-coloring. This problem was introduced by Grünbaum [7] during a conference in 1968. A positive solution of this conjecture would generalize the dual form of the Four Color Theorem to every $S_n$, $n \geq 0$. The conjecture holds true for $S_0$, as follows from results of Tait [10] and Appel and Haken [2]. For some special classes of graphs it was verified in [1, 3, 4]. Note that an analogous property does not hold for nonorientable surfaces, because the Petersen graph has a polyhedral embedding in the projective plane.

In this paper we construct counterexamples to the conjecture by using superposition techniques we introduced in [8, 9]. In particular, for any $S_n$, $n \geq 5$, we construct infinitely many non-3-edge-colorable cubic graphs with a polyhedral embedding in $S_n$.

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Nontrivial cubic graphs without a 3-edge-coloring are called \textit{snarks}. By nontrivial we mean cyclically 4-edge-connected and with girth (length of the shortest circuit) at least 5. Note that a graph is \textit{cyclically $k$-edge-connected} if deleting fewer than $k$ edges does not disconnect the graph into components so that at least two of these components have circuits. The best known snark is the Petersen graph (shown in part (a) of Figure 1). Snarks represent an important class of graphs, because the smallest counterexamples to many conjectures about graphs must be snarks (see, e.g., \cite{8,9}).

If $G$ is a graph, then $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. If $v$ is a vertex of $G$, then $\omega_G(v)$ denotes the set of edges having one end $v$ and the other end from $V(G) \setminus \{v\}$. By a \textit{nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow} in $G$ we mean a mapping $\varphi : E(G) \to \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\varphi(e) \neq 0$ for each edge $e$ of $G$ and $\partial \varphi(v) = \sum_{e \in \omega_G(v)} \varphi(e) = 0$ for each vertex $v$ of $G$.

By a \textit{4-snark} we mean a graph without a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow. It is well known that nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows in a cubic graph $G$ correspond to 3-edge-colorings of $G$ by nonzero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (see, e.g., \cite{9}). Thus snarks form a proper subclass of 4-snarks. In \cite{9} we introduced a general method for constructing 4-snarks. It is based on the following two steps.

Suppose $v$ is a vertex of a graph $G$ and $G'$ arises from $G$ by the following process. Replace $v$ by a graph $H_v$ so that each edge $e$ of $G$ having one end $v$ now has one end from $H_v$. If $e$ is a loop and has both ends equal to $v$, then both ends of $e$ will now be from $H_v$. Thus constructed, $G'$ is called a \textit{vertex superposition} of $G$.

Suppose $e$ is an edge of $G$ with ends $u$ and $v$ and $G'$ arises from $G$ by the following process. Replace $e$ by a graph $H_e$ having at least two vertices, i.e., we delete $e$; pick two distinct vertices $u'$, $v'$ of $H_e$; and identify $u'$ with $u$ and $v'$ with $v$. Then $G'$ is called an \textit{edge superposition} of $G$. Furthermore, if $H_e$ is a 4-snark, then $G'$ is called a \textit{4-strong edge superposition} of $G$.

We say that a graph $G'$ is a (4-strong) \textit{superposition} of $G$ if $G'$ arises from $G$ after finitely many vertex and (4-strong) edge superpositions. The following statement was proved in \cite{8} Lemma 4.4.

\textbf{Lemma 2.1.} \textit{Let $G$ be a 4-strong superposition of a 4-snark. Then $G$ is a 4-snark.}

In other words, if we take a 4-snark and replace some of its vertices by arbitrary graphs and some of its edges by 4-snarks, we get a new 4-snark. For example,
part (a) of Figure 1 we show the Petersen graph. In parts (b) and (c) of the figure we depict a vertex and an edge superposition of the Petersen graph, respectively. Since the edge superposition is 4-strong (an edge is replaced by another copy of the Petersen graph) and the Petersen graph is a snark, all graphs from Figure 1 are 4-snarks by Lemma 2.1.

For more details about superposition techniques we refer to [8, 9].

3. Constructions

In part (a) of Figure 2 we show an embedding of the Petersen graph in the torus. In order to describe our constructions, it is convenient to present this embedding in the form indicated in part (b). If we identify the opposite segments of the square in part (a), we get a torus. If we identify these segments in part (b), we get a handle on the plane. The two embeddings correspond to each other, the only difference being that face $f_0$ is infinite in part (b) and finite in part (a). The following properties hold true.

(a) (b)

**Figure 2**

1. Any two faces $f_i$ and $f_j$, $i, j \in \{1, \ldots, 4\}$, share exactly one edge.
2. Face $f_0$ shares exactly two edges with each $f_i$, $i \in \{1, 2, 3\}$, and exactly three edges with $f_4$.

