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# THREEFOLDS CONTAINING BORDIGA SURFACES AS AMPLE DIVISORS

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Abstract. Let $L$ be an ample line bundle on a smooth complex projective variety $X$ of dimension three such that there exists a smooth member $Z$ of $|L|$. When the restriction $L_Z$ of $L$ to $Z$ is very ample and $(Z, L_Z)$ is a Bordiga surface, it is proved that there exists an ample vector bundle $\mathcal{E}$ of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$ and $3 \leq c_2(\mathcal{E}) \leq 10$ such that $(X, L) = (\mathbb{P}_{\mathbb{P}^2}(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle on the projective space bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ associated to $\mathcal{E}$.

## Introduction

In this paper varieties are always assumed to be defined over the field $\mathbb{C}$ of complex numbers.

Given a smooth projective variety $Z$, the classification of smooth projective varieties $X$ containing $Z$ as an ample divisor occupies an extremely important position in the theory of polarized varieties, and it is well-known that the structure of $Z$ imposes severe restrictions on that of $X$. Inspired by this philosophy, we set up the following condition $(*)$:

$(*)$ $L$ is an ample line bundle on a smooth projective variety $X$ such that there exists a smooth member $Z$ of $|L|$.

In this paper we treat Bordiga surfaces $(Z, L_Z)$ under the assumption $(*)$ when the restriction $L_Z$ of $L$ to $Z$ is very ample. Here $(Z, L_Z)$ with $L_Z$ very ample is called a Bordiga surface if $Z$ is a smooth projective surface obtained by the blowing-up $\sigma : Z \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $k$ distinct points $p_1, \ldots, p_k$ in general position $(0 \leq k \leq 10)$ and $L_Z = \sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \mathcal{O}_Z(\sum_{i=1}^k e_i)$, where $e_i = \sigma^{-1}(p_i)$ for $i = 1, \ldots, k$. When $L$ itself is very ample, if $(Z, L_Z)$ is a Bordiga surface, then it follows from [I, Theorem 4.2 and Proposition 4.7] and [LM2, Lemma 4] that there exists a very ample vector bundle $\mathcal{E}$ of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$ and $3 \leq c_2(\mathcal{E}) \leq 10$ such that $(X, L) = (\mathbb{P}_{\mathbb{P}^2}(\mathcal{E}), H(\mathcal{E}))$, where $H(\mathcal{E})$ is the tautological line bundle on the projective space bundle $\mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ associated to $\mathcal{E}$. The purpose of this paper is to generalize the above result when $L$ is simply supposed to be ample. The precise statement of our result is as follows:

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**Theorem.** *Let $L$ be an ample line bundle on a smooth projective variety $X$ of dimension three such that there exists a smooth member $Z$ of $|L|$. Assume that the restriction $L_Z$ of $L$ to $Z$ is very ample and that $(Z, L_Z)$ is a Bordiga surface. Then there exists an ample vector bundle $\mathcal{E}$ of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$ and $3 \leq c_2(\mathcal{E}) \leq 10$ such that $(X, L) = (\mathbb{P}_{\mathbb{P}^2}(\mathcal{E}), H(\mathcal{E}))$.*

This paper is organized as follows. In Section 1 we collect necessary material that will be used later. Sections 2 and 3 are devoted to the proof of the theorem. Concretely, in Section 2, under the assumption in the theorem we prove that there exists an ample vector bundle $\mathcal{E}$ of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$ and $1 \leq c_2(\mathcal{E}) \leq 10$ such that $(X, L) = (\mathbb{P}_{\mathbb{P}^2}(\mathcal{E}), H(\mathcal{E}))$. In Section 3 we show that $c_2(\mathcal{E}) \geq 3$.

When $\mathcal{E}$ is an ample vector bundle of rank $n - 2 \geq 2$ on a smooth projective variety $X$ of dimension $n$ such that there exists a global section $s$ of $\mathcal{E}$ whose zero locus $Z = (s)_0$ is a smooth surface on $X$ and $H$ is an ample line bundle on $X$ such that $H_Z$ is very ample, the triplets $(X, \mathcal{E}, H)$ are completely classified in [LM1] and [LM2] under the assumption that $(Z, H_Z)$ is a Bordiga surface. Consequently the theorem is regarded as a result when $n = 3$ and $\mathcal{E} = H$.

