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COMPACT FACTORIZATION OF DIFFERENTIABLE MAPPINGS

RAFFAELLA CILIA, JOAQUÍN M. GUTIÉRREZ, AND GIUSEPPE SALUZZO

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Abstract. Results on factorization (through linear operators) of polynomials and holomorphic mappings between Banach spaces have been obtained in recent years by several authors. In the present paper, we obtain a factorization result for differentiable mappings through compact operators. Namely, we prove that a mapping $f : X \to Y$ between real Banach spaces is differentiable and its derivative f' is a compact mapping with values in the space $\mathcal{K}(X,Y)$ of compact operators from X into Y if and only if f may be written in the form $f = g \circ S$, where the intermediate space is normed, S is a precompact operator, and g is a Gâteaux differentiable mapping with some additional properties. We also show that if f' is uniformly continuous on bounded sets and takes values in $\mathcal{K}(X,Y)$, then f' is compact if and only if f is weakly uniformly continuous on bounded sets.

1. Introduction

It was proved in [GG1, Theorem 16] that a holomorphic mapping $f : X \to Y$ between complex Banach spaces is weakly uniformly continuous on bounded subsets if and only if there exist a Banach space Z, a compact (linear) operator $S : X \to Z$, and a holomorphic mapping $g : Z \to Y$ such that $f = g \circ S$.

It is also well-known [A, Theorem 1.7] that a holomorphic function $f : X \to \mathbb{C}$ is weakly uniformly continuous on bounded subsets if and only if its derivative $f' : X \to X^*$ is a compact mapping.

In the present paper, we give similar results in the setting of differentiable mappings, thus giving a partial answer to a question of [GG2, page 338]. In Section 2, we prove that, given a mapping $f : X \to Y$ between real Banach spaces, f is differentiable and its derivative f' is a compact mapping with values in the space $\mathcal{K}(X,Y)$ of compact operators from X into Y if and only if f may be written in the form $f = g \circ S$, where the intermediate space is normed, S is a precompact operator, and g is a Gâteaux differentiable mapping with some additional properties. In Section 3, we show that if f' is uniformly continuous on bounded sets and

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takes values in $\mathcal{K}(X,Y)$, then f' is compact if and only if f is weakly uniformly continuous on bounded sets.

Throughout this paper, X and Y denote real Banach spaces, and $U \subseteq X$ is an open subset. The symbol B_X stands for the closed unit ball of X. Given $x \in X$ and $r > 0$, $B(x,r)$ represents the open ball of radius r centered at x. If $B \subset X$ is a subset, the notation $\mathrm{co}(B)$ is used for the convex hull of B. Given $x, y \in X$, we write $I(x,y)$ for the segment with bounds x and y, that is, $I(x,y) := \mathrm{co}(\{x,y\})$. By $\mathcal{L}(X,Y)$ we denote the space of all (bounded linear) operators from X into Y, endowed with the supremum norm. Its subspace of all compact operators is represented by $\mathcal{K}(X,Y)$. The sequence $(e_n)_{n=1}^{\infty}$ is the unit vector basis of c_0.

We say that a subset $B \subset U$ is *U-bounded* if it is bounded and the distance $\mathrm{dist}(B, \partial U)$ between B and the boundary ∂U of U is strictly positive. In the case $U = X$, a subset $B \subset X$ is U-bounded if and only if it is bounded.

A mapping $f : U \to Y$ is *compact* if it takes U-bounded subsets of U into relatively compact subsets of Y.

We denote by $C_{\mathrm{wbu}}(U,Y)$ the space of all mappings $f : U \to Y$ which are weakly uniformly continuous on U-bounded subsets of U, that is, for each U-bounded set $B \subset U$ and each $\epsilon > 0$, there are $\varphi_1, \ldots, \varphi_k \in X^*$ and $\delta > 0$ such that if $x, y \in B$ satisfy $|\varphi_i(x-y)| < \delta$ $(i = 1, \ldots, k)$, then $\|f(x) - f(y)\| < \epsilon$. If the space Y is omitted, it is understood to be the real field $\mathbb{R}$. An operator is weakly uniformly continuous on bounded sets if and only if it is compact [AP, Proposition 2.5].

