

ASYMPTOTIC ANALYSIS OF A DIRICHLET PROBLEM FOR THE HEAT EQUATION ON A COATED BODY

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ABSTRACT. Of concern is the protection from overheating of an isotropically conducting body by an anisotropically conducting coating which is thin compared to the scale of the body. We assume either that the whole thermal tensor of the coating is small or that it is small in the directions normal to the body (a case we call “optimally aligned coating”). We study the asymptotic behavior of the solution to the heat equation with Dirichlet boundary conditions on the outer surface of the coating, as the thickness of the coating shrinks. We obtain the exact scaling relations between the thermal tensor and the thickness of the coating so that the effective (limiting) condition on the boundary of the body is of Dirichlet, Robin or Neumann type, with the last condition indicating good insulation.

1. INTRODUCTION

In [9], we studied the scenario of protecting an isotropically conducting body from overheating by using an insulating coating which is allowed to be anisotropic (e.g. a space shuttle painted with a nanocomposite insulator). Our interest is in finding the right scaling for the thickness of the coating so that the body will be protected most efficiently.

The mathematical model is as follows. Let the domain $\Omega_1 \subset \mathbb{R}^n$ (representing the isotropic body) be surrounded by another domain $\Omega_2 \subset \mathbb{R}^n$ (representing the anisotropic coating). See Figure 1. Let $\Omega = \overline{\Omega}_1 \cup \Omega_2$. Assume that the thickness of Ω_2 is uniform and is equal to δ . Let the thermal tensor $A(x)$ of Ω be given by

$$(1.1) \quad A(x) = (a_{ij}(x))_{n \times n} = \begin{cases} kI_{n \times n}, & x \in \Omega_1, \\ \sigma(\overline{a}_{ij}(x))_{n \times n}, & x \in \Omega_2, \end{cases}$$

where $k > 0$ is the constant thermal conductivity of Ω_1 , I is the identity matrix, σ is a small positive parameter, the matrix $(\overline{a}_{ij}(x))$ is symmetric and positive definite at every $x \in \Omega_2$, with its smallest eigenvalue bounded from below by a positive constant, and each \overline{a}_{ij} is a bounded measurable function on Ω_2 . σ being small means that the thermal conductivity of Ω_2 is small in all directions (we shall also consider the case where the thermal conductivity of Ω_2 is small only in the direction normal to $\partial\Omega_1$). See [1] and [11] for the homogenization theory that gives rise to

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such anisotropic thermal tensors; see [12] for explicit formulas for these tensors. Suppose the whole system Ω is subject to high exterior temperature H (as in the case, for example, of a returning space shuttle). Then the temperature function $Q(x, t)$ is the solution of a Dirichlet boundary value problem for the heat equation:

$$(1.2) \quad \begin{cases} Q_t - \nabla \cdot (A\nabla Q) = 0, & x \in \Omega, t > 0, \\ Q = H, & x \in \partial\Omega, t > 0, \\ Q = Q_0(x), & x \in \Omega, t = 0, \end{cases}$$

where the constant H is large compared to the values of $Q_0(x)$.

The eigenfunction expansion of Q is given by the following explicit formula:

$$(1.3) \quad Q(x, t) = H + \sum_{m \geq 1} e^{-\lambda_m t} \phi_m(x) \int_{\Omega} \phi_m(x') (Q_0(x') - H) dx',$$

where (λ_m, ϕ_m) are the eigenvalues and normalized eigenfunctions of the elliptic operator $u \rightarrow -\nabla \cdot (A\nabla u)$ under Dirichlet boundary conditions on $\partial\Omega$. Recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \rightarrow \infty$ and that $\phi_1(x)$ can be taken to be positive in the interior of Ω . In particular, $Q(x, t) \rightarrow H$ as $t \rightarrow \infty$. Thus to protect the body Ω_1 from overheating, we need to slow down the rate of convergence of Q to H . To achieve this, it was argued in [9] that (a) the first Dirichlet eigenvalue λ_1 should be suppressed; (b) the first eigenfunction $\phi_1(x)$ should be made to take large values on Ω_1 ; (c) as many $\{\lambda_m\}$ as possible should be suppressed; (d) the higher eigenfunctions should take small values on Ω_1 . It was proved in [9] that both (a) and (b) occur if the coating is “not too thin” in the sense of

$$(1.4) \quad \frac{\sigma}{\delta} \rightarrow 0 \text{ as } \delta \rightarrow 0^+$$

and that (a)-(d) all occur if the coating is “optimally thick” in the sense of

$$(1.5) \quad \frac{\sigma}{\delta^2} \rightarrow 0 \text{ as } \delta \rightarrow 0^+.$$

(Friedman [4] and Panasenko [8] also obtained related results on the first eigenpair.)