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## 1. Preliminaries

We use the standard notation from algebraic geometry. The tensor products of line bundles are denoted additively. The pullback $i^*\mathcal{E}$ of a vector bundle $\mathcal{E}$ on $X$ by an embedding $i : Y \hookrightarrow X$ is denoted by $\mathcal{E}_Y$. In particular, for a closed subvariety $V$ of $\mathbb{P}^N$, $(\mathcal{O}_{\mathbb{P}^N}(1))_V$ is denoted by $\mathcal{O}_V(1)$. For a vector bundle $\mathcal{E}$ on a projective variety $X$, the tautological line bundle on the projective space bundle $\mathbb{P}_X(\mathcal{E})$ associated to $\mathcal{E}$ is denoted by $H(\mathcal{E})$. A vector bundle $\mathcal{E}$ on a projective variety $X$ is said to be *ample* (respectively *very ample*) if $H(\mathcal{E})$ is ample (respectively very ample). We denote by $K_X$ the canonical bundle of a smooth variety $X$. A *polarized manifold* is a pair $(X, L)$ consisting of a smooth projective variety $X$ and an ample line bundle $L$ on $X$. The *sectional genus* $g(X, L)$ of a polarized manifold $(X, L)$ is defined by the formula $2g(X, L) - 2 = (K_X + (n-1)L)L^{n-1}$, where $n = \dim X$. A polarized manifold $(X, L)$ is called a *scroll* over a smooth projective variety $W$ if $(X, L) = (\mathbb{P}_W(\mathcal{E}), H(\mathcal{E}))$ for some ample vector bundle $\mathcal{E}$ on $W$. A polarized manifold $(X, L)$ is called a *Del Pezzo manifold* if $K_X + (\dim X - 1)L = \mathcal{O}_X$. A pair $(X, L)$ with $L$ very ample is called a *Bordiga surface* if $X$ is a smooth projective surface obtained by the blowing-up $\sigma : X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $k$ distinct points $p_1, \ldots, p_k$ in general position $(0 \leq k \leq 10)$ and $L = \sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \mathcal{O}_X(\sum_{i=1}^k e_i)$, where $e_i = \sigma^{-1}(p_i)$ for $i = 1, \ldots, k$.

First let us recall some numerical properties of adjoint bundles.

**Lemma 1.** *Let $L$ be an ample line bundle on a smooth projective variety $X$ of dimension $n \geq 1$.*

(i) *If $t \geq n + 1$, then $K_X + tL$ is always nef.*
(ii) *If $K_X + nL$ is not nef, then $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.*
(iii) *Assume that $K_X + nL$ is nef and that $n \geq 2$. If $K_X + (n-1)L$ is not nef, then $(X, L)$ is one of the following:*
(iii-1) *$X$ is a quadric hypersurface $\mathbb{Q}^n$ in $\mathbb{P}^{n+1}$, and $L = \mathcal{O}_{\mathbb{Q}^n}(1)$;*
(iii-2) *$(X, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$;*
(iii-3) *$(X, L)$ is a scroll over a smooth projective curve.*

(iv) *Assume that* $K_X + (n-1)L$ *is nef and that* $n \geq 3$. *If* $K_X + (n-2)L$ *is not nef, then* $(X, L)$ *is one of the following*:

(iv-1) *there exists an effective divisor* $E$ *on* $X$ *such that* $(E, L_E, (\mathcal{O}_X(E))_E) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1), \mathcal{O}_{\mathbb{P}^{n-1}}(-1))$;

(iv-2) $(X, L)$ *is a Del Pezzo manifold*;

(iv-3) $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$;

(iv-4) $(X, L) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$;

(iv-5) $(X, L) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$;

(iv-6) $X$ *is a* $\mathbb{P}^2$*-bundle over a smooth projective curve* $C$, *and* $L_F = \mathcal{O}_{\mathbb{P}^2}(2)$ *for any fiber* $F$ *of the bundle projection* $X \to C$;

(iv-7) *there exists a surjective morphism* $\pi : X \to C$ *onto a smooth projective curve* $C$ *with Picard number* $\rho(C) = \rho(X) - 1$ *such that any fiber* $D$ *of* $\pi$ *is a quadric hypersurface in* $\mathbb{P}^n$ *with* $L_D = \mathcal{O}_D(1)$;

(iv-8) $(X, L)$ *is a scroll over a smooth projective surface.*

*Proof.* We refer the reader to [F, Theorems 11.2, 11.7 and 11.8]. □

Second we need the following:

**Lemma 2.** *Assume that* $(X, L) = (\mathbb{P}_C(\mathcal{E}), H(\mathcal{E}))$ *for some* (*not necessarily ample*) *vector bundle* $\mathcal{E}$ *of rank* $n$ *on a smooth projective curve* $C$. *Then* $g(X, L) = g(C)$, *where* $g(C)$ *is the genus of* $C$.

*Proof.* Let $\pi : X \to C$ be the bundle projection. Then $K_X + nL = \pi^*(K_C + \det \mathcal{E})$. Furthermore, the Wu-Chern relation tells us that $L^n - L^{n-1}\pi^*(\det \mathcal{E}) = 0$. Thus

$$\begin{aligned} 2g(X, L) - 2 &= (K_X + (n-1)L)L^{n-1} = (-L + \pi^*(K_C + \det \mathcal{E}))L^{n-1} \\ &= -L^n + L^{n-1}\pi^*(K_C + \det \mathcal{E}) = -L^{n-1}\pi^*(\det \mathcal{E}) + L^{n-1}\pi^*(K_C + \det \mathcal{E}) \\ &= L^{n-1}\pi^* K_C = \deg K_C = 2g(C) - 2, \end{aligned}$$

i.e., $g(X, L) = g(C)$. □

Let $(X, L)$ be a Bordiga surface, that is to say, $L$ is a very ample line bundle on a smooth projective surface $X$ obtained by the blowing-up $\sigma : X \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $k$ distinct points $p_1, \ldots, p_k$ in general position ($0 \leq k \leq 10$), and $L = \sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \mathcal{O}_X(\sum_{i=1}^k e_i)$, where $e_i = \sigma^{-1}(p_i)$ for $i = 1, \ldots, k$. For $k \geq 1$, the $(-1)$-curves $e_i$ satisfy $Le_i = 1$. Conversely, we also need the following:

**Lemma 3.** *Let* $(X, L)$ *be a Bordiga surface as above, and let* $l$ *be a* $(-1)$*-curve on* $X$ *with* $Ll = 1$. *Then* $l = e_i$ *for some* $i$.