Given a mapping $f : U \to Y$ and a class $\mathcal{M}$ of subsets of U such that every singleton belongs to $\mathcal{M}$, we say that f is $\mathcal{M}$-differentiable at $x \in U$ if there exists an operator $f'(x) \in \mathcal{L}(X,Y)$ such that

$$\lim_{\epsilon \to 0} \frac{f(x+\epsilon y) - f(x) - f'(x)(\epsilon y)}{\epsilon} = 0$$

uniformly with respect to y on each member of $\mathcal{M}$ [Y, §1.2]. In this case, we write $f \in D_{\mathcal{M}}(x,Y)$.

We say that f is *(Fréchet) differentiable at x* if $f \in D_{\mathcal{M}}(x,Y)$, where $\mathcal{M}$ is the class of all bounded subsets of X. We say that f is *Gâteaux differentiable at x* if $f \in D_{\mathcal{M}}(x,Y)$, where $\mathcal{M}$ is the class of all single-point subsets of X.

We say that f is *differentiable (respectively, Gâteaux differentiable) on U* if it is differentiable (respectively, Gâteaux differentiable) at every point $x \in U$.

2. The factorization theorem

If K is a bounded subset of $\mathcal{L}(X,Y)$, we construct a normed space in the spirit of [CDDL]. We define a continuous seminorm on X by

$$\|x\|_K := \sup_{\phi \in K} \|\phi(x)\| \qquad \text{for all } x \in X\,.$$

Then the set

$$V_K := \{x \in X : \|x\|_K = 0\}$$

is a closed subspace of X. Let π be the canonical quotient map of X onto the quotient space X/V_K. We define a norm on X/V_K by

$$\|\pi(x)\| := \|x\|_K \qquad (x \in X)\,. \tag{2.1}$$

This construction was used in [BGV] to factor multilinear mappings and polynomials. We shall use it here to factor differentiable mappings.

Theorem 2.1. *Let $f : X \to Y$ be a mapping between real Banach spaces. Then the following assertions are equivalent:*

(a) *f is differentiable, and $f' : X \to \mathcal{K}(X,Y)$ is compact.*

(b) *There exist a normed space Z, a surjective operator $S : X \to Z$, and a mapping $g : Z \to Y$ such that:*

(i) *$f(x) = g(S(x))$ for all $x \in X$;*

(ii) *S is a precompact operator;*

(iii) *$g \in D_{\mathcal{M}}(S(x), Y)$ for every $x \in X$, where*

$$\mathcal{M} := \{S(B) :\ B \text{ is a bounded subset of } X\} ;$$

(iv) *g' is bounded on $S(nB_X)$ for all $n \in \mathbb{N}$;*

(v) *for every $n \in \mathbb{N}$, the set $\{g'(S(x)) \circ S :\ x \in nB_X\}$ is relatively compact in $\mathcal{K}(X,Y)$.*

Proof. (a) $\Rightarrow$ (b). Let

$$K := \bigcup_{n\in\mathbb{N}} \frac{\overline{f'(nB_X)}}{n\|f'\|_{nB_X}} \cup \{0\} .$$

As in the proof of [AS, Proposition 3.5], we observe that K is compact in $\mathcal{K}(X,Y)$ and its span contains the set $f'(X)$. Let $Z := X/V_K$ be endowed with the norm introduced in (2.1), and denote by $S : X \to Z$ the quotient map π.

First of all we prove that S is a precompact operator. Let $(x_n)_n$ be a bounded sequence in X and let $M := \sup_n \|x_n\|$. Fix $\epsilon > 0$. Since K is compact, there exist $\phi_1, \ldots, \phi_p \in K$ such that $K \subset \bigcup_{i=1}^{p} B(\phi_i, \epsilon/(4M))$. Since $\phi_1, \ldots, \phi_p$ are compact operators, we can find a subsequence of (x_n), which we still denote by (x_n), such that $(\phi_i(x_n))_{n=1}^{\infty}$ is a convergent sequence in Y for every $i = 1, \ldots, p$. So there exists $\nu \in \mathbb{N}$ such that

$$\|\phi_i(x_n) - \phi_i(x_m)\| < \frac{\epsilon}{2} \qquad (i = 1, \ldots, p;\ n, m > \nu) .$$

Given $\phi \in K$, there is $i \in \{1, \ldots, p\}$ such that $\|\phi_i - \phi\| \le \epsilon/(4M)$. If $n, m > \nu$, we have

$$\|\phi(x_n) - \phi(x_m)\| \le \|(\phi - \phi_i)(x_n - x_m)\| + \|\phi_i(x_n - x_m)\| \le \frac{\epsilon}{4M}\, 2M + \frac{\epsilon}{2} = \epsilon .$$

Therefore,

$$\|S(x_n - x_m)\| = \sup_{\phi\in K} \|\phi(x_n - x_m)\| \le \epsilon .$$

So $S(x_n)$ is a Cauchy sequence in Z and thus S is precompact, which proves (ii).

