INDUCED QUASI-ACTIONS: A REMARK

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ABSTRACT. We observe that the notion of an induced representation has an analog for quasi-actions and give some applications.

1. INTRODUCTION

In this paper we observe that the notion of an induced representation has an analog for quasi-actions and give some applications.

We will use the definitions and notation from [KL01].

1.1. Induced quasi-actions and their properties. Let $G$ be a group and \{\(X_i\)\}_{i \in I} be a finite collection of unbounded metric spaces.

**Definition 1.1.** A quasi-action $G \curvearrowright \prod_{i \in I} X_i$ preserves the product structure if each $g \in G$ acts by a product of quasi-isometries, up to a uniformly bounded error. Note that we allow the quasi-isometries $\rho(g)$ to permute the factors; i.e. $\rho(g)$ is uniformly close to a map of the form $(x_i) \mapsto (\phi_{\sigma^{-1}(i)}(x_{\sigma^{-1}(i)}))$ with a permutation $\sigma$ of $I$ and quasi-isometries $\phi_i : X_i \to X_{\sigma(i)}$.

Associated to every quasi-action $G \curvearrowright \prod_{i \in I} X_i$ preserving product structure is the action $G \curvearrowright I$ corresponding to the induced permutation of the factors; this is well-defined because the $X_i$’s are unbounded metric spaces. For each $i \in I$, the stabilizer $G_i$ of $i$ with respect to $\rho_I$ has a quasi-action $G_i \curvearrowright X_i$ by restriction of $\rho$. It is well-defined up to equivalence in the sense of [KL01, Definition 2.3].

If the permutation action $\rho_I$ is transitive, all factors $X_i$ are quasi-isometric to each other, and the restricted quasi-actions $G_i \curvearrowright X_i$ are quasi-conjugate (when identifying different stabilizers $G_i$ by inner automorphisms of $G$). The main result of this note is that in this case any of the quasi-actions $G_i \curvearrowright X_i$ determines $\rho$ up to quasi-conjugacy, and moreover any quasi-conjugacy class may arise as a restricted action.

**Theorem 1.2.** Let $G$ be a group, $H$ be a finite index subgroup, and $H \curvearrowright X$ be a quasi-action of $H$ on an unbounded metric space $X$. Then there exists a quasi-action $G \curvearrowright \prod_{i \in G/H} X_i$ preserving product structure, where

1. Each factor $X_i$ is quasi-isometric to $X$. 

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The associated action $G^\cdot G/H \actson G/H$ is the natural action by left multiplication.

The restriction of $\beta$ to a quasi-action of $H$ on $X_H$ is quasi-conjugate to $H^\cdot X_H$. Furthermore, there is a unique such quasi-action $\beta$ preserving the product structure, up to quasi-conjugacy by a product quasi-isometry. Finally, if $\alpha$ is an isometric action, then the $X_i$ may be taken isometric to $X$ and $\beta$ may be taken to be an isometric action.

**Definition 1.3.** Let $G$, $H$ and $H \actson X$ be as in Theorem 1.2. The quasi-action $\beta$ is called the quasi-action induced by $H \actson X$.

As a byproduct of the main construction, we get the following:

**Corollary 1.1.** If $G^\cdot X$ is an $(L, A)$-quasi-action on an arbitrary metric space $X$, then $\rho$ is $(L, 3A)$-quasi-conjugate to a canonically defined isometric action $G^\cdot X'$. 

1.2. **Applications.** The implication of Theorem 1.2 is that in order to quasi-conjugate a quasi-action on a product to an isometric action, it suffices to quasi-conjugate the factor quasi-actions to isometric actions. We begin with a special case:

**Theorem 1.4.** Let $G^\cdot X$ be a cobounded quasi-action on $X = \prod_i X_i$, where each $X_i$ is either an irreducible symmetric space of noncompact type, or a thick irreducible Euclidean building of rank at least two, with cocompact Weyl group. Then $\rho$ is quasi-conjugate to an isometric action on $X$, after suitable rescaling of the metrics on the factors $X_i$.