*Proof.* We refer the reader to [LM1, Proposition 0.2]. □

In addition, we quote the following from [LM1].

**Lemma 4.** *Let* $(X, L)$ *be a Bordiga surface as above, let* $\rho : X \to \mathbb{P}^1$ *be a* $\mathbb{P}^1$*-fibration, and let* $f$ *be a fiber of* $\rho$. *Then* $f \in |\sigma^*\mathcal{O}_{\mathbb{P}^2}(d) - \mathcal{O}_X(\sum_{i=1}^k m_i e_i)|$ *for some* $d > 0$ *and for some* $m_i \geq 0$. *Moreover,*

$$\sum_{i=1}^k m_i^2 = d^2 \quad \text{and} \quad \sum_{i=1}^k m_i = 3d - 2. \tag{1.1}$$

*Proof.* We refer the reader to [LM1, Lemma 0.3]. □

Finally we prove the following:

**Lemma 5.** *Let $\mathcal{E}$ be an ample vector bundle of rank $r$ on a smooth projective variety $X$ of dimension $n \geq 2$. Assume that $r \geq n$. If $K_X + \det \mathcal{E}$ is not ample, then either $K_X + \det \mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(-1)$ or $(K_X + \det \mathcal{E})^n = 0$.*

*Proof.* If $K_X + \det \mathcal{E}$ is not ample, then it follows from [F, Theorems 20.1 and 20.8] that $(X, \mathcal{E})$ is one of the following:

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)})$;
(2) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n})$;
(3) there exists a vector bundle $\mathcal{F}$ of rank $n$ on a smooth projective curve $C$ such that $X = \mathbb{P}_C(\mathcal{F})$, and $\mathcal{E}_F = \mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus n}$ for any fiber $F$ of the bundle projection;
(4) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n-1)})$;
(5) $(\mathbb{P}^n, T_{\mathbb{P}^n})$, where $T_{\mathbb{P}^n}$ is the tangent bundle of $\mathbb{P}^n$;
(6) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1)^{\oplus n})$.

In cases (1), (4), (5) and (6) we get $K_X + \det \mathcal{E} = \mathcal{O}_X$. In case (2) we obtain $K_X + \det \mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(-1)$. Suppose that case (3) holds. Then there exists a vector bundle $\mathcal{G}$ of rank $n$ on $C$ such that $\mathcal{E} = H(\mathcal{F}) \otimes \rho^*\mathcal{G}$, where $\rho : X \to C$ is the bundle projection. We have $K_X = -nH(\mathcal{F}) + \rho^*(K_C + \det \mathcal{F})$ and $\det \mathcal{E} = nH(\mathcal{F}) + \rho^*(\det \mathcal{G})$, so that $K_X + \det \mathcal{E} = \rho^*(K_C + \det \mathcal{F} + \det \mathcal{G})$. Hence $(K_X + \det \mathcal{E})^n = 0$, and the result is proved. □

## 2. Proof of the theorem: Part I

Let $(Z, L_Z)$ be a Bordiga surface. Then $Z$ is a smooth projective surface obtained by the blowing-up $\sigma : Z \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $k$ distinct points $p_1, \ldots, p_k$ in general position $(0 \leq k \leq 10)$, and $L_Z = \sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \mathcal{O}_Z(\sum_{i=1}^k e_i)$, where $e_i = \sigma^{-1}(p_i)$ for $i = 1, \ldots, k$. Since $(K_X + L)_Z = K_Z$ and $K_Z$ is not nef, we see that $K_X + L$ itself is not nef. Thus it follows from Lemma 1 that $(X, L)$ is one of the following:

(1) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$;
(2) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1))$;
(3) $(X, L)$ is a scroll over a smooth projective curve $C$;
(4) there exists an effective divisor $E$ on $X$ such that $(E, L_E, (\mathcal{O}_X(E))_E) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(-1))$;
(5) $(X, L)$ is a Del Pezzo manifold;
(6) $(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$;
(7) $(\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$;
(8) $X$ is a $\mathbb{P}^2$-bundle over a smooth projective curve $C$, and $L_F = \mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber $F$ of the bundle projection $X \to C$;
(9) there exists a surjective morphism $\pi : X \to C$ onto a smooth projective curve $C$ with Picard number $\rho(C) = \rho(X) - 1$ such that any fiber $D$ of $\pi$ is a quadric surface in $\mathbb{P}^3$ with $L_D = \mathcal{O}_D(1)$;
(10) $(X, L)$ is a scroll over a smooth projective surface $S$.

Furthermore, we have $K_Z + L_Z = (\sigma^*\mathcal{O}_{\mathbb{P}^2}(-3) + \mathcal{O}_Z(\sum_{i=1}^k e_i)) + (\sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \mathcal{O}_Z(\sum_{i=1}^k e_i)) = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$, so that $2g(Z, L_Z) - 2 = (K_Z + L_Z)L_Z = (\sigma^*\mathcal{O}_{\mathbb{P}^2}(1))(\sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \mathcal{O}_Z(\sum_{i=1}^k e_i)) = 4$. Hence $g(Z, L_Z) = 3$, and we conclude that $g(X,L)=3$. Moreover, the Lefschetz theorem tells us that $h^1(X,\mathcal{O}_X)=h^1(Z,\mathcal{O}_Z)=0$. Set $Z=(s)_0$ for some global section $s$ of $L$. Now let us deal with each of the cases (1)–(10) separately.

