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ON STRINGS OF CONSECUTIVE INTEGERS WITH A DISTINCT NUMBER OF PRIME FACTORS

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ABSTRACT. Let $\omega(n)$ be the number of distinct prime factors of n. For any positive integer k let $n = n_k$ be the smallest positive integer such that $\omega(n+1), \dots, \omega(n+k)$ are mutually distinct. In this paper, we give upper and lower bounds for n_k. We study the same quantity when $\omega(n)$ is replaced by $\Omega(n)$, the total number of prime factors of n counted with repetitions.

Let $\omega(n)$ and $\Omega(n)$ denote respectively the number of distinct prime factors of n and the total number of prime factors of n counted with repetitions. For any positive integer k let $n = n_k$ be the smallest positive integer n such that $\omega(n+1), \dots, \omega(n+k)$ are mutually distinct. We also let $m = m_k$ be the smallest positive integer m such that $\Omega(m+1), \dots, \Omega(m+k)$ are mutually distinct. Using a computer, we easily obtain that $n_2 = 4$, $n_3 = 27$, $n_4 = 416$, $n_5 = 14321$, $n_6 = 461889$, $n_7 = 46908263$ and $n_8 = 7362724274$, and also that $m_2 = 2$, $m_3 = 5$, $m_4 = 14$, $m_5 = 59$, $m_6 = 725$, $m_7 = 6317$, $m_8 = 189374$, $m_9 = 755967$ and $m_{10} = 683441870$. In this paper, we give upper and lower bounds for n_k and m_k. Let p_i be the i-th prime number. Let $n = n_k$. Since the set $\{\omega(n+j) : j = 1, \dots, k\}$ consists of k nonnegative integers, it follows that one of $n+j$ for $j = 1, \dots, k$ must have at least k distinct prime factors. Thus,

$$n + k \geq \prod_{i=1}^{k} p_i = \exp((1+o(1))p_k) = \exp((1+o(1))k \log k)$$

as $k \to \infty$ by the Prime Number Theorem; therefore

$$n_k \geq \exp((1+o(1))k \log k) \qquad \text{as } k \to \infty.$$

Similarly, letting $m = m_k$, we get that $\Omega(m+i) \geq k$ for some $i \in \{1, \dots, k\}$. Thus, $m + k \geq 2^k$, giving $m_k \geq \exp((\log 2 + o(1))k)$ as $k \to \infty$.

We start by improving these trivial estimates as follows.

Theorem 1. *The inequality*

$$n_k \geq \exp((2+o(1))k \log k)$$

holds as $k \to \infty$. Furthermore, the inequality

$$m_k \geq \exp((1/2+o(1))k \log k)$$

holds as $k \to \infty$.

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The problem of finding lower and upper bounds for n_k and m_k was raised in the recent book [1] by the first author. We remark that, after writing this paper, we noticed that the first of these bounds is essentially equivalent to one due to Erdős [2]. We were somewhat surprised that we could not find any other work on these problems.

Proof. We start with the first inequality. Assume that $\omega(n+1),\ldots,\omega(n+k)$ are mutually distinct. Let $\varepsilon\in(0,1)$ be arbitrarily small but fixed. Put $s=\lfloor k^{1-\varepsilon}\rfloor$. Let $i_1,\ldots,i_s$ be s distinct integers in $\{1,\ldots,k\}$ such that $\omega(n+i_j)\ge k-j$ for $j=1,\ldots,s$. Let $\mathcal{A}_{i_j}$ be the set of prime factors of $n+i_j$. Note that if $j\ne\ell$ and $p\in\mathcal{A}_{i_j}\cap\mathcal{A}_{i_\ell}$, then $p\mid(n+i_j)-(n+i_\ell)=(i_j-i_\ell)$ and $1\le|i_j-i_\ell|\le k-1$. Since $\omega(m)\ll\log m/\log\log m$ holds for all positive integers, we get that

$$\#\left(\mathcal{A}_{i_j}\cap\mathcal{A}_{i_\ell}\right)<c_1\frac{\log k}{\log\log k}$$

holds for all $j\ne\ell$ with some absolute constant c_1. By the Principle of Inclusion and Exclusion,

$$\begin{aligned}\#\left(\bigcup_{j=1}^{s}\mathcal{A}_{i_j}\right) &\ge \sum_{j=1}^{s}\#\mathcal{A}_{i_j}-\sum_{1\le j<\ell\le s}\#\left(\mathcal{A}_{i_j}\cap\mathcal{A}_{i_\ell}\right)\\ &\ge ks-\frac{s(s+1)}{2}-c_1\binom{s}{2}\frac{\log k}{\log\log k}>(1-\varepsilon)k^{2-\varepsilon}\end{aligned}$$

