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AN UPPER BOUND ON THE CHARACTERISTIC POLYNOMIAL OF A NONNEGATIVE MATRIX LEADING TO A PROOF OF THE BOYLE–HANDELMAN CONJECTURE

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(Communicated by Martin Lorenz)

Dedicated to the memory of our dear colleague Professor Israel Koltracht, 1949–2008

Abstract. In their celebrated 1991 paper on the inverse eigenvalue problem for nonnegative matrices, Boyle and Handelman conjectured that if A is an $(n+1)\times(n+1)$ nonnegative matrix whose **nonzero** eigenvalues are: $\lambda_0 \geq |\lambda_i|$, $i = 1, \ldots, r$, $r \leq n$, then for all $x \geq \lambda_0$,

$$(*) \qquad \prod_{i=0}^{r}(x - \lambda_i) \leq x^{r+1} - \lambda_0^{r+1}.$$

To date the status of this conjecture is that Ambikkumar and Drury (1997) showed that the conjecture is true when $2(r+1) \geq (n+1)$, while Koltracht, Neumann, and Xiao (1993) showed that the conjecture is true when $n \leq 4$ and when the spectrum of A is real. They also showed that the conjecture is asymptotically true with the dimension.

Here we prove a slightly stronger inequality than in $(*)$, from which it follows that the Boyle–Handelman conjecture is true. Actually, we do not start from the assumption that the λ_i's are eigenvalues of a nonnegative matrix, but that $\lambda_1, \ldots, \lambda_{r+1}$ satisfy $\lambda_0 \geq |\lambda_i|$, $i = 1, \ldots, r$, and the trace conditions:

$$(**) \qquad \sum_{i=0}^{r}\lambda_i^k \geq 0, \quad \text{for all } k \geq 1.$$

A strong form of the Boyle–Handelman conjecture, conjectured in 2002 by the present authors, says that $(*)$ continues to hold if the trace inequalities in $(**)$ hold only for $k = 1, \ldots, r$. We further improve here on earlier results of the authors concerning this stronger form of the Boyle–Handelman conjecture.

1. Introduction

Suppose that A is a nonnegative matrix of size $n+1$ and spectral radius 1. Ashley [2], Keilson and Styan [7], Vermes [7, pp. 457–458], and Fiedler [4] have shown independently and by different methods that if $\chi_A(x)$ is the characteristic

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polynomial of A, then

$$\chi_A(x) \ \le \ x^{n+1} - 1, \quad \text{for all } x \ge 1, \tag{1.1}$$

and that if an equality occurs for some $x > 1$, then $\chi_A(x) = x^{n+1} - 1$ as polynomials. In their celebrated paper [3], Boyle and Handelman make the following conjecture: Suppose further that A has a nonzero spectrum of size $r + 1 \ge 1$. Then,

$$\chi_A(x) \ \le \ x^{n+1} - x^{n-r}, \quad \text{for all } x \ge 1. \tag{1.2}$$

If $1 = \lambda_0, \lambda_1, \ldots, \lambda_n$ are the eigenvalues of A ordered by decreasing modulus, then (1.2) can be restated as

$$\prod_{i=0}^{r}(x - \lambda_i) \ \le \ x^{r+1} - 1 \quad \text{for all } x \ge 1. \tag{1.3}$$

In this paper, we shall refer to (1.2) as the conjecture of Boyle and Handelman. Notice that $\lambda_0, \ldots, \lambda_r$ form the *nonzero* spectrum of A. As explained in [3, p. 312], the minimal possible size of a matrix A with nonzero spectrum of size $r + 1$ is unbounded in terms of r. Therefore the bound conjectured by Boyle and Handelman is in some cases a big improvement over the known bound in (1.1).

For the sake of completeness it should be mentioned that in their celebrated paper, Boyle and Handelman make a very important contribution to the inverse eigenvalue problem for nonnegative matrices. They show that if a set of nonzero numbers satisfies some of the most obvious properties that the eigenvalues of nonnegative matrices must satisfy, see assumptions (a) and (b) in Theorem 1.1, then the set can be augmented by a sufficient number of 0's so that there exists a nonnegative matrix whose spectrum is the augmented set.