Now we define $g : Z \to Y$ by

$$g(S(x)) := f(x) \qquad (x \in X) .$$

To see that g is well defined, suppose that $\|S(x - y)\| = 0$. Then

$$\|f'(c)(x - y)\| = 0 \qquad (c \in X) ,$$

since the span of K contains the range of f'. By the Mean Value Theorem [C, Theorem 6.4],

$$\|f(x) - f(y)\| \le \sup_{c\in I(x,y)} \|f'(c)(x - y)\| = 0 .$$

So $f(x) = f(y)$, and (i) is proved.

Given $x, y \in X$, the following limit exists:

$$\lim_{t\to 0} \frac{g(S(x)+tS(y))-g(S(x))}{t} = \lim_{t\to 0} \frac{g(S(x+ty))-g(S(x))}{t} = \lim_{t\to 0} \frac{f(x+ty)-f(x)}{t} = f'(x)(y). \tag{2.2}$$

For fixed $x \in X$, the mapping

$$g'(S(x)) : Z = S(X) \longrightarrow Y$$

defined by

$$g'(S(x))(S(y)) := f'(x)(y) \qquad (y \in X)$$

is linear. Moreover, choosing $n \in \mathbb{N}$ such that $x \in nB_X$, we have

$$\begin{aligned} \|g'(S(x))(S(y))\| &= \|f'(x)(y)\| \\ &\leq n\|f'\|_{nB_X} \sup_{\phi\in K} \|\phi(y)\| \\ &= n\|f'\|_{nB_X} \|S(y)\| . \end{aligned} \tag{2.3}$$

It follows that $g'(S(x)) \in \mathcal{L}(Z, Y)$, so g is Gâteaux differentiable. Moreover, since f is differentiable, for every bounded subset B of X, the limit in formula (2.2) exists uniformly with respect to $S(y)$ in $S(B)$. Therefore $g \in D_{\mathcal{M}}(S(x), Y)$ where $\mathcal{M} := \{S(B) : \ B \subset X \text{ is bounded}\}$, and (iii) is proved. From the inequality (2.3), we have

$$\|g'(S(x))\| = \sup_{\|S(y)\|\leq 1} \|g'(S(x))(S(y))\| \leq n\|f'\|_{nB_X} \qquad (x \in nB_X)\, ,$$

and (iv) is proved.

Since f' is compact, from the equality

$$g'(S(nB_X)) \circ S = f'(nB_X) \qquad (n \in \mathbb{N})\, ,$$

(v) follows.

(b) $\Rightarrow$ (a). Suppose that there exist a normed space Z, an operator S from X onto Z, and a mapping $g : Z \to Y$ satisfying conditions (i)–(v) of (b). Then, clearly, f is Gâteaux differentiable. Moreover, it is Fréchet differentiable by (iii). For $x \in X$, since $f'(x) = g'(S(x)) \circ S$, we have $f'(x) \in \mathcal{K}(X, Y)$. Since

$$\{f'(kx) : \ x \in B_X\} = g'(S(kB_X)) \circ S$$

is a relatively compact subset of $\mathcal{K}(X, Y)$, f' takes bounded subsets of X into relatively compact subsets of $\mathcal{K}(X, Y)$. □

Remark 2.2. The mapping g in Theorem 2.1 is uniformly continuous on $S(nB_X)$ for every $n \in \mathbb{N}$. Indeed, for $x, y \in nB_X$, we have

$$\begin{aligned} \|g(S(x)) - g(S(y))\| &= \|f(x) - f(y)\| \\ &\leq \sup_{c\in I(x,y)} \|f'(c)(x-y)\| \\ &\leq \sup_{c\in nB_X} \|f'(c)(x-y)\| \\ &\leq n\|f'\|_{nB_X} \sup_{\phi\in K} \|\phi(x-y)\| \\ &= n\|f'\|_{nB_X} \|S(x-y)\| . \end{aligned}$$