In this paper, we study the asymptotic behavior of the solution of the heat equation (1.2) *directly*, and obtain the following result. Suppose

$$\lim_{\delta \rightarrow 0^+} \frac{\sigma}{\delta} \text{ exists and } = \alpha \in [0, \infty].$$

Then for any fixed, finite time $T > 0$, $Q \rightarrow v$ in $L^2(\Omega_1 \times (0, T))$ as $\delta \rightarrow 0^+$, where v is the solution of

$$(1.6) \quad \begin{cases} v_t - k\Delta v = 0, & x \in \Omega_1, t > 0, \\ \partial v / \partial \nu + (\alpha \nu \cdot \nu_{\overline{A}} / k)(v - H) = 0, & x \in \partial\Omega_1, t > 0, \\ v = Q_0, & x \in \Omega_1, t = 0, \end{cases}$$

with $\nu = (\nu_1, \dots, \nu_n)$ being the unit outer normal vector field on $\partial\Omega_1$ and $\nu_{\overline{A}}$ the conormal $\overline{a}_{ij}\nu_j$. If $\alpha = +\infty$, then the boundary condition is understood as the Dirichlet boundary condition $v = H$. See Theorem 2.2.

In other words, if $\alpha = 0$, i.e. if (1.4) holds, then on the finite time interval $[0, T]$, $\partial\Omega_1$ is effectively well-insulated; otherwise, if $0 < \alpha \leq \infty$, then $\partial\Omega_1$ is subject to either a high exterior temperature H or a large heat flux $\alpha \nu \cdot \nu_{\overline{A}}(H - v)$. These results confirm the rule: to protect Ω_1 from overheating, the coating should be at least as thick as required in (1.4). We emphasize that these results are available for a limited (i.e., fixed, finite) time only.

These results still hold even if the entire thermal tensor is not small over Ω_2 in the sense of (1.1), as long as it is small over Ω_2 in the direction normal to the body Ω_1 . This is the case of what we called “optimally aligned coating” in [9]: for $x \in \Omega_2$, let p be the projection of x on $\partial\Omega_1$; then the vector \overline{px} is an eigenvector corresponding to the smallest eigenvalue of the thermal tensor $(a_{ij}(x))$. See Theorem 2.3, which says that the magnitude of the other eigenvalues is not significant.

We remark that if the insulator Ω_2 is not uniformly thick but is thicker than δ , then we can compare the solution Q of (1.2) with the solution Q' of (1.2) with Ω_2 replaced by Ω'_2 , where Ω'_2 is contained in Ω_2 and has uniform thickness equal to δ : By the maximum principle for weak solutions (see Theorem 7.2 of [7]), we have $Q(x, t) \leq H$ for $x \in \overline{\Omega}_1 \cup \Omega_2$, $t \in (0, T)$; and then from the maximum principle again, we find that $Q(x, t) \leq Q'(x, t)$ for $x \in \overline{\Omega}_1 \cup \Omega'_2$, $t \in (0, T)$. Note that Q' approaches the solution v of (1.6) as $\delta \rightarrow 0$. Thus, roughly speaking, $Q(x, t)$ is either smaller than or close to $v(x, t)$ for $x \in \Omega_1$ and $t \in (0, T)$.

The “interior reinforcement problem” for elliptic and parabolic equations was first studied by Sanchez-Palencia [10]. The elliptic problem corresponding to our parabolic one was studied by Brezis, Caffarelli and Friedman [3] (see also Buttazzo and Kohn[2] for the case of rapidly oscillating thickness of the coating). The method of [3] is based on H^2 a priori estimates, while that of [2] is based on Γ -convergence. We prove our results using only the parabolic $W^{1,1}$ a priori estimates; we are unaware of any prior papers (except our [9]) that cover the case of optimally aligned coating. We have obtained parabolic $W^{2,1}$ a priori estimates, but as they are not needed for Theorems 2.2-2.3, we do not present them here. (However, $W^{2,1}$ estimates imply that the L^2 convergence in Theorems 2.2-2.3 can be strengthened to convergence in $C^{\frac{1}{2}, \frac{1}{4}}(\overline{\Omega}_1 \times [0, T])$ in the case $n = 1$.)