**Remarks.**
- Theorem [1.4] was stated incorrectly as Corollary 4.5 in [KL01]. The proof given there was valid only for quasi-actions which do not permute the factors.
- Rescaling of the factors is necessary, in general: if one takes the product of two copies of $H^2$ where the factors are scaled to have different curvature, then a quasi-action which permutes the factors will not be quasi-conjugate to an isometric action.

We now consider a more general situation. Let $G^\cdot \prod_{i \in I} X_i$ be a quasi-action, where each $X_i$ is one of the following four types of spaces:

1. An irreducible symmetric space of noncompact type.
2. A bounded irreducible Euclidean building of rank/dimension $\geq 2$, with cocompact Weyl group.
3. A bounded valence bushy tree in the sense of [MSW03]. We recall that a tree is bushy if each of its points lies within a uniformly bounded distance from a vertex having at least three unbounded complementary components.
4. A quasi-isometrically rigid Gromov hyperbolic space which is of coarse type I in the sense of [KKL98, Sec. 3] (see the remarks below). A space is quasi-isometrically rigid if every $(L, A)$-quasi-isometry is at distance at most $D = D(L, A)$ from a unique isometry. Examples include rank 1 symmetric spaces other than hyperbolic and complex hyperbolic spaces [Pan89], Fuchsian buildings [BP00, Xie06], and fundamental groups of hyperbolic $n$-manifolds with nonempty totally geodesic boundary, $n \geq 3$ [KKLS, BKM].
By [KKL98] Theorem B, the quasi-action preserves the product structure, and hence we have an induced permutation action $G \curvearrowright I$. Let $J \subset I$ be the set of indices $i \in I$ such that $X_i$ is either a real hyperbolic space $\mathbb{H}^k$ for some $k \geq 4$, a complex hyperbolic space $\mathbb{CH}^l$ for some $l \geq 2$, or a bounded valence bushy tree. Generalizing Theorem [KL97, Lee00] we obtain:

**Theorem 1.5.** If the quasi-action $G_j \curvearrowright X_j$ is cobounded for each $j \in J$, then $\alpha$ is quasi-conjugate by a product quasi-isometry to an isometric action $G \curvearrowright \prod_{i \in I} X'_i$, where for every $i$, $X'_i$ is quasi-isometric to $X_i$, and precisely one of the following holds:

1. If $X_i$ is not a bounded valence bushy tree, then $X'_i$ is isometric to $X_i'$ for some $i'$ in the $G$-orbit $G(i)$ of $i$.
2. If $X_i$ is a bounded valence bushy tree, then so is $X'_i$.

As in the previous corollary, it is necessary to permit $X'_i$ to be nonisometric to $X_i$. Moreover, there may be factors $X_i$ and $X_j$ of type (4) lying in the same $G$-orbit, but which are not even homothetic, so it is not sufficient to allow rescaling of factors.

**Proof.** We first assume that the action $G \curvearrowright I$ is transitive. Pick $n \in I$. Then the quasi-action $G_n \curvearrowright X_n$ is quasi-conjugate to an isometric action $G_n \curvearrowright X'_n$, where $X'_n$ is isometric to $X_n$ unless $X_n$ is a bounded valence bushy tree, in which case $X'_n$ is a bounded valence bushy tree but not necessarily isometric to $X_n$; this follows from:

- [Hin90, Gab92, CJ94, Mar06] when $X_n$ is $\mathbb{H}^2$. Note that any quasi-action on $\mathbb{H}^2$ is quasi-conjugate to an isometric action.
- [Sul81, Gro, Tuk86, Pan89, Cho96] when $X_n$ is a rank 1 symmetric space other than $\mathbb{H}^2$. Note that Sullivan’s theorem implies that any quasi-action on $\mathbb{H}^3$ is quasi-conjugate to an isometric action. Also, the proof given in Chow’s paper on the complex hyperbolic case covers arbitrary cobounded quasi-actions, even though it is only stated for discrete cobounded quasi-actions.
- [KKL97, Lee00] when $X_n$ is an irreducible symmetric space or Euclidean building of rank at least 2.
- [MSW03] when $X_n$ is a bounded valence bushy tree.