Remark 2.3. If we denote by

$$S_Y^* : \mathcal{L}(Z,Y) \longrightarrow \mathcal{K}(X,Y)$$

the operator given by

$$S_Y^*(T) := T \circ S \quad \text{for all } T \in \mathcal{L}(Z,Y)\,,$$

then we clearly have

$$f'(x) = g'(S(x)) \circ S = S_Y^* \circ g' \circ S(x) \qquad (x \in X)\,,$$

so the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\;f'\;} & \mathcal{K}(X,Y) \\ {\scriptstyle S}\big\downarrow & & \big\uparrow{\scriptstyle S_Y^*} \\ Z & \xrightarrow[\;g'\;]{} & \mathcal{L}(Z,Y) \end{array}$$

In the scalar-valued case ($Y = \mathbb{R}$), S_Y^* is none other than the adjoint S^* of S, which is compact, and condition (v) of Theorem 2.1 becomes superfluous.

The following example shows that the mapping g in Theorem 2.1 is not necessarily continuous.

Example 2.4. Let $h \in C^1(\mathbb{R})$ be given by

$$h(x) := \begin{cases} x^2/2\,, & \text{if } x \geq 0 \\ -x^2/2\,, & \text{if } x < 0\,. \end{cases}$$

Then $h'(x) = |x|$.

Define $f : c_0 \to \mathbb{R}$ by

$$f(x) := \sum_{n=1}^{\infty} \frac{h(x_n)}{2^n} \qquad (x = (x_n)_{n=1}^{\infty} \in c_0)\,.$$

It is easy to see that f is differentiable and that

$$f'(x) = \left(\frac{|x_n|}{2^n}\right)_{n=1}^{\infty} \in \ell_1\,.$$

Let $k \in \mathbb{N}$. Then

$$\sup_{x \in kB_{c_0}} \sum_{m=n}^{\infty} |\langle f'(x), e_m \rangle| = \sup_{x \in kB_{c_0}} \sum_{m=n}^{\infty} \frac{|x_m|}{2^m} \leq k \sum_{m=n}^{\infty} \frac{1}{2^m} \xrightarrow[n\to\infty]{} 0\,,$$

so $f' : c_0 \to \ell_1$ is compact.

Let

$$K := \bigcup_{n \in \mathbb{N}} \overline{\frac{f'(nB_{c_0})}{n\|f'\|_{nB_{c_0}}}} \cup \{0\} \subset \ell_1\,.$$

For each $p \in \mathbb{N}$,

$$M_p := \|f'\|_{pB_{c_0}} = \sup_{x \in pB_{c_0}} \sum_{n=1}^{\infty} \frac{|x_n|}{2^n} > \frac{p}{2}\,.$$

Factor $f = g \circ S$ as in the proof of Theorem 2.1. Suppose that g is continuous at the origin. Then, given $\epsilon > 0$, there is $\delta > 0$ such that

$$\|S(y) - S(0)\| \leq \delta \quad \Rightarrow \quad |f(y) - f(0)| = |g(S(y)) - g(S(0))| \leq \epsilon\,.$$

Choose $r \in \mathbb{N}$ with

$$\frac{1}{r} < \frac{\delta}{4} \qquad \text{and} \qquad \frac{2^{r-1}}{r^2} > \epsilon\,.$$

Let

$$y := \frac{2^r}{r}\, e_r \in c_0\,,$$

and let $\phi \in K$. Then there are $s \in \mathbb{N}$ and $x \in sB_{c_0}$ such that

$$\left\| \phi - \frac{f'(x)}{sM_s} \right\| < \frac{\delta}{2}\,\frac{r}{2^r}\,.$$