Returning to the Boyle–Handelman conjecture, Ambikkumar and Drury [1] proved (1.2) in the case where $2(r + 1) \ge n + 1$. A well-known necessary condition for $\lambda_0, \ldots, \lambda_r$ to be a nonzero spectrum of a nonnegative matrix A is that the quantities

$$\mathcal{S}_k \ := \ \sum_{j=0}^{r} \lambda_j^k \ \ge \ 0 \quad \text{for all integers } k \ge 1. \tag{1.4}$$

Suppose that the complex numbers $1 = \lambda_0, \ldots, \lambda_r$ ordered by decreasing modulus satisfy (1.4). Koltracht, Neumann, and Xiao [8] prove that (1.3) is true in the following three cases: (i) if $n + 1 \le 5$, (ii) if all λ_i are real, and (iii) the bound in (1.3) is true in an asymptotic sense; i.e., there exists a sequence c_r such that $c_r^r \sim 9r^2/16$ (hence $c_r \to 1$ as $r \to \infty$) and such that (1.3) holds for all $x \ge c_r$. Even though this asymptotic result may look very close to proving the conjecture, it should be noted that the sides of the inequality (1.3) vary significantly in the interval $1 \le x \le c_r$, so there is still a true gap.

In this paper we will prove the following theorem:

Theorem 1.1. *Suppose that $\lambda_0 = 1, \lambda_1, \ldots, \lambda_n$ are complex numbers satisfying:*

(a) $\lambda_0 \ge \max_{1 \le i \le n} |\lambda_i|$,
(b) $\mathcal{S}_k := \sum_{i=0}^{n} \lambda_i^k \ge 0$, *for all integers $k \ge 1$.*

Then

$$\prod_{i=0}^{n}(x - \lambda_i) \ \le \ x^{n+1} - 1, \quad \text{for all } x \ge 1. \tag{1.5}$$

Furthermore, if an equality occurs for some $x > 1$, *then there is an equality for all* x, *and both sides of the inequality are equal as polynomials. In particular, this theorem implies the conjecture of Boyle and Handelman.*

Our methods, as in [8], involve the Newton Identities. These identities give us information on how the polynomial $\prod_{i=0}^{n}(x-\lambda_i)$ varies as we vary the traces $\mathcal{S}_k$. This provides a natural proof in the case of what we call "initially positive" polynomials. In the complementary case, violation of (1.5) seems more unlikely, and we concentrate our efforts into turning this intuition to a proof.

Finally, let us mention that in [5] a stronger result than Theorem 1.1 was conjectured. Namely, Theorem 1.1 is believed to be true if only we assume in (b) that $\mathcal{S}_k \geq 0$, for $k \leq n$. Our methods in this paper fall short for proving this conjecture. However, we still prove:

Theorem 1.2. *Suppose that the complex numbers* λ_i, $0 \leq i \leq n$, *satisfy*

(a) $\lambda_0 = 1$,

(b) $|\lambda_i| \leq 1$, *for all* i, *and*

(c) $\mathcal{S}_k = \sum_{i=0}^{n} \lambda_i^k \geq 0$ *for all* $1 \leq k \leq n$.

Then

$$\prod_{i=0}^{n}(x-\lambda_i) \quad \leq \quad x^{n+1}-1, \quad \textit{for all } x \;\geq\; \sqrt[n+1]{2n+2}.$$

Furthermore, if an equality occurs for any such x, *then both sides agree as polynomials in* x.

This theorem improves the main result in [5, Theorem 1.4], which shows that the above inequality holds for $x \geq 6.75$.