In case (1) we have $2g(X,L)-2=(K_X+2L)L^2=(\mathcal{O}_{\mathbb{P}^3}(-4)+\mathcal{O}_{\mathbb{P}^3}(2))\mathcal{O}_{\mathbb{P}^3}(1)^2=-2$, so that $g(X,L)=0$, which contradicts the fact that $g(X,L)=3$.

In case (2) we obtain $2g(X,L)-2=(K_X+2L)L^2=(\mathcal{O}_{\mathbb{Q}^3}(-3)+\mathcal{O}_{\mathbb{Q}^3}(2))\mathcal{O}_{\mathbb{Q}^3}(1)^2=-2$, and then $g(X,L)=0$. This is also impossible.

Assume that case (3) holds. Then $h^1(C,\mathcal{O}_C)=h^1(X,\mathcal{O}_X)=0$, i.e., $C=\mathbb{P}^1$. Combining this with Lemma 2, we have $g(X,L)=g(C)=0$. This is absurd because $g(X,L)=3$.

We treat case (4) after case (9). In case (5) we get $2g(X,L)-2=(K_X+2L)L^2=0$, so that $g(X,L)=1$, which is also impossible.

In case (6) we have $2g(X,L)-2=(K_X+2L)L^2=(\mathcal{O}_{\mathbb{P}^3}(-4)+\mathcal{O}_{\mathbb{P}^3}(6))\mathcal{O}_{\mathbb{P}^3}(3)^2=18$, and hence $g(X,L)=10$. This is absurd.

In case (7) we obtain $2g(X,L)-2=(K_X+2L)L^2=(\mathcal{O}_{\mathbb{Q}^3}(-3)+\mathcal{O}_{\mathbb{Q}^3}(4))\mathcal{O}_{\mathbb{Q}^3}(2)^2=8$, and so $g(X,L)=5$. This is also absurd.

Now we consider case (8). Then $h^1(C,\mathcal{O}_C)=h^1(X,\mathcal{O}_X)=0$. Thus $C=\mathbb{P}^1$. This directly indicates that $\mathrm{Pic}(X)=\mathbb{Z}\oplus\mathbb{Z}$, and the Lefschetz theorem tells us that $k\geq 1$. We can write $X=\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, where $\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(a_1)\oplus\mathcal{O}_{\mathbb{P}^1}(a_2)$ with $a_1,a_2\geq 0$. Let $\rho:X\to\mathbb{P}^1$ be the bundle projection, and let $H$ denote the tautological line bundle $H(\mathcal{E})$ on $X$. Then $H$ is spanned. Since $L_F=\mathcal{O}_{\mathbb{P}^2}(2)$ for any fiber $F$ of $\rho$, we have $L=2H+b\rho^*\mathcal{O}_{\mathbb{P}^1}(1)$ for some $b$. Combining [BS, Lemma 3.2.4] with the ampleness of $L$ gives $b>0$. We have $1=L_Ze_1=(2H_Z+b\rho_Z^*\mathcal{O}_{\mathbb{P}^1}(1))e_1=2H_Ze_1+b(\rho_Z^*\mathcal{O}_{\mathbb{P}^1}(1))e_1$. Since $H_Z$ and $\rho_Z^*\mathcal{O}_{\mathbb{P}^1}(1)$ are spanned, we obtain $H_Ze_1=0$ and $b=(\rho_Z^*\mathcal{O}_{\mathbb{P}^1}(1))e_1=1$. Therefore $L=2H+\rho^*\mathcal{O}_{\mathbb{P}^1}(1)$. Let us compute the sectional genus $g(X,L)$. Since $K_X=-3H+\rho^*\mathcal{O}_{\mathbb{P}^1}(c_1(\mathcal{E})-2)$, we get $4=2g(X,L)-2=(K_X+2L)L^2=(-3H+\rho^*\mathcal{O}_{\mathbb{P}^1}(c_1(\mathcal{E})-2)+4H+\rho^*\mathcal{O}_{\mathbb{P}^1}(2))(2H+\rho^*\mathcal{O}_{\mathbb{P}^1}(1))^2=(H+\rho^*\mathcal{O}_{\mathbb{P}^1}(c_1(\mathcal{E})))(4H^2+4H\rho^*\mathcal{O}_{\mathbb{P}^1}(1))=4H^3+4+4c_1(\mathcal{E})=8c_1(\mathcal{E})+4$. Hence $c_1(\mathcal{E})=0$, i.e., $a_1=a_2=0$, so that $\mathcal{E}=\mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$. Consequently $X=\mathbb{P}^2\times\mathbb{P}^1$ and $L=\mathcal{O}(2,1)$. However, we can regard $(X,L)$ as $(\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2}),H(\mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2}))$. From this, case (8) is included in case (10).