2. RESULTS AND PROOFS

Let $u(x, t) = Q(x, t) - H$ and $\varphi(x) = Q_0(x) - H$; then (1.2) with a given source term is transformed to

$$(2.1) \quad \begin{cases} u_t - \nabla \cdot (A\nabla u) = f, & (x, t) \in Q_T, \\ u = 0, & (x, t) \in S_T, \\ u = \varphi(x), & x \in \Omega, t = 0, \end{cases}$$

where $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$. Denote by $W_2^{1,1}(Q_T)$ the Hilbert space

$$\{u : u, \nabla_x u, u_t \in L^2(Q_T)\}$$

equipped with the norm

$$\|u\|_{W_2^{1,1}(Q_T)}^2 := \|u\|_{L^2(Q_T)}^2 + \|\nabla_x u\|_{L^2(Q_T)}^2 + \|u_t\|_{L^2(Q_T)}^2.$$

If $\partial\Omega$ is C^1 smooth, it is well-known that (i) the set of C^∞ smooth functions is dense in $W_2^{1,1}(Q_T)$, and (ii) $W_2^{1,1}(Q_T)$ is continuously embedded in $C([0, T]; L^2(\Omega))$. Denote by $W_{2,0}^{1,1}(Q_T)$ the closure in $W_2^{1,1}(Q_T)$ of C^∞ smooth functions that vanish near $\overline{S_T}$. Then it is easy to prove that $W_{2,0}^{1,1}(Q_T)$ is continuously embedded in $C([0, T]; L^2(\Omega))$ without assuming any smoothness of $\partial\Omega$.

The weak solution of (2.1) is a function u satisfying (i) $u \in W_{2,0}^{1,1}(Q_T)$ and $u(\cdot, t) \rightarrow \varphi$ in $L^2(\Omega)$ as $t \rightarrow 0^+$; (ii) for any $v \in W_{2,0}^{1,1}(Q_T)$, we have

$$(2.2) \quad \int_0^T \left[\int_{\Omega} (u_t v + a_{ij}(x) u_{x_i} v_{x_j}) dx \right] dt = \int_0^T \int_{\Omega} f v dx dt.$$

By using eigenexpansions, we can prove that if $f \in L^2(Q_T)$ and $\varphi \in H_0^1(\Omega)$, then (2.1) has one and only one weak solution u which is also in $C([0, T]; H_0^1(\Omega))$. The weak solution has the following properties, which we present for the case where (1.1) holds: (a) the “transmission condition” on $\partial\Omega_1 \times (0, T]$ in the weak sense, i.e.

$$(2.3) \quad u_1 = u_2, \quad k \frac{\partial u_1}{\partial \nu} = \sigma \frac{\partial u_2}{\partial \nu_A},$$

where u_i is the restriction of u on $\Omega_i \times (0, T)$, and the conormal derivative is given by $\frac{\partial u}{\partial \nu_A} = u_{x_i} \bar{a}_{ij} \nu_j$; (b) if both f and φ are bounded, then so is u (Theorem 7.1 of [7]); (c) if $\partial\Omega$ is C^2 smooth, f is bounded, and φ is Hölder continuous on $\bar{\Omega}$ and vanishes on $\partial\Omega$, then u is also Hölder continuous on \bar{Q}_T (see Theorem 10.1 of [7]); (d) if $\bar{a}_{ij} \in C^1(\bar{\Omega}_2)$, both $\partial\Omega_i (i = 1, 2)$ are C^2 smooth, and both f and f_t are bounded on Q_T , then $\nabla_x u$ is Hölder continuous in each $\bar{\Omega}_i \times (0, T]$ and the “transmission condition” holds on $\partial\Omega_1 \times (0, T]$ in the classical sense; furthermore, u_t is continuous on $\bar{\Omega} \times (0, T]$ and $u_{x_k x_j}, u_{x_k t}$ are in each $L^2(\Omega_i \times (0, T))$ (see Theorem 7 of [6]).

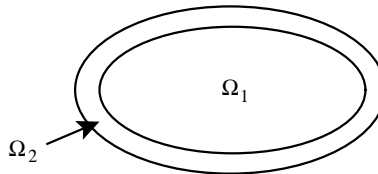


FIGURE 1. $\Omega = \Omega_2 \cup \bar{\Omega}_1$, Ω_2 is an open set, and the distance between $\partial\Omega_1$ and $\partial\Omega$ is a positive constant.