2. Newton Identities

Our proof will be based on the well-known Newton Identities. Let $\xi_1, \ldots, \xi_n$ be indeterminates. We define the following indeterminates:

$$s_k \;:=\; \sum_{i=1}^{n} \xi_i^k, \quad 1 \leq k < \infty, \tag{2.1}$$

$$\sigma_k \;:=\; \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \xi_{i_1}\xi_{i_2}\ldots\xi_{i_k}, \quad 1 \leq k \leq n. \tag{2.2}$$

The Newton Identities relate the s_k's with the σ_k's, for $1 \leq k \leq n$, in the following way: For every power series (or a polynomial) $f(t) = \sum_{i=0}^{\infty} c_i t^i \in \mathbb{C}[[t]]$, define $[f(t)]_r = \sum_{i=0}^{r} c_i t^i$ to be the truncation at degree r. We then have

(2.3)

$$\prod_{i=1}^{n}(1-\xi_i t) = 1-\sigma_1 t+\sigma_2 t^2-\cdots+(-1)^n\sigma_n t^n = \left[\exp\left(-s_1 t - s_2\frac{t^2}{2} - \cdots - s_n\frac{t^n}{n}\right)\right]_n.$$

Denote this polynomial by $P_n(t,s)$ and view it as a function of the indeterminates $s = (s_1, \ldots, s_n)$. It is also true that as a power series in t,

$$P_n(t,s) \;=\; 1-\sigma_1 t+\sigma_2 t^2-\cdots+(-1)^n\sigma_n t^n \;=\; \exp\left(-\sum_{j=1}^{\infty} s_j \frac{t^j}{j}\right). \tag{2.4}$$

Continuing, for every $r \leq n$, let $P_r(t,s) = [P_n(t,s)]_r$. Note that $P_r(t,s)$ depends only on $s_1, \ldots, s_r$ and therefore this notation agrees with the original definition of $P_n(t,s)$. For a discussion of Newton's Identities, see MacDonald [9, Chap 1, §§3-5].

We will now give a formula for the s–derivatives of $P_n(t,s)$.

Lemma 2.1.

$$\frac{\partial P_n(t,s)}{\partial s_j} = -\frac{t^j}{j} P_{n-j}(t,s). \tag{2.5}$$

Proof. Using (2.3),

$$\frac{\partial}{\partial s_j}\left[\exp\left(-\sum_{k=1}^{n}\frac{s_k t^k}{k}\right)\right]_n = \left[\frac{\partial}{\partial s_j}\exp\left(-\sum_{k=1}^{n}\frac{s_k t^k}{k}\right)\right]_n$$
$$= \left[-\frac{t^j}{j}\exp\left(-\sum_{k=1}^{n}\frac{s_k t^k}{k}\right)\right]_n = -\frac{t^j}{j}\left[\exp\left(-\sum_{k=1}^{n}\frac{s_k t^k}{k}\right)\right]_{n-j} = -\frac{t^j}{j} P_{n-j}(t,s).$$

□

3. Proof of Theorems 1.1 and 1.2

We begin by reformulating Theorem 1.1 by the removal of the root $\lambda_0 = 1$ and by the change of variable $\tau = 1/x$. Then Theorem 1.1 reads:

Theorem 3.1. *Suppose that $\lambda_1, \ldots, \lambda_n$ are complex numbers satisfying:*

(a) $1 \geq \max_{1 \leq i \leq n} |\lambda_i|$,
(b) $\mathcal{S}_k := \sum_{i=1}^{n} \lambda_i^k \geq -1$, *for all integers $k \geq 1$.*

Then

$$\prod_{i=1}^{n}(1-\lambda_i\tau) \leq 1 + \tau + \tau^2 + \cdots + \tau^n, \quad \text{for all } 0 \leq \tau \leq 1. \tag{3.1}$$

Furthermore, if an equality occurs for some $0 < \tau \leq 1$, *then there is an equality for all τ and both sides of the inequality are equal as polynomials.*

Remark 3.2. This is slightly stronger than Theorem 1.1, because the second assertion applies also to $\tau = 1$, which was excluded there.

To prove our results we shall require the following notion:

Definition 3.3. A polynomial $\sum_{i=0}^{d} c_i t^i$ is *initially positive at $t = \tau$* (resp. *strictly initially positive*), if $\sum_{i=0}^{r} c_i \tau^i \geq 0$ (resp. $\sum_{i=0}^{r} c_i \tau^i > 0$) for every $0 \leq r \leq d$.