Suppose that $(X,L)$ is as in case (9). Then $h^1(C,\mathcal{O}_C)\leq h^1(X,\mathcal{O}_X)=0$, so that $C=\mathbb{P}^1$. Hence $\rho(X)=2$. By the Lefschetz theorem, the restriction homomorphism $\mathrm{Pic}(X)\to\mathrm{Pic}(Z)$ is injective. Moreover, $\mathrm{Pic}(Z)$ is torsion free because $Z$ is rational. Thus $\mathrm{Pic}(X)=\mathbb{Z}\oplus\mathbb{Z}$, and hence $k\geq 1$. Set $N=-(K_X+L)$. Then $N_F=-(K_F+L_F)=\mathcal{O}_{\mathbb{Q}^2}(1)$ for a general fiber $F=\mathbb{Q}^2$ of $\pi$, and so $\mathrm{Pic}(X)$ is generated by $N$ and $\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Set $L=aN+b\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ for some integers $a,b$. Then, since $L_D=\mathcal{O}_D(1)$ for any fiber $D$ of $\pi$, we obtain $a=1$, so that $L=N+b\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. We know that $Z=(s)_0$ for some global section $s$ of $L$. Let $s_F$ denote the restriction of $s$ to a general fiber $F$. Then $s_F\in\Gamma(\mathbb{Q}^2,\mathcal{O}_{\mathbb{Q}^2}(1))$, so that $Z\cap F=(s_F)_0\neq\emptyset$. This implies that the restriction $\pi_Z:Z\to\mathbb{P}^1$ of $\pi$ to $Z$ is surjective. Now $Z\cap F$ is a conic in $\mathbb{P}^2$ for a general $F$, which indicates that $\pi_Z$ is a $\mathbb{P}^1$-fibration. Set $f=Z\cap F$ for a general $F$. Then $L_Z=N_Z+b\pi_Z^*\mathcal{O}_{\mathbb{P}^1}(1)=-K_Z+b\pi_Z^*\mathcal{O}_{\mathbb{P}^1}(1)=-K_Z+b\mathcal{O}_Z(f)$. In addition, by Lemma 4 we know that $f\in|\sigma^*\mathcal{O}_{\mathbb{P}^2}(d)-\mathcal{O}_Z(\sum_{i=1}^k m_ie_i)|$ for some $d>0$ and for some $m_i\geq 0$. Since $-K_Z=\sigma^*\mathcal{O}_{\mathbb{P}^2}(3)-\mathcal{O}_Z(\sum_{i=1}^k e_i)$, we obtain $L_Z=-K_Z+b\mathcal{O}_Z(f)=(\sigma^*\mathcal{O}_{\mathbb{P}^2}(3)-\mathcal{O}_Z(\sum_{i=1}^k e_i))+b(\sigma^*\mathcal{O}_{\mathbb{P}^2}(d)-\mathcal{O}_Z(\sum_{i=1}^k m_ie_i))=\sigma^*\mathcal{O}_{\mathbb{P}^2}(3+bd)-\mathcal{O}_Z(\sum_{i=1}^k(1+bm_i)e_i)$. On the other hand,

$L_Z = \sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \mathcal{O}_Z(\sum_{i=1}^k e_i)$. Consequently $3 + bd = 4$ and $1 + bm_i = 1$ for any $i$, so that $bd = 1$ and $bm_i = 0$ for every $i$. Therefore $b = d = 1$ and $m_i = 0$ for any $i$. But then, since $d = 1$, it follows from (1.1) that $m_i = 1$ for some $i$. This is a contradiction.

Let us consider case (4). Let $f : X \to X'$ be the blowing-down of $E$ to a point $p \in X'$. Then there exists a line bundle $L'$ on $X'$ such that $L = f^*L' - \mathcal{O}_X(E)$. It follows from [F, Lemma 7.16] that $L'$ is ample on $X'$. Set $l = (s_E)_0 = Z \cap E$. Since $s_E \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, $l$ is a linear subspace of $E$ with $\dim l \geq 1$. If $Z \cap E = Z$, then $Z = \mathbb{P}^2$, so that $L_Z = (L_E)_Z = \mathcal{O}_{\mathbb{P}^2}(1)$. Thus $K_Z + L_Z = \mathcal{O}_{\mathbb{P}^2}(-2)$, which contradicts the nefness of $K_Z + L_Z = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$. Hence $l = Z \cap E \subsetneq Z$, and the irreducibility of $Z$ gives $\dim l \leq 1$. Therefore $\dim l = 1$, i.e., $l = \mathbb{P}^1$. Moreover, $(\mathcal{O}_Z(l))_l = ((\mathcal{O}_X(E))_Z)_l = ((\mathcal{O}_X(E))_E)_l = \mathcal{O}_{\mathbb{P}^1}(-1)$. This directly implies that $l$ is a $(-1)$-curve on $Z$. Set $Z' = f(Z)$. Then $Z'$ is a smooth projective surface, and $Z'$ is also a smooth member of $|L'|$. It should be emphasized that $L_Z = f_Z^* L'_{Z'} - \mathcal{O}_Z(l)$, so that $L_Z l = 1$. Combining this with Lemma 3 leads us to the conclusion that $l = e_i$ for some $i$. Consequently $(Z', L'_{Z'})$ is again a Bordiga surface, and $(X', L')$ satisfies the same assumption as that in the theorem. We have $K_X + 2L = f^*(K_{X'} + 2L')$ because $K_X = f^*K_{X'} + 2\mathcal{O}_X(E)$. Since we are in case (iv-1) of Lemma 1, $K_X + 2L$ is nef, so that $K_{X'} + 2L'$ is also nef. Moreover, since $(Z', L'_{Z'})$ is a Bordiga surface, we see that $K_{X'} + L'$ is not nef. Therefore $(X', L')$ is as in cases (4)–(10). However, we know that cases (5)–(9) do not occur when $(Z', L'_{Z'})$ is a Bordiga surface (we should keep in mind that case (8) is included in case (10)). Suppose that $(X', L')$ is as in case (10). Then there exists a smooth rational curve $C$ on $X'$ passing through $p$ such that $L'C = 1$. Let $\widetilde{C}$ be the strict transform of $C$ by $f$. Then $L\widetilde{C} = (f^*L' - \mathcal{O}_X(E))\widetilde{C} = L'C - \mathcal{O}_X(E)\widetilde{C} = 0$, which contradicts the ampleness of $L$. Thus $(X', L')$ must be as in case (4) again. We apply the same argument as above to $X'$, $L'$ and $Z'$, and continue in this manner. This procedure must come to an end after a finite number of repetitions, and we obtain $(\widetilde{X}, \widetilde{L})$ satisfying the same assumption as in the theorem such that $K_{\widetilde{X}} + \widetilde{L}$ is nef. For the corresponding smooth projective surface $\widetilde{Z}$, $K_{\widetilde{Z}}$ is nef. This contradicts the fact that $\widetilde{Z}$ is rational, and case (4) does not occur.