For any small positive δ , define F by

$$(2.4) \quad \begin{aligned} F : (p, \tau) \in \partial\Omega_1 \times [-\delta, \delta] &\rightarrow (x_1, \dots, x_n) \in \mathbb{R}^n, \quad n > 1, \\ (x_1, \dots, x_n) &= F(p, \tau) = p + \tau \nu(p). \end{aligned}$$

By Lemma 14.16 of [5], if $\partial\Omega_1 \in C^2$ and δ is small enough, then F is a diffeomorphism; moreover, if we take

$$\Omega_2 = F(\partial\Omega_1 \times (0, \delta)),$$

then $\partial\Omega_2 \in C^2$. For the sake of simplicity, we assume $\partial\Omega_1$ is connected.

We need the following a priori estimates.

Lemma 2.1. *Suppose that (1.1) holds, $\partial\Omega_1 \in C^2$, $f \in L^2(Q_T)$ and $\varphi \in H_0^1(\Omega)$. Then the weak solution u of (2.1) satisfies*

$$(2.5) \quad \max_{0 \leq t \leq T} \int_{\Omega} u^2(x, t) dx \leq C \left(\int_{\Omega} \varphi^2(x) dx + \int \int_{Q_T} f^2(x, t) dx dt \right),$$

$$(2.6) \quad \int \int_{Q_T} |u_t|^2 dx dt + \max_{0 \leq t \leq T} \left[k \int_{\Omega_1} |\nabla u|^2 dx + \sigma \int_{\Omega_2} |\nabla u|^2 dx \right] \\ \leq C \left(\int_{\Omega} (\varphi^2(x) + |\nabla \varphi|^2(x)) dx + \int \int_{Q_T} f^2(x, t) dx dt \right),$$

where C is independent of σ and δ .

Proof. Estimates (2.5) and (2.6) are part of the standard L^2 parabolic theory. \square

Theorem 2.2. *Suppose that all the conditions in Lemma 2.1 hold, with*

$$\int \int_{\Omega_2 \times (0, T)} f^2(x, t) dx dt$$

and

$$\int_{\Omega_2} (\varphi^2 + |\nabla \varphi|^2) dx$$

being bounded by a constant independent of Ω_2 (f and φ are fixed on Ω_1 but on Ω_2 they are allowed to vary with δ), and assume that $\bar{a}_{ij} \in C^1(\bar{\Omega}_2)$ (\bar{a}_{ij} does not vary with respect to δ) and that σ remains bounded and

$$\frac{\sigma}{\delta} \rightarrow \alpha \quad \text{as } \delta \rightarrow 0^+,$$

where $\alpha \in [0, +\infty]$. Then the weak solution of (2.1) satisfies $u(x, t) \rightarrow w(x, t)$ strongly in $L^2(\Omega_1 \times (0, T))$ as $\delta \rightarrow 0^+$, where $w(x, t)$ is the solution of

$$(2.7) \quad \begin{cases} w_t - k\Delta w = f, & x \in \Omega_1, t > 0, \\ \partial w / \partial \nu + \alpha \nu \cdot \nu_{\bar{A}} w / k = 0, & x \in \partial\Omega_1, t > 0, \\ w = \varphi(x), & x \in \Omega_1, t = 0. \end{cases}$$

If $\alpha = +\infty$, then the boundary condition is understood as the Dirichlet boundary condition.

Proof. It follows from Lemma 2.1 that for all small $\delta > 0$,

$$(2.8) \quad \|u\|_{W_2^{1,1}(\Omega_1 \times (0, T))} \leq C.$$

Thus there exists $w \in W_2^{1,1}(\Omega_1 \times (0, T))$ such that $u(x, t) \rightarrow w(x, t)$ weakly in $W_2^{1,1}(\Omega_1 \times (0, T))$ and strongly in $L^2(\Omega_1 \times (0, T))$, after passing to a subsequence of $\delta \rightarrow 0$. Recall that $W_2^{1,1}(\Omega_1 \times (0, T))$ is continuously imbedded in $C([0, T], L^2(\Omega_1))$; thus u and w belong to the latter space. By Lemma 2.1, $\|u(\cdot, t)\|_{H^1(\Omega_1)}$ is bounded for all small δ , uniformly for $t \in [0, T]$. So by the compact embedding theorem, at each fixed t , $u(\cdot, t) \rightarrow w(\cdot, t)$ after passing to a subsequence of $\delta \rightarrow 0^+$. On the other hand, since $\|u_t\|_{L^2(\Omega_1 \times (0, T))}$ is bounded, $u(\cdot, t)$ is equicontinuous on $[0, T]$ with respect to the $L^2(\Omega_1)$ norm. Now by the Arzela-Ascoli theorem, $u \rightarrow w$ strongly in $C([0, T], L^2(\Omega_1))$ after passing to a subsequence of $\delta \rightarrow 0^+$. Hence the initial condition in (2.7) is satisfied by w in the sense of $w(\cdot, t) \rightarrow \varphi$ in $L^2(\Omega_1)$ as $t \rightarrow 0^+$.