provided that $k>k_\varepsilon$. Thus, using the Prime Number Theorem once more, we have

$$\begin{aligned}(n+k)^s &\ge \prod_{j=1}^{s}(n+i_j)\ge\prod_{1\le i<(1-\varepsilon)k^{2-\varepsilon}}p_i\\ &\ge \exp\left((2-\varepsilon+o(1))ks\log k\right)\end{aligned}$$

as $k\to\infty$. This leads to $n\ge\exp((2-\varepsilon+o(1))k\log k)$ as $k\to\infty$, which implies the desired conclusion since $\varepsilon\in(0,1)$ was arbitrary.

We now deal with the second inequality. Let $m=m_k$. For any given prime number p and positive integer n we let $\nu_p(n)$ be the exact exponent with which p appears in the prime factorization of n. For each $p\le k$ let $i_p\in\{1,\ldots,k\}$ be such that

$$\nu_p(m+i_p)=\max_{1\le i\le k}\nu_p(m+i). \tag{1}$$

If more than one value for $i_p\in\{1,\ldots,k\}$ exists for which equality (1) is satisfied, we simply pick one of them. Clearly, the set $\mathcal{I}$ of indices i_p so chosen satisfies

$$\#\mathcal{I}\le\pi(k). \tag{2}$$

An elementary argument (see, for example, Lemma 2 in [3]) shows that if we write

$$m+i=a_ib_i,$$

where the largest prime factor of a_i is $\le k$ and the smallest prime factor of b_i exceeds k, then

$$\prod_{\substack{1\le i\le k\\ i\notin\mathcal{I}}}a_i\le k^k.$$

In particular,

$$\sum_{\substack{1\le i\le k\\ i\notin\mathcal{I}}}\Omega(a_i)=\Omega\left(\prod_{\substack{1\le i\le k\\ i\notin\mathcal{I}}}a_i\right)<\frac{k\log k}{\log 2}<2k\log k. \tag{3}$$

Let

$$\mathcal{J}=\{i\notin\mathcal{I}:\Omega(a_i)>k^{1/2}\}.$$

Then inequality (3) shows that

$$\#\mathcal{J}<2k^{1/2}\log k. \tag{4}$$

Finally, let

$$\mathcal{K}=\{i\notin\mathcal{I}\cup\mathcal{J}:\Omega(m+i)\le k^{2/3}\}.$$

Since the numbers $\Omega(m+j)$ are distinct for $j=1,\dots,k$, it follows that

$$\#\mathcal{K}\le k^{2/3}. \tag{5}$$

Let $\mathcal{S}=\{1,\dots,k\}-(\mathcal{I}\cup\mathcal{J}\cup\mathcal{K})$ and put $s=\#\mathcal{S}$. Let $\varepsilon>0$ be fixed. Estimates (2), (4) and (5) show that

$$s\ge k-\pi(k)-2k^{1/2}\log k-k^{2/3}>(1-\varepsilon)k,$$

provided that $k>k_\varepsilon$. Note that if $i\in\mathcal{S}$, then

$$\Omega(a_i)\le k^{1/2}=(k^{2/3})^{3/4}\le\Omega(m+i)^{3/4},$$

so that

$$\Omega(b_i)=\Omega(m+i)-\Omega(a_i)\ge\Omega(m+i)-\Omega(m+i)^{3/4}\ge(1-\varepsilon)\Omega(m+i)$$

for all $i\in\mathcal{S}$, assuming that $k>k_\varepsilon$. Thus, since the $\Omega(m+i)$ are distinct,

$$\begin{aligned}(m+k)^s &\ge \prod_{i\in\mathcal{S}}b_i>k^{\sum_{i\in\mathcal{S}}\Omega(b_i)}>\left(k^{\sum_{i\in\mathcal{S}}\Omega(m+i)}\right)^{(1-\varepsilon)}\\ &> \left(k^{\sum_{j=1}^{s}j}\right)^{(1-\varepsilon)}>\exp((1/2-\varepsilon)s^2\log k).\end{aligned}$$

Hence,

$$m_k\ge\exp((1/2-\varepsilon)s\log k)>\exp((1/2-2\varepsilon)k\log k).$$

Since $\varepsilon>0$ is arbitrary, we get the desired conclusion. □

We next turn our attention to upper bounds for n_k and m_k. We have the following result.