In what follows, we shall understand an inequality $v \leq w$ between two vectors $v = (v_i)$ and $w = (w_i)$ to mean $v_i \leq w_i$ for all i. In addition, for every scalar c, $v = c$ means that $v_i = c$ for all i, and $v + c$ means the vector $(v_i + c)$. Let us now prove the following lemma:

Lemma 3.4. *Suppose that $P_r(t,\mathcal{S})$ is initially positive at $t = \tau \geq 0$, and that $\mathcal{S}_i \geq -1$, for all $1 \leq i \leq r$. Then*

$$P_r(\tau,\mathcal{S}) \leq 1 + \tau + \tau^2 + \cdots + \tau^r \tag{3.2}$$

and equality occurs if and only if $P_r(t,\mathcal{S}) = 1 + t + t^2 + \cdots + t^r$ as polynomials in t. The polynomial function $s \mapsto P_r(\tau, s)$ is monotonically decreasing as a function of each s_i in the box $-1 \leq s \leq \mathcal{S}$.

Proof. We change variables $\rho_i = \mathcal{S}_i - s_i$. With respect to this change,

$$\left.\frac{\partial P_r(t,s)}{\partial \rho_j}\right|_{t=\tau, s=\mathcal{S}} \;=\; \frac{\tau^j}{j} P_{r-j}(\tau,\mathcal{S}) \geq 0.$$

Moreover, since $P_r(t,\mathcal{S})$ is initially positive at $t=\tau$, then all partial derivatives of $P_r(\tau,s)$ with respect to the ρ_i are nonnegative. Expanding $P_r(\tau,s)$ as a Taylor series with center at $\rho=0$ shows that $F(\rho) := P_r(\tau,s)$ is a polynomial function in the ρ_i with nonnegative coefficients. It follows that

$$P_r(\tau,\mathcal{S}) \;=\; F(0) \;\leq\; F(\mathcal{S}+1) \;=\; P_r(\tau,-1,\ldots,-1) \;=\; 1+\tau+\cdots+\tau^r.$$

This proves (3.2) and the monotonicity claim. Suppose now that there is an equality in (3.2). If $\mathcal{S}_i > -1$, for some $i \leq r$, we shall connect the point $\mathcal{S}$ (i.e. $\rho = 0$) with the point $\mathcal{S}^{-1} := (-1,\ldots,-1)$ (i.e. $\rho = \mathcal{S}+1 \geq 0$) by a straight line segment. Computing the directional derivative of $F(\rho)$ in the direction $\mathcal{S}+1 \geq 0$ in the box $0 \leq \rho \leq \mathcal{S}+1$ yields the nonnegative quantity $\sum_j \partial F/\partial\rho_j(\mathcal{S}_j+1)$. Moreover, this directional derivative must be strictly positive at $\rho = \mathcal{S}+1$, since $\mathcal{S}_i + 1 > 0$ for some i and since $P_r(t,-1,\ldots,-1)$ is strictly initially positive at $t=\tau$. This shows that $F(\mathcal{S}+1)$ must be strictly greater than $F(0)$, leading to a contradiction. As a consequence, we must obtain that $\mathcal{S} = -1$, and an equality in (3.2) holds if and only if both sides agree as polynomials in t. □

Corollary 3.5. *Theorem* 3.1 *is correct for any point* $t=\tau$ *where* $P_n(t,\mathcal{S})$ *is initially positive.*

To prove Theorem 3.1 we need to study the case where $P_n(t,s)$ is not initially positive at some point. Intuitively speaking, the fact that some $P_r(\tau,\mathcal{S}) < 0$ makes the violation of (3.1) less likely to happen. This, in a sense, will be the idea behind the rest of the proof.