By the trace theorem (see Lemma 3.4 of [7]) and (2.8), we have

$$(2.9) \quad \|u_1\|_{W_2^{\frac{1}{2}, \frac{1}{4}}(\partial\Omega_1 \times (0, T))} \leq C.$$

So $u_1 \rightarrow w$ strongly in $L^2(\partial\Omega_1 \times (0, T))$ after passing to a subsequence of $\delta \rightarrow 0$.

We shall eventually prove that w is a weak solution of (2.7). Since the weak solution of (2.7) is unique, all the convergence statements made above hold without passing to a subsequence of $\delta \rightarrow 0$.

For any point $p \in \partial\Omega_1$, let l be the ray in the conormal direction $\nu_{\overline{A}}$ initiated at p . It intersects $\partial\Omega$ at a point q^δ . Denote by q the projection of q^δ on $\partial\Omega_1$ so that $q^\delta = q + \delta\nu(q)$ (see Figure 2). Denote by $h(p)$ the distance between p and q^δ .

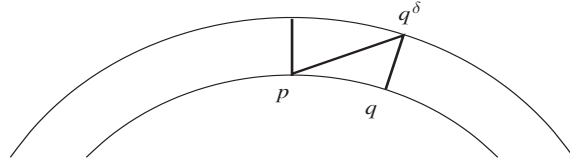


FIGURE 2

Claim. $\frac{h(p)}{\delta} \rightarrow \frac{|\nu_{\overline{A}}(p)|}{\nu(p) \cdot \nu_{\overline{A}}(p)}$ uniformly for p on $\partial\Omega_1$ as $\delta \rightarrow 0$. Let $\beta = h(p)/|\nu_{\overline{A}}|$. Then

$$(2.10) \quad \beta\nu_{\overline{A}}(p) = q - p + \delta\nu(q).$$

Now taking the dot-product of (2.10) with $\nu(p)$, we deduce

$$(2.11) \quad \beta\nu_{\overline{A}}(p) \cdot \nu(p) = (q - p) \cdot \nu(p) + \delta\nu(q) \cdot \nu(p).$$

Recall that $\tau \in C^2$, $\tau \equiv 0$ on $\partial\Omega_1$, and $\nabla\tau(p) = \nu(p)$. So Taylor's theorem yields

$$(q - p) \cdot \nu(p) = O(|q - p|^2).$$

Combining this with (2.11), we have $\beta = O(|q - p|^2 + \delta)$. Now going back to (2.10), we see that $|q - p| = O(|q - p|^2 + \delta)$ and hence $|q - p| = O(\delta)$. This and (2.11) lead to

$$\beta\nu_{\overline{A}}(p) \cdot \nu(p) = O(\delta^2) + \delta\nu(q) \cdot \nu(p),$$

from which the claim follows.

Next we need a convenient coordinate system. Reparameterize Ω_2 by

$$(2.12) \quad (x_1, \dots, x_n) = \overline{F}(p, s) = p + s \frac{\nu_{\overline{A}}}{|\nu_{\overline{A}}|}(p),$$

with $s \in (0, h(p))$. For any $q \in \partial\Omega_1$, the portion of $\partial\Omega_1$ near q can be parameterized as

$$x_i = p_i(r), \quad r = (r_1, \dots, r_{n-1}), \quad 1 \leq i \leq n,$$

with $p(0) = q$, and with $\frac{\partial p}{\partial r_i}(0)$ and $\nu(q)$ forming an orthonormal basis for \mathbb{R}^n . Write

$$\frac{\partial(p_1, \dots, p_n)}{\partial(r_1, \dots, r_{n-1})} := \begin{pmatrix} \frac{\partial p_1}{\partial r_1} & \dots & \frac{\partial p_n}{\partial r_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial p_1}{\partial r_{n-1}} & \dots & \frac{\partial p_n}{\partial r_{n-1}} \end{pmatrix}.$$