Hence,

$$\begin{aligned} |\phi(y)| &\le \left| \phi(y) - \frac{f'(x)(y)}{sM_s} \right| + \left| \frac{f'(x)(y)}{sM_s} \right| \\ &\le \left\| \phi - \frac{f'(x)}{sM_s} \right\| \|y\| + \left| \frac{f'(x)(y)}{sM_s} \right| \\ &\le \frac{\delta}{2}\,\frac{r}{2^r}\,\frac{2^r}{r} + \left| \frac{\langle f'(x), e_r\rangle}{sM_s} \right| \,\frac{2^r}{r} \\ &\le \frac{\delta}{2} + \frac{2^r}{r}\,\frac{1}{s}\,\frac{2}{s}\,\frac{|x_r|}{2^r} \qquad (\text{since } |x_r| \le s) \\ &\le \frac{\delta}{2} + \frac{1}{r}\,\frac{2}{s} \\ &< \delta\,. \end{aligned}$$

Taking suprema over $\phi \in K$, we have

$$\|S(y) - S(0)\| = \|S(y)\| \le \delta\,.$$

Therefore, $|f(y) - f(0)| \le \epsilon$, which implies

$$\epsilon \ge |f(y) - f(0)| = \frac{2^{r-1}}{r^2}\,,$$

a contradiction.

We do not know if f admits a compact factorization through a Banach space.

3. Weakly continuous differentiable mappings

In this section, we give characterizations of the differentiable mappings whose restrictions to bounded subsets are weakly uniformly continuous, in terms of their derivatives.

Theorem 3.1. *Let $U \subseteq X$ be an open convex subset, and let $f : U \to Y$ be a differentiable mapping such that f' is uniformly continuous on U-bounded sets. Then the following assertions are equivalent:*

(a) *$f \in C_{\mathrm{wbu}}(U, Y)$ and f' is $\mathcal{K}(X, Y)$-valued;*

(b) *$f' \in C_{\mathrm{wbu}}(U, \mathcal{K}(X, Y))$;*

(c) *$f' : U \to \mathcal{K}(X, Y)$ is a compact mapping;*

(d) *f' is $\mathcal{K}(X, Y)$-valued and, for every U-bounded subset $B \subset U$, there exists a compact set $K \subset \mathcal{K}(X, Y)$ such that*

$$\|f(x) - f(y)\| \le \|x - y\|_K \qquad (x, y \in B)\,.$$

Proof. (a) $\Rightarrow$ (b). This is shown by an adaptation of the proof of [AP, Proposition 3.6].

Given a U-bounded subset $B \subset U$ and $\epsilon > 0$, let $d := \min\{1, \mathrm{dist}(B, \partial U)\}$. The set

$$B' := B + \frac{d}{2} B_X \subset U$$

is U-bounded. The uniform continuity of f' on U-bounded subsets guarantees that there is $0 < \delta_1 < d/2$ such that if $x_1, x_2 \in B'$ with $\|x_1 - x_2\| < \delta_1$, then

$$\|f'(x_1) - f'(x_2)\| < \frac{\epsilon}{4} .$$

Since f is weakly uniformly continuous on B', there exist $\delta > 0$ and $\varphi_1, \dots, \varphi_k \in X^*$ such that whenever $x, y \in B'$ satisfy $|\varphi_i(x - y)| < \delta$ $(i = 1, \dots, k)$, then

$$\|f(x) - f(y)\| < \frac{\epsilon\delta_1}{16} .$$

Take $x, y \in B$ with $|\varphi_i(x - y)| < \delta$ $(i = 1, \dots, k)$. For $h \in B_X$, we have

$$\begin{aligned}
&\|f'(x)(h) - f'(y)(h)\| \\
&\quad = \frac{4}{\delta_1} \left\| f'(x)\left(\frac{\delta_1 h}{4}\right) - f'(y)\left(\frac{\delta_1 h}{4}\right) \right\| \\
&\quad \le \frac{4}{\delta_1} \left\| f'(x)\left(\frac{\delta_1 h}{4}\right) - f\left(x + \frac{\delta_1 h}{4}\right) + f(x) \right\| \\
&\qquad + \frac{4}{\delta_1} \left\| f\left(y + \frac{\delta_1 h}{4}\right) - f(y) - f'(y)\left(\frac{\delta_1 h}{4}\right) \right\| \\
&\qquad + \frac{4}{\delta_1} \left\| f\left(x + \frac{\delta_1 h}{4}\right) - f\left(y + \frac{\delta_1 h}{4}\right) \right\| + \frac{4}{\delta_1} \|f(x) - f(y)\| .
\end{aligned}$$