Theorem 2. *The inequalities*

$$n_k\le\exp((6/\log 2+o(1))k^2(\log k)^2)$$

and

$$m_k\le\exp((4/\log 2+o(1))k^2(\log k)^2)$$

hold as $k\to\infty$.

Proof. We assume that $k \geq 2$. Again, we deal first with n_k. We let A be a positive integer depending on k, to be determined later. We let $q_1 < q_2 < \cdots < q_m < \cdots$ be all the consecutive prime numbers exceeding k. For $j = 1, \ldots, k$, we put $T_j = j(j-1)/2$ and

$$M_j = \prod_{\ell = T_j A + 1}^{T_{j+1} A} q_\ell.$$

Put $M = \prod_{j=1}^{k} M_j$ and let N be the smallest positive integer such that M_j divides $N + j$ for each j with $1 \leq j \leq k$. Such an integer N exists by the Chinese Remainder Theorem. Note that $N + k < M$. Indeed, if not, then $N = M - i$ for some $i \in \{1, \ldots, k\}$, and by taking some $j \neq i \in \{1, \ldots, k\}$ (which exists because $k \geq 2$), we would get that $M_j \mid N + j = M + (j - i)$; therefore $M_j \mid j - i$, which is impossible. Let $n = M\lambda + N$ be a positive integer with $\lambda \in [M, 2M]$. Note that

$$n + j = M\lambda + (N + j) = M_j \left((M/M_j)\lambda + (N + j)/M_j \right), \qquad j = 1, \ldots, k.$$

By setting $A_j = (N + j)/M_j$ and $B_j = M/M_j$, it follows that

$$jA = T_{j+1}A - T_j A = \omega(M_j) \leq \omega(n + j) \leq jA + \omega(B_j \lambda + A_j),$$

so that if λ is such that

$$\omega(B_j \lambda + A_j) < A, \qquad \text{for all } j = 1, \ldots, k - 1, \tag{6}$$

then

$$jA \leq \omega(n + j) < jA + A \leq \omega(n + j + 1) \qquad \text{for all } j = 1, \ldots, k - 1.$$

Hence, we certainly have that $\omega(n + 1), \ldots, \omega(n + k)$ are pairwise distinct.

It now remains to estimate A and M such that we can guarantee the existence of a positive integer $\lambda \in [M, 2M]$ with the property that all of the inequalities (6) hold.

We claim that A_j and B_j are coprime. Indeed, to see this, note first that

$$B_j = M/M_j = \prod_{\substack{1 \leq \ell \leq k \\ \ell \neq j}} M_\ell.$$

If there exists a prime $p \mid (A_j, B_j)$, we then get that $p \mid M_\ell$ for some $\ell \neq j$. Since $M_\ell \mid N + \ell$, we get that $p \mid N + \ell$. But obviously $p \mid A_j \mid N + j$; therefore $p \mid (N + \ell) - (N + j) = (\ell - j)$, and $1 \leq |\ell - j| < k$. Thus $p < k$, which is impossible because all prime factors of M exceed k, proving the claim.

Now note that since $N + k \leq M$, we have

$$B_j \lambda + A_j \leq \frac{1}{M_j} (M\lambda + N + k) < \frac{2M\lambda}{M_j} \leq \frac{4M^2}{M_j} < M^2$$

for all $\lambda \in [M, 2M]$ and $j = 1, \ldots, k$, when $k \geq 3$, because in this case all primes dividing M exceed 4 and $N + k < M$. Thus, writing $\tau(m)$ for the number of divisors of m, we obtain

$$\tau(B_j \lambda + A_j) \leq 2 \sum_{\substack{d \mid B_j \lambda + A_j \\ d \leq M}} 1.$$

Summing the above inequality over all $\lambda \in [M, 2M]$ and changing the order of summation, we find that

$$\begin{aligned}
\sum_{\lambda\in[M,2M]} \tau(B_j\lambda + A_j) &\leq 2\sum_{\lambda\in[M,2M]}\sum_{\substack{d|B_j\lambda+A_j\\ d\leq M}} 1 \leq 2\sum_{d\leq M}\sum_{\substack{\lambda\in[M,2M]\\ B_j\lambda+A_j\equiv 0 \pmod d}} 1\\
&\leq 2\sum_{d\leq M}\left(\left\lfloor\frac{M}{d}\right\rfloor + 1\right) \leq 4M\sum_{d\leq M}\frac{1}{d}\\
&\leq 4M(\log M + 1). \qquad (7)
\end{aligned}$$