Then the surface element on $\partial\Omega_1$ at q is

$$(2.13) \quad dS_p = \left(\det \frac{\partial(p_1, \dots, p_n)}{\partial(r_1, \dots, r_{n-1})} \left(\frac{\partial(p_1, \dots, p_n)}{\partial(r_1, \dots, r_{n-1})} \right)^\top \right)^{1/2} \Big|_{r=0} dr_1 \cdots dr_{n-1} \\ = dr_1 \cdots dr_{n-1},$$

where “ \top ” means transpose. Also, the volume element on Ω_2 at $\bar{F}(q, s)$ is

$$(2.14) \quad dx_1 \cdots dx_n = \left| \det \frac{\partial \bar{F}}{\partial(r_1, \dots, r_{n-1}, s)} \right|_{r=0} dr_1 \cdots dr_{n-1} ds,$$

where

$$\frac{\partial \bar{F}}{\partial(r_1, \dots, r_{n-1}, s)} = \begin{pmatrix} \frac{\partial p_1}{\partial r_1} + s \frac{\partial}{\partial r_1} \left(\frac{\nu_{\bar{A}_1}}{|\nu_{\bar{A}}|} \right) & \cdots & \frac{\partial p_n}{\partial r_1} + s \frac{\partial}{\partial r_1} \left(\frac{\nu_{\bar{A}_n}}{|\nu_{\bar{A}}|} \right) \\ \vdots & \ddots & \vdots \\ \frac{\partial p_1}{\partial r_{n-1}} + s \frac{\partial}{\partial r_{n-1}} \left(\frac{\nu_{\bar{A}_1}}{|\nu_{\bar{A}}|} \right) & \cdots & \frac{\partial p_n}{\partial r_{n-1}} + s \frac{\partial}{\partial r_{n-1}} \left(\frac{\nu_{\bar{A}_n}}{|\nu_{\bar{A}}|} \right) \\ \frac{\nu_{\bar{A}_1}}{|\nu_{\bar{A}}|} & \cdots & \frac{\nu_{\bar{A}_n}}{|\nu_{\bar{A}}|} \end{pmatrix}.$$

Let

$$G(r) = \begin{pmatrix} \frac{\partial p_1}{\partial r_1} & \cdots & \frac{\partial p_n}{\partial r_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial p_1}{\partial r_{n-1}} & \cdots & \frac{\partial p_n}{\partial r_{n-1}} \\ \frac{\nu_{\bar{A}_1}}{|\nu_{\bar{A}}|} & \cdots & \frac{\nu_{\bar{A}_n}}{|\nu_{\bar{A}}|} \end{pmatrix}.$$

Since $\partial\Omega_1 \in C^2$ and $\bar{a}_{ij} \in C^1(\bar{\Omega}_2)$, we have

$$(2.15) \quad \left| \det \frac{\partial \bar{F}}{\partial(r_1, \dots, r_{n-1}, s)} \right| = |\det G(r)| + O(1)s.$$

On the other hand, by using the orthonormality of the vectors $\frac{\partial p}{\partial r_i}(0)$ and $\nu(q)$, we can easily compute

$$\det G(0) = \frac{\nu(q) \cdot \nu_{\bar{A}}(q)}{|\nu_{\bar{A}}(q)|}.$$

This, together with (2.13), (2.14) and (2.15), implies that the volume element on Ω_2 at $\bar{F}(p, s)$ is

$$dx_1 \cdots dx_n = \left(\frac{\nu \cdot \nu_{\bar{A}}}{|\nu_{\bar{A}}|} + O(1)s \right) dS_p ds.$$

We take

$$\phi(x) := \begin{cases} 1 - \frac{\tau}{\delta}, & 0 \leq \tau \leq \delta, p \in \partial\Omega_1, \\ 1, & \text{otherwise,} \end{cases}$$

where we have used the coordinates introduced in (2.4). Observe that $\nabla\phi(x) = \frac{\partial\phi}{\partial\tau}\nu(p) = -\frac{\nu(p)}{\delta}$. For any $\zeta(x, t) \in C^1(\bar{\Omega}_1 \times [0, T])$, extend $\zeta(x, t)$ along $\nu_{\bar{A}}$ to Ω

by $\zeta(p, s, t) = \zeta(p, t)$ for every $p \in \partial\Omega_1$; then it is easy to verify that $\zeta(x, t) \in W_2^{1,1}(\Omega \times (0, T))$. By (2.2), we have

$$(2.16) \quad \int_0^T \left[\int_{\Omega_1} u_t \zeta dx + \int_{\Omega_2} u_t \phi \zeta dx + k \int_{\Omega_1} \nabla u \cdot \nabla \zeta dx + \sigma \int_{\Omega_2} \bar{a}_{ij}(x) u_{x_i} (\phi \zeta)_{x_j} dx \right] dt = \int_0^T \left[\int_{\Omega_1} f \zeta dx + \int_{\Omega_2} f \phi \zeta dx \right] dt.$$