Since

$$x,\ y,\ x + \frac{\delta_1 h}{4},\ y + \frac{\delta_1 h}{4} \in B'$$

and

$$\left| \varphi_i\left(x + \frac{\delta_1 h}{4}\right) - \varphi_i\left(y + \frac{\delta_1 h}{4}\right) \right| = |\varphi_i(x) - \varphi_i(y)| < \delta \qquad (i = 1, \dots, k) ,$$

we have

$$\left\| f\left(x + \frac{\delta_1 h}{4}\right) - f\left(y + \frac{\delta_1 h}{4}\right) \right\| < \frac{\epsilon\delta_1}{16} .$$

For every

$$z \in I\left(x, x + \frac{\delta_1 h}{4}\right) ,$$

we have

$$z \in B' \qquad \text{and} \qquad \|z - x\| < \frac{\delta_1}{4} .$$

By a consequence of the Mean Value Theorem (see [Die, (8.6.2)]), we can write

$$\begin{aligned}
\left\| f'(x)\left(\frac{\delta_1 h}{4}\right) - f\left(x + \frac{\delta_1 h}{4}\right) + f(x) \right\| &\le \sup_{z \in I\left(x, x + \frac{\delta_1 h}{4}\right)} \|f'(z) - f'(x)\| \left\| \frac{\delta_1 h}{4} \right\| \\
&< \frac{\epsilon}{4} \frac{\delta_1}{4} .
\end{aligned}$$

The same is true upon replacing x by y.

Therefore

$$\|f'(x)(h) - f'(y)(h)\| < \frac{4}{\delta_1}\,\frac{\epsilon}{4}\,\frac{\delta_1}{4} + \frac{4}{\delta_1}\,\frac{\epsilon}{4}\,\frac{\delta_1}{4} + \frac{4}{\delta_1}\,\frac{\epsilon\delta_1}{16} + \frac{4}{\delta_1}\,\frac{\epsilon\delta_1}{16} = \epsilon\,.$$

Taking suprema over $h \in B_X$ we obtain

$$\|f'(x) - f'(y)\| \leq \epsilon\,.$$

(b) $\Rightarrow$ (c) by [AP, Lemma 2.2].

(c) $\Rightarrow$ (d). Let $B \subset U$ be a U-bounded subset, and let $K := \overline{f'(\mathrm{co}(B))}$. Since $\mathrm{co}(B)$ is also U-bounded, K is a compact subset of $\mathcal{K}(X, Y)$. If $x, y \in B$, by the Mean Value Theorem [C, Theorem 6.4] we have

$$\begin{aligned} \|f(x) - f(y)\| &\leq \sup_{c \in I(x,y)} \|f'(c)(x-y)\| \\ &\leq \sup_{c \in \mathrm{co}(B)} \|f'(c)(x-y)\| \\ &\leq \sup_{\phi \in K} \|\phi(x-y)\| \\ &= \|x-y\|_K\,. \end{aligned}$$

(d) $\Rightarrow$ (a). Let $B \subset U$ be a U-bounded set, and let $(x_\alpha) \subset B$ be a weak Cauchy net. Take $M > 0$ such that $\|x_\alpha\| \leq M$ for all α. Let $K \subset \mathcal{K}(X, Y)$ be the compact set associated with B by (d). Given $\epsilon > 0$, there exist $\phi_1, \ldots, \phi_k \in K$ such that $K \subset \bigcup_{i=1}^k B(\phi_i, \epsilon/(4M))$. For $\phi \in K$, we can find $i \in \{1, \ldots, k\}$ such that $\|\phi - \phi_i\| < \epsilon/(4M)$. Then we have

$$\begin{aligned} \|\phi(x_\alpha) - \phi(x_\beta)\| &\leq \|(\phi - \phi_i)(x_\alpha - x_\beta)\| + \|\phi_i(x_\alpha - x_\beta)\| \\ &\leq \frac{\epsilon}{4M}\,\|x_\alpha - x_\beta\| + \|\phi_i(x_\alpha - x_\beta)\| \\ &\leq \frac{\epsilon}{2} + \max_{1 \leq j \leq k} \|\phi_j(x_\alpha - x_\beta)\| \xrightarrow[\alpha,\beta]{} 0\,. \end{aligned}$$

Using (d), we obtain

$$\|f(x_\alpha) - f(x_\beta)\| \leq \|x_\alpha - x_\beta\|_K = \sup_{\phi \in K} \|\phi(x_\alpha - x_\beta)\| \xrightarrow[\alpha,\beta]{} 0\,.$$

Since bounded sets are weakly precompact [J, Corollary 8.1.6], we obtain by [GG1, Theorem 5] that f is weakly uniformly continuous on U-bounded sets. □

Clearly, if we take $U = X$ in Theorem 3.1, then the corresponding assertions of this theorem are equivalent to f having a factorization as in Theorem 2.1,(b).