In the above chain of inequalities, we used the fact that, since A_j and B_j are coprime, the congruence $B_j\lambda + A_j \equiv 0 \pmod d$ has at most $\lfloor M/d\rfloor + 1$ solutions $\lambda \in [M, 2M]$. This is true assuming that d and B_j are coprime. When d and B_j are not coprime, then this congruence has no integer solution λ. Thus, if λ is such that $\omega(B_j\lambda + A_j) \geq A$, then $\tau(B_j\lambda + A_j) \geq 2^A$ and inequality (7) shows that

$$\#\{\lambda \in [M, 2M] : \omega(B_j\lambda + A_j) \geq A\} \leq \frac{4M(\log M + 1)}{2^A}.$$

Summing the above inequality over $j = 1, \ldots, k-1$, we get that

$$\sum_{j=1}^{k-1} \#\{\lambda \in [M, 2M] : \omega(B_j\lambda + A_j) \geq A\} \leq \frac{4(k-1)M(\log M + 1)}{2^A}.$$

Hence, assuming that

$$M > \frac{4(k-1)M(\log M + 1)}{2^A}, \qquad (8)$$

we see that there exists a number $\lambda \in [M, 2M]$ such that all inequalities (6) are satisfied, and therefore

$$n < n + 1 = M\lambda + N + 1 < 2M^2 + M < 3M^2. \qquad (9)$$

It remains to estimate the size of the minimal integer A depending on k such that inequality (8) holds. Clearly, M has $Ak(k+1)/2$ prime factors, which are all the consecutive primes starting with the first one exceeding k. Thus, by the Prime Number Theorem,

$$M = \exp((1/2 + o(1))k^2 A(\log k^2 A))$$

as $k \to \infty$ uniformly in $A \geq 1$. Thus, inequality (8) is fulfilled when

$$A\log 2 > \log(4(k-1)) + \log(\log M + 1) = (3 + o(1))\log k + O(\log\log k + \log A).$$

This shows that given $\varepsilon > 0$, we may choose $A = \lfloor(3/\log 2 + \varepsilon)\log k\rfloor$, and then inequality (8) is fulfilled once $k > k_\varepsilon$. With this choice of A, we have that

$$M < \exp((3/\log 2 + 2\varepsilon)k^2(\log k)^2)$$

provided that k is sufficiently large, and now inequality (9) shows that

$$n < \exp((6/\log 2 + 5\varepsilon)k^2(\log k)^2)$$

if k is sufficiently large with respect to ε, which implies the desired estimate as $k \to \infty$, since $\varepsilon \in (0, 1)$ may be chosen arbitrarily small.

We now turn our attention to the upper bound for m_k. We follow the same line of attack, based on the Chinese Remainder Theorem, although the details are somewhat different.

We assume again that $k \geq 2$; we take $M_0 = (k!)^2$, $M_j = q_j^{jA}$ for $j = 1, \ldots, k$, and let N be the smallest positive integer m in the arithmetic progression

$$m + j \equiv 0 \pmod{M_j}, \qquad j = 0, \ldots, k.$$

Here,

$$M = \prod_{j=0}^{k} M_j = (k!)^2 \prod_{j=1}^{k} q_j^{jA} = \exp((1/2 + o(1))k^2 A \log k) \tag{10}$$

as $k \to \infty$. Let $m = M\lambda + N$ again be such that $\lambda \in [M, 2M]$. Then

$$m + i = iM_i \left(\frac{M}{iM_i}\lambda + \frac{N+i}{iM_i} \right), \qquad \text{for all } i = 1, \ldots, k,$$

so that if we set $A_i = (N+i)/(iM_i)$ and $B_i = M/(iM_i)$, we have

$$\Omega(m+i) = \Omega(i) + \Omega(M_i) + \Omega(B_i\lambda + A_i).$$

Now since $i \leq k$, it follows that $\Omega(i) \leq (\log k)/\log 2$. Furthermore, $\Omega(M_i) = iA$. Thus, if

$$\Omega(B_i\lambda + A_i) < A - (\log k)/\log 2, \qquad \text{for all } i = 1, \ldots, k-1, \tag{11}$$

then

$$\Omega(m+i) < A(i+1) = \Omega(M_{i+1}) < \Omega(m+i+1), \qquad \text{for all } i = 1, \ldots, k-1,$$

which certainly shows that $\Omega(m+1), \ldots, \Omega(m+k)$ are pairwise distinct.