Finally we consider case (10). Let $\rho : X \to S$ be the scroll projection, and let $F$ be an arbitrary fiber of $\rho$. Then $L_F = \mathcal{O}_{\mathbb{P}^1}(1)$. This indicates that $Z \cap F$ is either a point or all of $F$. In particular, $\rho_Z : Z \to S$ is surjective, so that $\rho_Z$ is generically finite. Hence $\rho_Z$ is birational. The Lefschetz theorem tells us that the restriction homomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ is injective, so that $\rho_Z$ is not an isomorphism. Thus there exists a positive dimensional fiber $e$ of $\rho_Z$. Since $e = Z \cap F$ for some fiber $F$ of $\rho$ and $Z \cap F$ is all of $F$, we have $e = \mathbb{P}^1$. We can write $(X, L) = (\mathbb{P}_S(\mathcal{E}), H(\mathcal{E}))$ for some ample vector bundle $\mathcal{E}$ of rank 2 on $S$, and we obtain $K_X + 2L = \rho^*(K_S + \det \mathcal{E})$, so that $K_Z + L_Z = \rho_Z^*(K_S + \det \mathcal{E})$. Therefore $0 = (\rho_Z^*(K_S + \det \mathcal{E}))e = K_Z e + L_Z e = K_Z e + L_F e = K_Z e + 1$, i.e., $K_Z e = -1$. This directly implies that $e$ is a $(-1)$-curve on $Z$. Moreover, since $L_Z e = 1$, it follows from Lemma 3 that $e = e_i$ for some $i$. From this, we can conclude that $S$ is also a smooth projective surface with the Bordiga polarization, so that $\sigma$ factors through $\rho_Z$. Let us recall that $K_Z + L_Z = \rho_Z^*(K_S + \det \mathcal{E})$. Since $K_Z + L_Z = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$ is nef and big, we see that $K_S + \det \mathcal{E}$ is also nef and big. Hence by Lemma 5, $K_S + \det \mathcal{E}$ is ample. We get $(K_Z + L_Z)e_i = 0$ for any $i$ because $K_Z + L_Z = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$, so that $0 = (K_Z + L_Z)e_i = (\rho_Z^*(K_S + \det \mathcal{E}))e_i$

for any $i$. Combining this with the ampleness of $K_S + \det \mathcal{E}$ implies that $\rho_Z(e_i)$ is a point of $S$ for every $i$. Therefore $S = \mathbb{P}^2$ and $\rho_Z = \sigma$. Thus $K_X + 2L = \rho^*(\mathcal{O}_{\mathbb{P}^2}(-3)+\det \mathcal{E})$, which indicates that $K_Z+L_Z = \rho_Z^*(\mathcal{O}_{\mathbb{P}^2}(-3)+\det \mathcal{E})$. We know that $K_Z + L_Z = \sigma^*\mathcal{O}_{\mathbb{P}^2}(1)$. Hence $\mathcal{O}_{\mathbb{P}^2}(-3) + \det \mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1)$, i.e., $\det \mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(4)$. Thus $c_1(\mathcal{E}) = 4$. By the Wu-Chern relation, we have $L^2 - L\rho^*c_1(\mathcal{E}) + \rho^*c_2(\mathcal{E}) = 0$, and hence $L^3 = L^2\rho^*c_1(\mathcal{E}) - L\rho^*c_2(\mathcal{E}) = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = 16 - c_2(\mathcal{E})$. On the other hand, $L^3 = L_Z^2 = (\sigma^*\mathcal{O}_{\mathbb{P}^2}(4) - \mathcal{O}_Z(\sum_{i=1}^k e_i))^2 = 16 - k$. Consequently $c_2(\mathcal{E}) = k$. Since $\rho_Z = \sigma$ is not an isomorphism, we obtain $k \geq 1$, and we conclude that $1 \leq c_2(\mathcal{E}) \leq 10$. In Section 3 we show that $c_2(\mathcal{E}) \geq 3$.

## 3. Proof of the theorem: Part II

Let $X, L$ and $Z$ be as in the theorem. Then we know that there exists an ample vector bundle $\mathcal{E}$ of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$ and $1 \leq c_2(\mathcal{E}) \leq 10$ such that $(X, L) = (\mathbb{P}_{\mathbb{P}^2}(\mathcal{E}), H(\mathcal{E}))$. Let us consider the vector bundle $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)$. Then $c_1(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)) = c_1(\mathcal{E}) + 2c_1(\mathcal{O}_{\mathbb{P}^2}(-2)) = 0$, so that $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)$ is normalized in the sense of [OSS, p. 165].