By Theorem [KKL97, Lee00] the associated induced quasi-action of $G$ is quasi-conjugate to the original quasi-action $G \curvearrowright \prod_{i \in I} X_i$ by a product quasi-isometry, and we are done.

In the general case, for each orbit $G(i) \subset I$ of the action $G \curvearrowright I$, we have a well-defined associated quasi-action $G \curvearrowright \prod_{j \in G(i)} X_j$ for which the theorem has already been established, and we obtain the desired isometric action $G \curvearrowright \prod_{i \in I} X'_i$ by taking products.

**Corollary 1.2.** Let $\{X_i\}_{i \in I}$ be as above, and suppose $G$ is a finitely generated group quasi-isometric to the product $\prod_{i \in I} X_i$. Then $G$ admits a discrete, cocompact, isometric action on a product $\prod_{i \in I} X'_i$, where for every $i$, $X'_i$ is quasi-isometric to $X_i$, and precisely one of the following holds:

1. $X_i$ is not a bounded valence bushy tree, and $X'_i$ is isometric to $X_i'$ for some $i'$ in the $G$-orbit $G(i) \subset I$ of $i$.
2. Both $X_i$ and $X'_i$ are bounded valence bushy trees.
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Proof. Such a group $G$ admits a discrete, cobounded quasi-action on $\prod_{i \in I} X_i$. Theorem 1.5 furnishes the desired isometric action $G \acts \prod_i X_i$. □

Remarks. • Corollary 1.2 refines earlier results [Ahl02, KL01, MSW03].
• A proper Gromov hyperbolic space with cocompact isometry group is of coarse type I unless it is quasi-isometric to $\mathbb{R}$ [KKL98, Sec. 3].
• The classification of the four different types of spaces above is quasi-isometry invariant, with one exception: a space of type (1) will also be a space of type (4) iff it is a quasi-isometrically rigid rank 1 symmetric space (i.e. a quaternionic hyperbolic space or the Cayley hyperbolic plane [Pan89]). See Lemma 3.1.
• Two irreducible symmetric spaces are quasi-isometric iff they are isometric, up to rescaling [Mos73, Pan89, KL97]. Two Euclidean buildings as in (2) above are quasi-isometric iff they are isometric up to rescaling [KL97, Lee00].

2. The Construction of Induced Quasi-actions

The construction of induced quasi-actions is a direct imitation of one of the standard constructions of induced representations. We now review this for the convenience of the reader.

Let $H$ be a subgroup of some group $G$, and suppose $\alpha : H \acts V$ is a linear representation. Then we have an action $H \acts G \times V$ where $(h, (g, v)) = (gh^{-1}, hv)$. Let $E := (G \times V)/H$ be the quotient. There is a natural projection map $\pi : E \to G/H$ whose fibers are copies of $V$; this would be a vector bundle over the discrete space $G/H$ if $V$ were endowed with a topology. The action $G \acts G \times V$ by left translation on the first factor descends to $E$, and commutes with the projection map $\pi$. Moreover, it preserves the linear structure on the fibers. Hence there is a representation of $G$ on the vector space of sections $\Gamma(E)$, and this is the representation of $G$ induced by $\alpha$.

We use the terminology of [KL01, Sec. 2]. (However, we replace quasi-isometrically conjugate by the shorter and more accurate term quasi-conjugate.)

We will work with generalized metrics taking values in $[0, +\infty]$. A finite component of a generalized metric space is an equivalence class of points with pairwise finite distances. Clearly, quasi-isometries respect finite components.