Lemma 3.6. *Suppose that* $P_n(t,\mathcal{S})$ *is* not *initially positive at* $t=\tau$, $0 \leq \tau \leq 1$. *Write* $\Delta\mathcal{S}_i = \mathcal{S}_i + 1 \geq 0$. *Then* (a)

$$\sum_{j=1}^{n} \frac{\tau^j}{j}\Delta\mathcal{S}_j \;>\; 1. \tag{3.3}$$

(b) *More strongly,*

$$\sum_{j=1}^{n} \frac{\tau^j+\cdots+\tau^n}{1+\cdots+\tau^n}\frac{\Delta\mathcal{S}_j}{j} \;>\; 1.$$

Proof. Let d be the integer such that $P_d(\tau,\mathcal{S}) < 0$, and $P_r(\tau,\mathcal{S}) \geq 0$ for all $r < d$. Such an integer certainly exists since $P_0(\tau,\mathcal{S}) = 1$. Define a polynomial function: $F(\mu) = P_d(\tau,\mathcal{S}-\mu(\mathcal{S}+1))$. We have that

$$F'(\mu) \;=\; \sum_{j=1}^{d} \frac{\tau^j\Delta\mathcal{S}_j}{j} P_{d-j}(\tau,\mathcal{S}-\mu(\mathcal{S}+1)). \tag{3.4}$$

Thus $F(0) = P_d(\tau,\mathcal{S}) < 0$ since $P_d(t,\mathcal{S})$ is no longer initially positive at τ. We also have that $F(1) = 1+\tau+\cdots+\tau^d$. By the Lagrange Mean Value Theorem,

$F(1)-F(0)=F'(c)$, for some $0<c<1$. By (3.4), the initial positiveness at τ of $P_r(t,\mathcal{S})$, for $r<d$, and the monotonicity property of Lemma 3.4, we have

$$(3.5)\quad 1+\tau+\cdots+\tau^d \;<\; F(1)-F(0) \;=\; F'(c) \\ \le\; \sum_{j=1}^{d}\frac{\tau^j}{j}\Delta\mathcal{S}_j(1+\tau+\cdots+\tau^{d-j}) \;\le\; \sum_{j=1}^{d}\frac{\tau^j}{j}\Delta\mathcal{S}_j(1+\tau+\cdots+\tau^{d}).$$

Canceling out $1+\tau+\cdots+\tau^d$, we conclude that

$$\sum_{j=1}^{n}\frac{\tau^j}{j}\Delta\mathcal{S}_j \;\ge\; \sum_{j=1}^{d}\frac{\tau^j}{j}\Delta\mathcal{S}_j \;>\; 1.$$

This proves part (a).

To prove part (b), we look more carefully at (3.5) to deduce that

$$\sum_{j=1}^{d}\left(\frac{\tau^j+\cdots+\tau^d}{1+\cdots+\tau^d}\cdot\frac{\Delta\mathcal{S}_j}{j}\right) \;>\; 1,$$

which implies (b), since

$$\frac{\tau^j+\cdots+\tau^n}{1+\cdots+\tau^n} \;\ge\; \frac{\tau^j+\cdots+\tau^d}{1+\cdots+\tau^d},$$

for all $0\le\tau\le 1$.

□

Using this lemma, we can prove the following proposition.

Proposition 3.7. *Let us now assume that* $\mathcal{S}_k=\sum_{i=1}^n\lambda_i^k\ge -1$, *for* all $k\ge 1$, *and that in* $P_n(t,\mathcal{S})=\prod(1-\lambda_i t)$, *all* $|\lambda_i|\le 1$. *Suppose that for some* $0\le\tau\le 1$, $\sum_{j=1}^n\frac{\Delta\mathcal{S}_j}{j}\tau^j>1$. *Then*

$$(3.6)\qquad 1+\tau+\cdots+\tau^n \;>\; \mathrm{e}(1-\tau^{n+1})P_n(\tau,\mathcal{S}),\quad \mathrm{e}:=\exp(1).$$

An immediate corollary to Proposition 3.7 and Lemma 3.6 is the following:

Corollary 3.8. *Theorem* 3.1 *holds for every* $\tau<(1-1/\mathrm{e})^{1/(n+1)}$.

Proof of Proposition 3.7. For $\tau=1$ the proposition is trivially true. So we assume from now on that $0\le\tau<1$. Notice that then $P_n(\tau,\mathcal{S})>0$ since the numbers $1-\lambda_i\tau$ are either positive, or come in complex conjugate pairs.