First assume that $\mathcal{E}$ is not semistable. Then [OSS, Chapter II, Lemma 1.2.5] tells us that $H^0(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) \neq 0$. Take a nonzero global section $t \in H^0(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3))$. If $(t)_0 = \emptyset$, then we have an exact sequence $(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3))^\vee \to \mathcal{O}_{\mathbb{P}^2} \to 0$, where $(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3))^\vee$ is the dual of $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)$. Hence the sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{\mathbb{P}^2}(-2) \to 0$ is exact, so that $0 \to \mathcal{O}_{\mathbb{P}^2}(3) \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(1) \to 0$ is exact. Now $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(3)) = \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(2)) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) = 0$. Therefore $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(3) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, and $c_2(\mathcal{E}) = 3$. On the other hand, when $(t)_0 \neq \emptyset$, we take a line $l$ in $\mathbb{P}^2$ such that $(t_l)_0 = (t)_0 \cap l$ is a nonempty finite set. Then we can write $\mathcal{E}_l = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(4 - a)$ for some integer $a$. Taking the ampleness of $\mathcal{E}$ and the symmetry into account, we can assume that $a \geq 4 - a \geq 1$, so that $2 \leq a \leq 3$. Now $(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3))_l = \mathcal{O}_{\mathbb{P}^1}(a - 3) \oplus \mathcal{O}_{\mathbb{P}^1}(1 - a)$. If $a = 3$, then $(t_l)_0 = l$, which is contrary to our assumption. If $a = 2$, then $(t_l)_0$ is also $l$. This is still absurd.

Next assume that $\mathcal{E}$ is not stable but semistable. Then by [OSS, Chapter II, Lemma 1.2.5] we get $H^0(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$ and $H^0(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)) \neq 0$. Take a nonzero global section $t \in H^0(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2))$. If $(t)_0 = \emptyset$, then we obtain an exact sequence $(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2))^\vee \to \mathcal{O}_{\mathbb{P}^2} \to 0$, which induces an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}_{\mathbb{P}^2} \to 0$. As a consequence, the sequence $0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(2) \to 0$ is exact. We have $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^2}(2)) = \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$. Hence $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2}$ and $c_2(\mathcal{E}) = 4$. When $(t)_0 \neq \emptyset$, the case where $\dim(t)_0 = 1$ is impossible because $H^0(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$. Thus $\dim(t)_0 = 0$. Take an arbitrary line $l$ in $\mathbb{P}^2$ such that $(t)_0 \cap l \neq \emptyset$. With the same notation as above we have $(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2))_l = \mathcal{O}_{\mathbb{P}^1}(a-2) \oplus \mathcal{O}_{\mathbb{P}^1}(2-a)$. We should keep in mind that $2 \leq a \leq 3$ by the ampleness of $\mathcal{E}$. Thus $a = 3$, and $(t)_0 \cap l$ is a single point $p$ of $\mathbb{P}^2$. Therefore $c_2(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)) = 1$, and the Koszul complex gives rise to an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to (\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2))^\vee = \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{I}_p \to 0$, where $\mathcal{I}_p$ is the ideal sheaf of $p$. Consequently the sequence $0 \to \mathcal{O}_{\mathbb{P}^2}(2) \to \mathcal{E} \to \mathcal{I}_p \otimes \mathcal{O}_{\mathbb{P}^2}(2) \to 0$ is exact, and $c_2(\mathcal{E}) = 5$.

Finally we assume that $\mathcal{E}$ is stable. Then it follows from [OSS, Chapter II, Lemma 1.2.5] that $H^0(\mathbb{P}^2, \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)) = 0$. We apply the Riemann-Roch theorem to $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$. Now $\det(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = \mathcal{O}_{\mathbb{P}^2}(2)$ and $c_2(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-1)) = c_2(\mathcal{E}) - 3$.

The Riemann-Roch theorem tells us that

$$\begin{aligned}\chi(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)) &= \frac{1}{2}(\det(\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)) - K_{\mathbb{P}^2})\det(\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)) \\ &\qquad - c_2(\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)) + 2\chi(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \\ &= 10 - c_2(\mathcal{E}).\end{aligned}$$