By using Hölder's inequality and Lemma 2.1, we have

$$\begin{aligned} \left| \int_0^T \int_{\Omega_2} f \phi \zeta \right| &\leq \left(\int_0^T \int_{\Omega_2} f^2 \right)^{1/2} \left(\int_0^T \int_{\Omega_2} \zeta^2 \right)^{1/2} = o(1), \\ \left| \int_0^T \int_{\Omega_2} u_t \phi \zeta \right| &\leq \left(\int_0^T \int_{\Omega_2} u_t^2 \right)^{1/2} \left(\int_0^T \int_{\Omega_2} \zeta^2 \right)^{1/2} = o(1), \\ \left| \sigma \int_0^T \int_{\Omega_2} \bar{a}_{ij}(x) u_{x_i} \zeta_{x_j} \phi \right| &\leq \sigma^{1/2} \left(\sigma \int_0^T \int_{\Omega_2} \bar{a}_{ij}(x) u_{x_i} u_{x_j} \right)^{1/2} \left(\int_0^T \int_{\Omega_2} \bar{a}_{ij}(x) \zeta_{x_i} \zeta_{x_j} \right)^{1/2} = o(1). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\sigma \int_0^T \int_{\Omega_2} \bar{a}_{ij} u_{x_i} \phi_{x_j} \zeta dx dt \\ &= -\frac{\sigma}{\delta} \int_0^T \int_{\Omega_2} \bar{a}_{ij} u_{x_i} \nu_j \zeta dx dt \\ &= -\frac{\sigma}{\delta} \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} [\bar{a}_{ij}(p, s) - \bar{a}_{ij}(p, 0)] u_{x_i} \nu_j \zeta \left(\frac{\nu \cdot \nu_{\bar{A}}}{|\nu_{\bar{A}}|} + O(s) \right) ds dS_p dt \\ &\quad - \frac{\sigma}{\delta} \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} \bar{a}_{ij}(p, 0) u_{x_i} \nu_j \zeta \left(\frac{\nu \cdot \nu_{\bar{A}}}{|\nu_{\bar{A}}|} + O(s) \right) ds dS_p dt. \end{aligned}$$

Observe that $|\bar{a}_{ij}(p, s) - \bar{a}_{ij}(p, 0)| \leq Cs$, $\bar{a}_{ij}(p, 0) \nu_j = \nu_{\bar{A}}$, $\zeta(p, s, t) = \zeta(p, 0, t)$, and

$$u_2(p, t) = - \int_0^{h(p)} \frac{1}{|\nu_{\bar{A}}|} \cdot \frac{\partial u_2}{\partial \nu_{\bar{A}}} ds,$$

for $t \in [0, T]$. Observe also that

$$\begin{aligned} \frac{\sigma}{\delta} \int_0^T \int_{\partial\Omega_1} \int_0^{h(p)} s |\zeta \nabla u_2| ds dS_p dt &\leq C\sigma \left(\int_0^T \int_{\Omega_2} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_0^T \int_{\Omega_2} |\zeta|^2 \right)^{\frac{1}{2}} \\ &= o(1), \end{aligned}$$

where we have used the claim.

Therefore it follows from (2.16) that

$$\begin{aligned} &\int_0^T \left[\int_{\Omega_1} u_t \zeta dx + k \int_{\Omega_1} \nabla u \cdot \nabla \zeta dx + \frac{\sigma}{\delta} \int_{\partial\Omega_1} \nu \cdot \nu_{\bar{A}} u(p, t) \zeta(p, t) dS_p \right] dt \\ &= \int_0^T \int_{\Omega_1} f \zeta dx dt + o(1). \end{aligned}$$

By this and the fact that $u_1 \rightarrow w$ weakly in $W_2^{1,1}(\Omega_1 \times (0, T))$ and strongly in $L^2(\partial\Omega_1 \times (0, T))$, we deduce that

$$\int_0^T \left[\int_{\Omega_1} w_t \zeta dx + k \int_{\Omega_1} \nabla w \cdot \nabla \zeta dx + \int_{\partial\Omega_1} \alpha \nu \cdot \nu_{\overline{A}} w \zeta dS_p \right] dt = \int_0^T \int_{\Omega_1} f \zeta dx dt.$$

This completes the proof. \square

We say the coating Ω_2 is *optimally aligned* if at every $x \in \Omega_2$, the vector \overline{px} is an eigenvector of the thermal tensor $(a_{ij}(x))$ corresponding to the smallest eigenvalue $\sigma_{\min}(x)$, where p is the projection of x onto $\partial\Omega_1$, i.e., $x = p + \tau\nu(p)$ ($\tau > 0$). An example is the 2D disk of radius $1 - \delta$ coated by an annulus of thickness δ , the annulus having thermal tensor

$$A(x) = \begin{pmatrix} \sigma x^2 + y^2 & xy(-1 + \sigma) \\ xy(-1 + \sigma) & x^2 + \sigma y^2 \end{pmatrix}.$$

The eigenvalues are $x^2 + y^2$ and $\sigma(x^2 + y^2)$ with eigenvectors angular and radial respectively.