Continuing, we compute $\log(1+\tau+\cdots+\tau^n)-\log P_n(\tau,\mathcal{S})$, for $0\le\tau<1$. We have by (2.4) that

$$(3.7)\qquad \log(1+\tau+\cdots+\tau^n) \;=\; \sum_{i=1}^{\infty}\frac{\tau^i}{i} \;-\; \sum_{i=1}^{\infty}\frac{\tau^{i(n+1)}}{i}$$

and

$$(3.8)\qquad \log P_n(\tau,\mathcal{S}) \;=\; -\sum_{i=1}^{\infty}\mathcal{S}_i\frac{\tau^i}{i}$$

as an absolutely convergent power series. Therefore,

$$\begin{aligned}\log(1+\tau+\cdots+\tau^n) &- \log P_n(\tau,\mathcal{S})\\ &= \sum_{i=1}^{\infty}\Delta\mathcal{S}_i\frac{\tau^i}{i} \;-\; \sum_{i=1}^{\infty}\frac{\tau^{i(n+1)}}{i}\\ &> 1 \;-\; \sum_{i=1}^{\infty}\frac{\tau^{(n+1)i}}{i} \;=\; 1+\log(1-\tau^{n+1}).\end{aligned}$$

Exponentiating, this completes the proof of the proposition. □

Let us pause shortly to see what we can obtain from the above proof under the weaker assumption that $\mathcal{S}_j \geq -1$, for $1 \leq j \leq n$ only.

Theorem 3.9. *Suppose that $\mathcal{S}_j \geq -1$, for $1 \leq j \leq n$, and that $P_n(t,\mathcal{S}) = \prod_i(1-\lambda_i t)$ for $|\lambda_i| \leq 1$. Then for all $0 \leq \tau \leq 1/\sqrt[n+1]{2n+2}$,*

$$P_n(\tau,\mathcal{S}) \;\;\leq\;\; 1+\tau+\cdots+\tau^n.$$

An equality at $0 < \tau \leq 1/\sqrt[n+1]{2n+2}$ occurs if and only if both sides are equal as polynomials.

Proof. The case $n = 0$ is trivial. So from here on, let us assume that $n > 0$. If $P_n(t,\mathcal{S})$ is initially positive at $t = \tau$, we complete the proof by Lemma 3.4. Otherwise, we argue as follows. We compute the difference $D = \log(1+\tau+\cdots+\tau^n) - \log P_n(\tau,\mathcal{S})$. As in the proof of Proposition 3.7, we obtain that

$$D \;=\; \sum_{i=1}^{\infty}\Delta\mathcal{S}_i\frac{\tau^i}{i} \;-\; \sum_{i=1}^{\infty}\frac{\tau^{i(n+1)}}{i} \;>\; 1+\sum_{i=n+1}^{\infty}\Delta\mathcal{S}_i\frac{\tau^i}{i} \;-\; \sum_{i=1}^{\infty}\frac{\tau^{i(n+1)}}{i}.$$

The inequality itself is obtained by Lemma 3.6(a). We must next show that $D > 0$. By the assumption that all $|\lambda_i| \leq 1$ and the traces $\mathcal{S}_j \geq -n$ for all j, we have

$$D \;>\; 1-\frac{n}{n+1}\sum_{i=n+1}^{\infty}\tau^i - \sum_{i=1}^{\infty}\tau^{i(n+1)} \;>\; 1-2\sum_{i=n+1}^{\infty}\tau^i \;=\; 1-2\frac{\tau^{n+1}}{1-\tau}.$$

We will have $D > 0$ if we can show that $2\tau^{n+1} \leq 1-\tau$. Since the left-hand side is increasing in τ and the right-hand side is decreasing, it is enough to inspect this last inequality for $\tau = 1/\sqrt[n+1]{2n+2}$. This amounts to $1/(2m) \leq (1-1/m)^m$ for $m = n+1$. But this is indeed true for $m \geq 2$, since it is true for $m = 2$, and since the left-hand side is decreasing in m, while the right-hand side is increasing. □

Corollary 3.10. *Theorem* 1.2 *readily follows.*

We continue with the proof of Theorem 3.1. We shall require the following lemma, which is an exercise in calculus and so its proof is omitted.