![Figure 3](https://example.com/figure3.png)

**Figure 3**

Replacing edge $e$ by another copy of the Petersen graph, we get the graph $G_{18}$ drawn in Figure 3. It is shown as an embedding in the plane with two handles. By Lemma 2.1, $G_{18}$ is a 4-snark, because it is a 4-strong edge superposition of the Petersen graph. Replacing each vertex of degree 5 in $G_{18}$ by a path of length 2, we get the graph $G_{22}$ in Figure 4. By Lemma 2.1, $G_{22}$ is a 4-snark. Since it is cubic,
cyclically 4-edge-connected, and has girth 5, we can conclude that

(3) $G_{22}$ is a snark.

The boundary of the infinite face $f_0$ is a circuit $C$, which is composed of two paths $P_1$ and $P_2$ with ends $u$ and $v$. The faces $f_i$ and $f_{i+4}$, $i = 1, \ldots, 4$, indicated in Figure 4 are copies of the face $f_i$ from Figure 2. Thus, using (1) and (2), we can prove the following properties.

(4) Any two faces $f_i$ and $f_j$, $i, j \in \{1, \ldots, 8\}$, share at most one edge.

(5) Face $f_0$ shares exactly two edges with each $f_i$, $i \in \{1, \ldots, 8\}$, so that $P_1$ and $P_2$ each contain exactly one of these edges.

By (4) and (5), the embedding of $G_{22}$ into $S_2$ can be transformed into a polyhedral one if we divide $f_0$ by a new edge with ends $u$ and $v$. Therefore, by taking a nonpolyhedral embedding of a snark in an orientable surface and replacing some of its edges by copies of $G_{22}$ so that the ends of replaced edges are identified with copies of $u$ and $v$, we can get a 4-snark with a polyhedral embedding in an orientable surface. In order to get a snark from this graph, we need to replace vertices of degrees higher than 4 by “suitable” graphs. We will use this idea in the proof of the following statement.

**Theorem 3.1.** For every $n \geq 5$, there exist infinitely many snarks with a polyhedral embedding in $S_n$.

![Figure 4](image4)

*Figure 4*

![Figure 5](image5)

*Figure 5*

**Proof.** In Figure 5 is a snark $G_{26}$ constructed by dot product (see, e.g., [9, Section 4]) from three copies of the Petersen graph. It was explicitly constructed in Belcastro and Kaminski [4, Figure 8]. The embedding in $S_1$ shown in the figure is not
polyhedral, because the pairs of faces $a_1, a_2$ and $b_1, b_2$ have two edges in common (i.e., the geometric dual has two pairs of parallel edges). We apply 4-strong edge superpositions so that we replace edges $e_1$ and $e_2$ by copies of $G_{22}$, getting graph $G_{66}$, indicated in Figure 6. It is shown as an embedding in $S_5$, the surface arising from the torus after adding four handles.

Replacing all vertices of degree 5 in $G_{66}$ by paths of order 3, we get the graph $G_{74}$ in Figure 7. It has order 74 and, by Lemma 2.1, is a snark. Furthermore, by (4) and (5), any two faces of this graph have at most one edge in common (i.e., the pairs of faces $a_1, a_2$ and $b_1, b_2$ are “separated” by copies of graph $G_{22}$). Thus the geometric dual of $G_{74}$ is a simple graph, i.e., $G_{74}$ is a snark with a polyhedral embedding in $S_5$.

In $G_{66}$, replace vertices of degree 5 by paths or by graphs $H_{i,5}$, $i \geq 1$, such as those in Figure 8. Dashed lines indicate the edges incident in $G_{66}$ with a vertex of degree 5. The resulting graphs are vertex superpositions of $G_{66}$; hence, by Lemma 2.1 they are snarks. In this way we get snarks of any even order $\geq 74$ with a polyhedral embedding in $S_5$.

In order to get snarks with embeddings in $S_n$ for $n > 5$, we replace some vertices of degree 5 in $G_{66}$ by “suitable” graphs with polyhedral embeddings in $S_n$, $n \geq 1$. For example, we can use embeddings of the graphs shown in Figure 9. This proves the theorem.

**Theorem 3.2.** For every $n \geq 9$, there exist infinitely many cyclically 5-edge-connected snarks with a polyhedral embedding in $S_n$. \[\square\]
Proof. Replace edges $e_1,\ldots,e_4$ of $G_{26}$ by copies of $G_{22}$ and then replace vertices of degrees 5 and 7 by paths of length 2 and 4, respectively. We get a cyclically 5-edge-connected snark of order 122 with a polyhedral embedding in $S_9$, indicated in Figure 10. Similarly as in the proof of Theorem 3.1 we can construct cyclically 5-edge-connected snarks of any even order $\geq 122$ with a polyhedral embedding in $S_9$ and cyclically 5-edge-connected snarks with polyhedral embeddings in $S_n$ for each $n \geq 9$. This proves the theorem. \hfill $\Box$

Figure 8

Figure 9

Figure 10
SNARKS IN ORIENTABLE SURFACES

References


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