Theorem 2.3. *Assume that $\partial\Omega_1 \in C^2$, and let f and φ be as given in Theorem 2.2. Suppose that the coating Ω_2 is optimally aligned, that*

$$\sigma_{\min}(x) = \sigma\gamma(x)$$

where σ is a small parameter and $\gamma(x)$ is a positive Lipschitz continuous function in $\overline{\Omega_2}$ (γ does not vary with δ), and that σ remains bounded and

$$\frac{\sigma}{\delta} \rightarrow \alpha \quad \text{as } \delta \rightarrow 0^+,$$

where $\alpha \in [0, +\infty]$. Then the solution $u \rightarrow w$ as $\delta \rightarrow 0^+$ strongly in $L^2(\Omega_1 \times (0, T))$, where $w(x, t)$ is the solution of

$$\begin{cases} w_t - k\Delta w = f, & x \in \Omega_1, t > 0, \\ \partial w / \partial \nu + \alpha \gamma w / k = 0, & x \in \partial\Omega_1, t > 0, \\ w = \varphi(x), & x \in \Omega_1, t = 0. \end{cases}$$

If $\alpha = +\infty$, then the boundary condition is the Dirichlet condition $w = 0$.

Proof. The proof is a slight modification of that of Theorem 2.2 without reparameterizing Ω_2 . First note that Lemma 2.1 still holds. From (2.4), it is clear that the volume element on Ω_2 at $F(p, \tau)$ is

$$dx_1 \cdots dx_n = (1 + O(1)\tau) dS_p d\tau.$$

For any $\zeta \in C^1(\overline{\Omega_1} \times [0, T])$, extend ζ along $\nu(p)$ by $\zeta(p, \tau, t) = \zeta(p, t)$. Using the same test function $\phi\zeta$ and observing that $\nabla\phi(x) = \frac{\partial\phi}{\partial\tau}\nu(p) = -\frac{\nu(p)}{\delta}$ is an eigenvector

of $a_{ij}(x)$ corresponding to $\sigma_{\min}(x)$, we infer that

$$\begin{aligned} & \int_0^T \int_{\Omega_2} a_{ij} u_{x_i} \phi_{x_j} \zeta dx dt \\ &= -\frac{\sigma}{\delta} \int_0^T \int_{\Omega_2} \gamma(x) \zeta \frac{\partial u}{\partial \nu} dx dt \\ &= -\frac{\sigma}{\delta} \int_0^T \int_{\partial\Omega_1} \int_0^\delta \gamma(p, \tau) \zeta(p, \tau, t) \frac{\partial u}{\partial \tau} (1 + O(\tau)) d\tau dS_p dt \\ &= -\frac{\sigma}{\delta} \int_0^T \int_{\partial\Omega_1} \int_0^\delta [\gamma(p, \tau) - \gamma(p, 0)] \zeta(p, \tau, t) \frac{\partial u}{\partial \tau} (1 + O(\tau)) d\tau dS_p dt \\ &\quad - \frac{\sigma}{\delta} \int_0^T \int_{\partial\Omega_1} \int_0^\delta \gamma(p, 0) \zeta(p, \tau, t) \frac{\partial u}{\partial \tau} (1 + O(\tau)) d\tau dS_p dt. \end{aligned}$$

Noticing that $\gamma(p, \tau) - \gamma(p, 0) = O(\tau)$, $\zeta(p, \tau, t) = \zeta(p, 0, t)$ and $u(p, t) = -\int_0^\delta \frac{\partial u}{\partial \tau} d\tau$, we have

$$\int_0^T \int_{\Omega_2} a_{ij} u_{x_i} \phi_{x_j} \zeta dx dt = \frac{\sigma}{\delta} \int_0^T \int_{\partial\Omega_1} \zeta(p, 0, t) \gamma(p, 0) u(p, 0, t) dS_p dt + o(1).$$

The other modifications are obvious. \square

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