Lemma 3.11. *The function*

$$h(t) \;=\; \frac{t^j+\cdots+t^n}{1+\cdots+t^n}$$

is increasing in the interval $t \in [0,1]$.

The next lemma is the key feature needed to complete the proof.

Lemma 3.12. *Suppose that* $f(t) = \prod_{i=1}^{n}(1-\lambda_i t)$ *is a real polynomial, with* $|\lambda_i| \leq 1$, *for all* i. *Then,*

$$\frac{f'(\tau)}{f(\tau)} < \frac{n}{2\tau}, \quad \textit{for all } 0 < \tau < 1. \tag{3.9}$$

Proof. The Möbius transformation $z \mapsto -z/(1-z)$ maps the open unit disc bijectively onto the domain $D = \{z \mid \operatorname{Re}(z) < \frac{1}{2}\}$. Thus,

$$\frac{f'(\tau)}{f(\tau)} \;=\; \operatorname{Re}\frac{f'(\tau)}{f(\tau)} \;=\; \frac{1}{\tau}\sum_{j=1}^{n}\operatorname{Re}\frac{-\lambda_i\tau}{1-\lambda_i\tau} < \frac{n}{2\tau},$$

since $|\lambda_j\tau| < 1$, for all j.

We are ready now to prove Theorem 3.1.

Proof of Theorem 3.1. For $\tau = 0$, the inequality (3.1) is trivial. So assume that $0 < \tau \leq 1$. If $P_n(t, \mathcal{S})$ is initially positive at τ, then the theorem is true at τ. So for the remainder of the proof, we will assume that $P_n(t, \mathcal{S})$ is not initially positive at τ. Let $\theta = (1 - 1/\mathrm{e})$ and $\alpha = \theta^{2/(n+1)}$. We will show that

$$P_n(\tau, \mathcal{S}) \;<\; 1 + \tau + \cdots + \tau^n. \tag{3.10}$$

If $\tau \leq \alpha$, we know that this is true by Corollary 3.8. Therefore we will assume that $\tau > \alpha$.

Next, we will show that $\sum_{j=1}^{n} \frac{\Delta\mathcal{S}_j}{j}\alpha^j > 1$. Suppose that this is not the case. Then we have

$$\begin{aligned}\sum_{j=1}^{n}\frac{\tau^j+\cdots+\tau^n}{1+\cdots+\tau^n}\frac{\Delta\mathcal{S}_j}{j} \;&\leq\; \sum_{j=1}^{n}\frac{n+1-j}{n+1}\frac{\Delta\mathcal{S}_j}{j}\\ &\leq\; \sum_{j=1}^{n}\mathrm{e}^{-j/(n+1)}\frac{\Delta\mathcal{S}_j}{j} \;\leq\; \sum\alpha^j\frac{\Delta\mathcal{S}_j}{j} \;\leq\; 1.\end{aligned}$$

We have used here Lemma 3.11, the inequality that $\mathrm{e}^x \geq 1 + x$, and the fact that $\alpha^{n+1} = \theta^2 > 1/\mathrm{e}$. But now, Lemma 3.6(b) implies that $P_n(t, \mathcal{S})$ is in fact initially positive at $t = \tau$, contrary to our assumption.

Now we are in the case where $\sum_{j=1}^{n} \frac{\Delta\mathcal{S}_j}{j}\alpha^j > 1$. We apply Proposition 3.7 to deduce that

$$(1 + \alpha + \cdots + \alpha^n) \;>\; P_n(\alpha, \mathcal{S})\mathrm{e}(1 - \alpha^{n+1}) \;=\; P_n(\alpha, \mathcal{S})\left(2 - \frac{1}{\mathbf{e}}\right). \tag{3.11}$$

We have by the Lagrange Theorem and by Lemma 3.12 that for some $\alpha < c < \tau$,

$$\log P_n(\tau, \mathcal{S}) - \log P_n(\alpha, \mathcal{S}) \;=\; \frac{P_n'(c, \mathcal{S})}{P_n(c, \mathcal{S})}(\tau - \alpha) \leq \frac{n}{2c}(\tau - \alpha) \leq \frac{n}{2\alpha}(\tau - \alpha). \tag{3.12}$$