Now let $i \in \{1, \ldots, k\}$. As in the analysis of the n_k case, one shows that A_i and B_i are coprime and that $B_i\lambda + A_i < M\lambda + N + k < 2M^2 + M < 3M^2$. Furthermore, since M_0/i is a divisor of B_i for all $i = 1, \ldots, k$ and $M_0/i = (k!)^2/i$ is divisible by all primes $p \leq k$, it follows that the smallest prime factor of $B_i\lambda + A_i$ exceeds k. Write

$$B_i\lambda + A_i = U_i V_i,$$

where all prime factors of U_i are $\leq M^{1/2}$ and all prime factors of V_i are $> M^{1/2}$. Clearly, $\Omega(V_i) \leq 4$ because $M > 9$. We will now bound from above the number of λ such that U_i is not squarefree for some $i = 1, \ldots, k$. There exists a prime $p \in [k, M^{1/2}]$ such that $B_i\lambda + A_i \equiv 0 \pmod{p^2}$. For a fixed prime p, the number of integers $\lambda \in [M, 2M]$ for which the above congruence holds is at most $\lfloor M/p^2 \rfloor + 1 \leq 2M/p^2$. Thus,

$$\begin{aligned} \#\{\lambda \in [M, 2M] : p^2 \mid B_i\lambda + A_i \text{ for some } p \in [k, M^{1/2}]\} \quad &\leq \quad 2M \sum_{p>k} \frac{1}{p^2} \\ &\ll \quad \frac{M}{k \log k} \end{aligned}$$

uniformly in $i \in \{1, \ldots, k\}$. Summing this over all $i \in \{1, \ldots, k\}$, we get that

$$\sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : U_i \text{ is not squarefree}\} \ll \frac{M}{\log k}.$$

In particular, if k is large, then

$$\sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : \Omega(B_i\lambda + A_i) > \omega(B_i\lambda + A_i) + 4\} < \frac{M}{2}.$$

Let λ be some number in $[M, 2M]$ such that $\Omega(B_i\lambda + A_i) \le \omega(B_i\lambda + A_i) + 4$. As we have seen, there are at least $M/2$ such values for λ. If there is such a positive integer λ with the additional property that

$$\omega(B_i\lambda + A_i) < A - \frac{\log k}{\log 2} - 4, \qquad \text{for all } i = 1, \ldots, k, \tag{12}$$

it follows that inequalities (11) are satisfied. So, let us look at the number of $\lambda \in [M, 2M]$ such that at least one of the inequalities (12) fails. The argument used in the proof of the upper bound for n_k (based on the fact that $\tau(m) \ge 2^{\omega(m)}$) shows that

$$\begin{aligned}\sum_{i=1}^{k} \#\{\lambda \in [M, 2M] : \omega(B_i\lambda + A_i) \ge A - (\log k)/\log 2 - 4\} \\ \le \frac{4(k-1)M(\log M)}{2^{A-(\log k)/(\log 2)-4}}.\end{aligned}$$

Thus, if

$$\frac{4(k-1)M(\log M)}{2^{A-(\log k)/(\log 2)-4}} < \frac{M}{2}, \tag{13}$$

then the number of $\lambda \in [M, 2M]$ such that at least one of the inequalities (12) fails is $< M/2$. Since we have $\ge M/2$ values of λ to choose from, it follows that one can indeed choose such a value of λ for which all inequalities in (11) hold. Clearly, with such a value of λ, we have that $m_k \le m = M\lambda + N < 3M^2$. Inequality (13) is equivalent, via estimate (10), to

$$A \log 2 - (\log k) - 4 \log 2 > \log(8(k-1)) + 2 \log k + O(\log\log k + \log A),$$

which holds if we first fix $\varepsilon > 0$, then take $k > k_\varepsilon$, and finally choose $A = \lfloor (4/\log 2 + \varepsilon) \log k \rfloor$. With this choice of A, we have

$$M < \exp((2/\log 2 + 2\varepsilon)k^2(\log k)^2)$$

once $k > k_\varepsilon$. Therefore,

$$m_k < 3M^2 < \exp((4/\log 2 + 5\varepsilon)k^2(\log k)^2)$$

if k is large with respect to ε, which implies the desired inequality since $\varepsilon > 0$ can be chosen arbitrarily small. $\square$

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