Let $\{X_i\}_{i \in I}$ be a finite collection of unbounded metric spaces in the usual sense; i.e. the metric on each $X_i$ takes only finite values. On their product $\prod_{i \in I} X_i$ we consider the natural ($L^2$-)product metric. On their disjoint union $\bigsqcup_{i \in I} X_i$ we consider the generalized metric which induces the original metric on each component $X_i$ and gives distance $+\infty$ to any pair of points in different components.

We observe that a quasi-isometry $\prod_{i \in I} X_i \to \prod_{i \in I} X'_i$ preserving the product structure gives rise to a quasi-isometry $\bigsqcup_{i \in I} X_i \to \bigsqcup_{i \in I} X'_i$, well-defined up to a bounded error, and vice versa. Thus equivalence classes of quasi-actions $\alpha : G \acts \prod_{i \in I} X_i$ preserving the product structure correspond one-to-one to quasi-actions $\beta : G \acts \bigsqcup_{i \in I} X_i$. In what follows we will prove the disjoint union analog of Theorem 1.2 (The index of $H$ can be arbitrary from now on.)

Lemma 2.1. Suppose that $Y$ is a generalized metric space and that $G \acts Y$ is a quasi-action such that $G$ acts transitively on the set of finite components of $Y$. Let
Y₀ be one of the finite components and H its stabilizer in G. Then the restricted action H ↷ Y₀ determines the action G ↷ Y up to quasi-conjugacy.

Proof. If G ↷ Y' is another quasi-action, Y₀' is a finite component with stabilizer H, then any quasi-conjugacy between H ↷ Y₀ and H ↷ Y₀' extends in a straightforward way to a quasi-conjugacy between G ↷ Y and G ↷ Y'. □

We will now show how to recover the G-quasi-action from the H-quasi-action by quasifying the construction of induced actions as described above.

**Definition 2.2.** An (L, A)-coarse fibration (Y, ℱ) consists of a (generalized) metric space Y and a family ℱ of subsets F ⊂ Y, the coarse fibers, with the following properties:

1. The union ∪_{F ∈ ℱ} F of all fibers has Hausdorff distance ≤ A from Y.
2. For any fibers F₁, F₂ ∈ ℱ we have

   \[ d_H(F₁, F₂) ≤ L \cdot d(y₁, y₂) + A \quad ∀ y₁ ∈ F₁. \]

We also say that ℱ is a coarse fibration of Y.

Note that the coarse fibers are not required to be disjoint.

It follows from part (2) of the definition that d_H(F₁, F₂) < +∞ if and only if F₁ and F₂ meet the same finite component of Y. We will equip the “base space” ℱ with the Hausdorff metric.

**Lemma 2.3.** If H ↷ Y is an (L, A)-quasi-action, then the collection of quasi-orbits O_{y} := H · y forms an (L, 3A)-coarse fibration of Y.

Proof. For h, h₁, h₂ ∈ H and y₁, y₂ ∈ Y we have

\[ d(hy₁, (hh₁^{-1}h₂)y₂) ≤ d((hh₁^{-1})(h₁y₁), (hh₂y₂)) + 2A \]
\[ ≤ L \cdot d(h₁y₁, h₂y₂) + 3A \]

and so

\[ d(O_{y₁}, O_{y₂}) ≤ L \cdot d(h₁y₁, O_{y₂}) + 3A. \]

Let (Y, ℱ) and (Y', ℱ') be coarse fibrations. We say that a map φ : Y → Y' quasi-respects the coarse fibrations if the image of each fiber F ∈ ℱ is uniformly Hausdorff close to a fiber F' ∈ ℱ', d_H(φ(F), F') ≤ C. The map φ then induces a map \( \bar{φ} : ℱ \to ℱ' \) which is well-defined up to a bounded error ≤ 2C. Observe that if φ is an (L, A)-quasi-isometry, then \( \bar{φ} \) is an \( (L, A + 2C) \)-quasi-isometry.