A polynomial version of Theorem 3.1 can be found in [D, Proposition 2.6].

Remark 3.2. Note that in Theorem 3.1 the convexity of U is used only in (c) $\Rightarrow$ (d), while the uniform continuity of f' on U-bounded sets is needed only in (a) $\Rightarrow$ (b).

Remark 3.3. If we drop the condition of uniform continuity of f' on U-bounded sets, Theorem 3.1 may fail. Consider, for instance, the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} x^2 \sin \frac{1}{x^3}\,, & \text{if } x > 0 \\ 0\,, & \text{if } x \leq 0\,, \end{cases}$$

and

$$g(x) := \begin{cases} x^2 \sin \frac{1}{x}\,, & \text{if } x > 0 \\ 0\,, & \text{if } x \leq 0\,. \end{cases}$$

The function f is uniformly continuous on bounded sets, so $f \in C_{\mathrm{wbu}}(\mathbb{R})$; but

$$\lim_k f'\left(\frac{1}{\sqrt[3]{\pi + 2k\pi}}\right) = +\infty\,,$$

so f' takes a bounded set into an unbounded set. Therefore, f satisfies (a) but not (b). Note that f' is not continuous at the origin.

On the other hand, g' is compact since it is bounded on bounded sets, but it is not continuous at the origin, so it is not weakly continuous on bounded sets. Therefore, g satisfies (c) but not (b).

We conclude with an observation which is a consequence of work by Hájek.

Corollary 3.4. *Let X be a Banach space containing no copy of ℓ_1, and let $f : X \to \mathbb{R}$ be a differentiable function such that $f' : X \to X^*$ is uniformly continuous on bounded sets. Then the following assertions are equivalent:*

(a) *f takes weak Cauchy sequences into norm convergent sequences,*

(b) *$f \in C_{\mathrm{wbu}}(X)$,*

(c) *f admits a factorization as in Theorem* 2.1,*(b).*

Proof. (a) $\Rightarrow$ (b). Let $U_n := n\overset{\circ}{B}_X$ for each $n \in \mathbb{N}$. By [H, Lemma 5], $f'(U_n)$ is relatively compact, so f' is compact. By Theorem 3.1, $f \in C_{\mathrm{wbu}}(X)$.

(b) $\Leftrightarrow$ (c) by Theorems 3.1 and 2.1.

(b) $\Rightarrow$ (a) is obvious. □

One reason for the interest in this corollary is its relation with the result of [FGL, Proposition 9], which states that if X contains no copy of ℓ_1, then every mapping on X taking weakly convergent sequences into convergent sequences is weakly continuous on bounded sets. The implication (a) $\Rightarrow$ (b) of Corollary 3.4 is a "uniform" version of this proposition.

Remark 3.5. If X contains a copy of ℓ_1, then Corollary 3.4 fails. Indeed, the polynomial $P : \ell_1 \to \mathbb{R}$ defined by $P(x) := \sum_{n=1}^{\infty} x_n^2$ obviously satisfies (a), but its derivative is not compact, so it does not satisfy (b).

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Dipartimento di Matematica, Facoltà di Scienze, Università di Catania, Viale Andrea Doria 6, 95125 Catania, Italy

E-mail address: cilia@dmi.unict.it

Departamento de Matemática Aplicada, ETS de Ingenieros Industriales, Universidad Politécnica de Madrid, C. José Gutiérrez Abascal 2, 28006 Madrid, Spain

E-mail address: jgutierrez@etsii.upm.es

Dipartimento di Matematica, Facoltà di Scienze, Università di Catania, Viale Andrea Doria 6, 95125 Catania, Italy

E-mail address: saluzzo@dmi.unict.it