Suppose that $c_2(\mathcal{E}) \le 9$. Then either $h^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)) > 0$ or $h^2(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)) > 0$. By Serre duality, the latter indicates that $0 < h^0(\mathbb{P}^2, K_{\mathbb{P}^2}\otimes\mathcal{E}^\vee\otimes\mathcal{O}_{\mathbb{P}^2}(1)) = h^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-6))$. However, since we know that $H^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-2)) = 0$, we obtain $h^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-6)) = 0$. Therefore $h^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)) > 0$. Take a nonzero global section $t \in H^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1))$. If $(t)_0 = \emptyset$, then we get an exact sequence $(\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1))^\vee \to \mathcal{O}_{\mathbb{P}^2} \to 0$. Thus the sequence $0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}(2) \to 0$ is exact, so that $0 \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^2}(3) \to 0$ is exact. We have $\operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}(3), \mathcal{O}_{\mathbb{P}^2}(1)) = \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(-2)) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-2)) = 0$, and so $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(3)\oplus\mathcal{O}_{\mathbb{P}^2}(1)$. However, we have $H^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-2)) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\oplus\mathcal{O}_{\mathbb{P}^2}(-1)) \ne 0$. This is a contradiction. Moreover, since $H^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-2)) = 0$, the case where $\dim(t)_0 = 1$ is also impossible. Thus $\dim(t)_0 = 0$. Set $Y = (t)_0$. Then $\deg Y = c_2(\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1)) = c_2(\mathcal{E}) - 3$. The Koszul complex induces an exact sequence $0 \to \mathcal{O}_{\mathbb{P}^2}(-2) \to (\mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-1))^\vee = \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{I}_Y \to 0$, where $\mathcal{I}_Y$ is the ideal sheaf of $Y$. Hence the sequence $0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{I}_Y\otimes\mathcal{O}_{\mathbb{P}^2}(1) \to 0$ is exact. Since $H^0(\mathbb{P}^2, \mathcal{E}\otimes\mathcal{O}_{\mathbb{P}^2}(-2)) = 0$ by assumption and $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$, we have $H^0(\mathbb{P}^2, \mathcal{I}_Y\otimes\mathcal{O}_{\mathbb{P}^2}(1)) = 0$. This means that $Y$ is not contained in a line. Hence $\deg Y \ge 3$, i.e., $c_2(\mathcal{E}) \ge 6$. Consequently, if $\mathcal{E}$ is stable, then we see that $c_2(\mathcal{E}) \ge 6$.

Thus we conclude that $c_2(\mathcal{E}) \ge 3$ when $\mathcal{E}$ is ample with $c_1(\mathcal{E}) = 4$. To sum up, under the assumption in the theorem, there exists an ample vector bundle $\mathcal{E}$ of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$ and $3 \le c_2(\mathcal{E}) \le 10$ such that $(X, L) = (\mathbb{P}_{\mathbb{P}^2}(\mathcal{E}), H(\mathcal{E}))$. We have completed the proof of the theorem. □

The argument developed in this section enables us to prove the following proposition. Statement (3) was proved in [M] when $\mathcal{E}$ is very ample.

**Proposition.** *Let $\mathcal{E}$ be an ample vector bundle of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$. Then*

(1) $c_2(\mathcal{E}) = 3$ *if and only if* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(3)\oplus\mathcal{O}_{\mathbb{P}^2}(1)$;
(2) $c_2(\mathcal{E}) = 4$ *if and only if* $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2}$;
(3) $c_2(\mathcal{E}) = 6$ *if and only if $\mathcal{E}$ is the cokernel of a bundle monomorphism* $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \to T_{\mathbb{P}^2}^{\oplus 2}$, *where $T_{\mathbb{P}^2}$ is the tangent bundle of $\mathbb{P}^2$.*

*Proof.* The argument developed in this section implies the following when $\mathcal{E}$ is an ample vector bundle of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$:

(i) $c_2(\mathcal{E}) = 3$ if and only if $\mathcal{E}$ is not semistable;
(ii) $c_2(\mathcal{E}) = 4$ or 5 if and only if $\mathcal{E}$ is not stable but semistable;
(iii) $c_2(\mathcal{E}) \ge 6$ if and only if $\mathcal{E}$ is stable.

(1) The "if" part is obvious. Assume that $c_2(\mathcal{E}) = 3$. Then $\mathcal{E}$ is not semistable, so that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(3)\oplus\mathcal{O}_{\mathbb{P}^2}(1)$.

(2) The "if" part is also obvious. If $c_2(\mathcal{E}) = 4$, then $\mathcal{E}$ is not stable but semistable. Thus we see that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2)^{\oplus 2}$.

(3) Assume that the sequence $0 \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \to T_{\mathbb{P}^2}^{\oplus 2} \to \mathcal{E} \to 0$ is exact. Then $\mathcal{E}$ is ample because $T_{\mathbb{P}^2}$ is ample. Moreover, $c_1(\mathcal{E}) = c_1(T_{\mathbb{P}^2}^{\oplus 2}) - c_1(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}) = 4$,

and $c_2(\mathcal{E}) = c_2(T_{\mathbb{P}^2}^{\oplus 2}) - c_1(\mathcal{E})c_1(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}) - c_2(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}) = 6$. Hence it suffices to prove the "only if" part.

Suppose that $\mathcal{E}$ is an ample vector bundle of rank two on $\mathbb{P}^2$ with $c_1(\mathcal{E}) = 4$ and $c_2(\mathcal{E}) = 6$. Then $\mathcal{E}$ is stable. Let us consider the vector bundle $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)$, which is also stable. Then $c_1(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)) = c_1(\mathcal{E}) + 2c_1(\mathcal{O}_{\mathbb{P}^2}(-2)) = 0$, and $c_2(\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)) = c_2(\mathcal{E}) + c_1(\mathcal{E})c_1(\mathcal{O}_{\mathbb{P}^2}(-2)) + c_1(\mathcal{O}_{\mathbb{P}^2}(-2))^2 = 2$. It follows from the Beilinson spectral sequence that $\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2)$ is the cokernel of a bundle monomorphism $\mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to (\Omega^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(1))^{\oplus 2}$ [OSS, Example 2, p. 248]. Since $\Omega^1_{\mathbb{P}^2} = T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)$, the sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2} \to (T_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-2))^{\oplus 2} \to \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \to 0$$

is exact. This directly leads us to the conclusion that $\mathcal{E}$ is the cokernel of a bundle monomorphism $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2} \to T_{\mathbb{P}^2}^{\oplus 2}$. □

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