Suppose by contradiction that $P_n(\tau, \mathcal{S}) \geq 1 + \tau + \cdots + \tau^n$. Combining this with (3.11) and (3.12) we obtain

$$\left(2 - \frac{1}{\mathrm{e}}\right)\frac{1 + \tau + \cdots + \tau^n}{1 + \alpha + \cdots + \alpha^n} \;<\; \exp\Big(\frac{n}{2\alpha}(\tau - \alpha)\Big). \tag{3.13}$$

Define a function $F(t) = (1 + t + \cdots + t^n)\exp\big((1 - t)\frac{n}{2\alpha}\big)$. Equation (3.13) can be rewritten as

$$\left(2 - \frac{1}{\mathrm{e}}\right)F(\tau) \;<\; F(\alpha). \tag{3.14}$$

We claim now that $F(t)$ is monotonically decreasing in $t \in [\alpha, 1]$. To see this, we calculate the logarithmic derivative

$$\frac{F'(t)}{F(t)} \;=\; \frac{(1+t+\cdots+t^n)'}{(1+t+\cdots+t^n)} - \frac{n}{2\alpha}.$$

By Lemma 3.12 this is less than or equal to $n/(2t) - n/(2\alpha) \le 0$. Consequently, (3.14) implies that

$$\left(2-\frac{1}{\mathrm{e}}\right)F(1) \;=\; \left(2-\frac{1}{\mathrm{e}}\right)(n+1) \;<\; F(\alpha). \tag{3.15}$$

To arrive at a contradiction, we will show that (3.15) cannot be true.

Let us show this for $n \ge 14$. A simple application of Lagrange's Theorem to the function $x \mapsto \theta^x$ on the interval $[0, 2/(n+1)]$ shows that $1-\alpha \le -\frac{2}{n}\log\theta$. Thus for $n \ge 14$,

$$F(\alpha) \le (n+1)\exp\left(\frac{n}{2\alpha}(1-\alpha)\right) \;\le\; (n+1)\exp(-\log\theta/\alpha) \tag{3.16}$$

$$=\; (n+1)\theta^{-1/\alpha} \le (n+1)\theta^{-\theta^{-2/15}} \;\le\; 1.63(n+1).$$

This is smaller than $(2-1/\mathrm{e})(n+1) \approx 1.632(n+1)$, so (3.15) is violated for $n \ge 14$. The cases $0 \le n \le 13$ can be tested by explicit calculations and likewise violate (3.15). Alternatively, one can show that the expression $F(\alpha)/(n+1)$ is decreasing as a function of n as n grows and thus avoid the separate treatment for $n \le 13$. The proof is complete. □

Theorem 1.1 has the following corollary in which, in particular, the nonzero entries of a positive semidefinite matrix can be viewed as a limit of the nonzero parts of the spectra of nonnegative matrices of larger size. It would be interesting if one could find an independent proof.

Corollary 3.13. *Let $H = (H_{i,j})$ be a positive semidefinite $n \times n$ Hermitian matrix. Assume that* $\max_{i,j}|H_{i,j}| = 1$. *Let $\nu(H)$ be the number of nonzero entries in H. Then*

$$\prod_{\{(i,j)\,|\,H_{i,j}\neq 0\}} (x - H_{i,j}) \quad\le\quad x^{\nu(H)} - 1, \quad \textit{for all } x \ge 1.$$

Proof. We first note that as H is positive semidefinite, its 2×2 principal submatrices have nonnegative determinants, and hence a maximal entry in modulus in H must occur on the diagonal of H. As the diagonal entries of H are nonnegative, such a maximal entry must equal 1.

Next, again using the fact that H is positive semidefinite, it follows, according to Fejér's Theorem (see Horn and Johnson [6, Corollary 7.5.4]), that

$$\sum_{\{(i,j)\,|\,H_{i,j}\neq 0\}} H_{i,j}^k \;=\; \sum_{i,j=1}^{n} H_{i,j}^k \ge 0, \quad \textit{for all } k \ge 1.$$

The conclusion now follows by applying our Theorem 1.1. □

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