We say that a quasi-action \( ρ : G ↷ Y \) quasi-respects a coarse fibration ℱ if all maps ρ(g) quasi-respect ℱ with uniformly bounded error. The quasi-action \( ρ \) then descends to a quasi-action \( \bar{ρ} : G ↷ ℱ \) which is unique up to equivalence (cf. [KL01, Definition 2.3]).

We apply these general remarks to the following situation in order to obtain our main construction.

Let G be a group, \( H < G \) a subgroup (of arbitrary index) and \( H ↷ X \) an (L, A)-quasi-action. Let Y = G × X where G is given the metric \( d(g₁, g₂) = +∞ \) unless \( g₁ = g₂ \). That is, Y consists of \( |G| \) finite components each of which is a copy of X. The quasi-action \( α \) gives rise to a product quasi-action \( H ↷ Y \) via

\[ ρ_H(h, (g, x)) = (gh^{-1}, hx). \]
We denote by $\mathcal{F}_H$ the coarse fibration of $Y$ by $H$-quasi-orbits. The isometric $G$-action given by

$$\hat{\rho}_G(g', (g, x)) = (g'g, x)$$

commutes with $\rho_H$. As a consequence, $\hat{\rho}_G$ descends to an isometric action

$$\hat{\beta} := \bar{\rho}_G : G \curvearrowright \mathcal{F}_H.$$ 

If $H = G$, then $\alpha$ is quasi-conjugate to $\hat{\beta}$ via the quasi-isometry $x \mapsto \rho_H(H) \cdot (e, x)$. This case is used to prove Corollary 1.1, where $X' = \mathcal{F}_H$.

In general, the finite components of $\mathcal{F}_H$ correspond to the left $H$-cosets in $G$. More precisely, $gH$ corresponds to $\bigcup_{x \in X} \rho_H(H) \cdot (g, x)$, that is, to the union of $\rho_H$-quasi-orbits contained in $gH \times X$. $H$ stabilizes the finite component $\bigcup_{x \in X} \rho_H(H) \cdot (e, x)$. The action of $H$ on this component is quasi-conjugate to $\alpha$.

As remarked in the beginning of this section, $\hat{\beta}$ is the unique $G$-quasi-action up to quasi-conjugacy such that $G$ acts transitively on finite components and such that $H$ is the stabilizer of a finite component and the restricted $H$-quasi-action is quasi-isometrically conjugate to $\alpha$.

Passing back from disjoint unions to products we obtain Theorem 1.2.

3. Quasi-isometries and the classification into types (1)-(4)

We now prove:

Lemma 3.1. Suppose $Y$ and $Y'$ are spaces of one of the types (1)-(4) as in Theorem 1.1. If $Y$ is quasi-isometric to $Y'$, then they have the same type, unless one is a quasi-isometrically rigid rank 1 symmetric space, and the other is of type (4).

Proof. First suppose one of the spaces is not Gromov hyperbolic. Since Gromov hyperbolicity is quasi-isometry invariant, both spaces must be higher rank spaces of either type (1) or (2). But by [KL97], two irreducible symmetric spaces or Euclidean buildings of rank at least two are quasi-isometric if they are homothetic. Thus in this case they must have the same type.

Now assume both spaces are Gromov hyperbolic. Then $\partial Y$ and $\partial Y'$ are homeomorphic.

If $Y$ is a bounded valence bushy tree, then it is well-known that $Y$ is quasi-isometric to a trivalent tree, and $\partial Y$ is homeomorphic to a Cantor set. Therefore $Y$ cannot be quasi-isometric to a space of type (1), since the boundary of a Gromov hyperbolic symmetric space is a sphere. Also, the quasi-isometry group of a trivalent tree $T$ has an induced action on the space of triples in $\partial T$ which is not proper, and hence it cannot be quasi-isometric to a space of type (4).

If $Y$ is a hyperbolic or complex hyperbolic space, then the induced action of $\text{QI}(X)$ on the space of triples in $\partial X$ is not proper, and hence $Y$ cannot be quasi-isometric to a space of type (4).

The lemma follows.

The references are

References


