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RESTRICTION AND EXTENSION OF FOURIER MULTIPLIERS BETWEEN WEIGHTED L^p SPACES ON $\mathbb{R}^n$ AND $\mathbb{T}^n$

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ABSTRACT. Weighted analogues of de Leeuw's restriction theorem for Fourier multipliers on $L^p(\mathbb{R}^n)$ are obtained. Weighted analogues of related extension theorems for multipliers on $L^p(\mathbb{T})$ are also considered.

1. INTRODUCTION

If X is either Euclidean n-space $\mathbb{R}^n$ or the n-dimensional torus $\mathbb{T}^n = \prod_{k=1}^n [0,1)$ and ω is a non-negative measurable weight function on X, denote by $L^p_\omega(X)$ the weighted L^p class of measurable functions f on X for which the norm $||f||_{L^p_\omega(X)}$ given by

$$||f||_{L^p_\omega(X)} = \begin{cases} \left(\int_X |f(x)|^p \omega(x)\, dx \right)^{1/p}, & 1 \le p < \infty \\ \omega\text{-ess.sup}_{x\in X} |f(x)|, & p = \infty \end{cases}$$

is finite. As usual, in the special case that $\omega(x) = 1$ a.e. we abbreviate $L^p_\omega(X)$ as $L^p(X)$ and $||f||_{L^p_\omega(X)}$ as $||f||_{L^p(X)}$.

A bounded measurable function ϕ on $\hat{X}$, where $\widehat{\mathbb{R}^n} = \mathbb{R}^n$ and $\widehat{\mathbb{T}^n} = \mathbb{Z}^n$ with $\mathbb{Z}$ denoting the set of all integers, is called an $L^p_\omega(X)$ multiplier if the operator T_ϕ defined initially on a suitable dense subset $S(X)$ of $L^p_\omega(X)$ by

$$\widehat{(T_\phi f)}(y) = \phi(y)\hat{f}(y), \qquad f \in S(X), y \in \hat{X} \tag{1.1}$$

extends to a bounded operator on $L^p_\omega(X)$, that is, T_ϕ satisfies

$$||T_\phi f||_{L^p_\omega(X)} \le C||f||_{L^p_\omega(X)}$$

for a constant C independent of $f \in S(X)$. The smallest such constant C is denoted by $||T_\phi||_{p,\omega}$. Here, as usual, $\hat{f}$ denotes the Fourier transform of f given by

$$\hat{f}(y) = \int_X e^{-i2\pi x\cdot y} f(x)\, dx, \qquad y \in \hat{X}.$$

We denote the class of all $L^p_\omega(X)$ multipliers by $M_{p,\omega}(X)$, abbreviated $M_p(X)$ in the case that $\omega(x) = 1$ a.e. Typically $S(\mathbb{R}^n)$ is taken to be the Schwartz space $\mathbb{S}(\mathbb{R}^n)$ of rapidly decreasing smooth functions on $\mathbb{R}^n$, and $S(\mathbb{T}^n)$ is typically taken to be

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the space $\mathbb{P}(\mathbb{T}^n)$ of all trigonometric polynomials of period one on $\mathbb{T}^n$. However, in certain situations, to be elaborated later, other choices for $S(X)$ are necessary.

The celebrated restriction theorem of de Leeuw [6] asserts that if $\phi \in M_p(\mathbb{R})$ is continuous on $\mathbb{R}$ and if $\phi|_{\mathbb{Z}}$ denotes the restriction of ϕ to $\mathbb{Z}$, then $\phi|_{\mathbb{Z}} \in M_p(\mathbb{T})$. This result has been generalized in various directions; see for example [1], [2], [3], [5], [7]. The main goal of this paper is to generalize de Leeuw's result to the context of weighted L^p spaces by determining conditions on the weight functions U and u so that ϕ continuous on $\mathbb{R}^n$ and $\phi \in M_{p,U}(\mathbb{R}^n)$ would imply $\phi|_{\mathbb{Z}^n} \in M_{p,u}(\mathbb{T}^n)$. We also briefly consider the corresponding extension problem in one dimension, namely, given $\phi \in M_{p,u}(\mathbb{T})$ we give a class of constructions by which the domain of ϕ may be extended from $\mathbb{Z}$ to $\mathbb{R}$ and for which the extended function, denoted by Φ, belongs to $M_{p,U}(\mathbb{R})$ for suitable U.

For the restriction problem, we consider first the case where the weight function on $\mathbb{R}^n$ is periodic. Our result is as follows.

Theorem 1.1. *Suppose $1 < p < \infty$ and that U is a non-negative periodic weight function of period one on $\mathbb{R}^n$. Let u denote the restriction of U to $\mathbb{T}^n$, and suppose that $u \in L^1(\mathbb{T}^n)$. If $\phi \in M_{p,U}(\mathbb{R}^n)$ is continuous on $\mathbb{R}^n$, then $\phi|_{\mathbb{Z}^n} \in M_{p,u}(\mathbb{T}^n)$ with $||T_{\phi|_{\mathbb{Z}^n}}||_{p,u} \leq ||T_\phi||_{p,U}$.*

In a recent lengthy paper, Berkson and Gillespie [3] obtained the conclusion of Theorem 1.1 in dimension one under the stronger hypothesis that u belongs to the Muckenhoupt class $A_p(\mathbb{T})$. As we shall see, the proof of Theorem 1.1 is comparatively short. Moreover, in the context of weighted $L^p_u(\mathbb{T}^n)$ spaces that contain the trigonometric polynomials, Theorem 1.1 is the best possible since such spaces contain the constant polynomial one and hence u is necessarily integrable on $\mathbb{T}^n$.

Our next two results for the restriction problem deal with non-periodic weights from the class of power weights $U(x) = |x|^\gamma$.

Theorem 1.2. *Let $1 < p < \infty$ and $-n < \gamma < n(p-1)$. If $U(x) = |x|^\gamma$ and $\phi \in M_{p,U}(\mathbb{R}^n)$ is continuous on $\mathbb{R}^n$, then $\phi|_{\mathbb{Z}^n} \in M_p(\mathbb{T}^n)$.*

Corollary 1.3. *Let $1 < p < \infty$ and $-n < \gamma < n(p-1)$. If $U(x) = |x|^\gamma$ and $\phi \in M_{p,U}(\mathbb{R}^n)$ is continuous on $\mathbb{R}^n$, then $\phi \in M_p(\mathbb{R}^n)$.*

In each of Theorem 1.1, Theorem 1.2, and Corollary 1.3, the hypothesis that ϕ is continuous on $\mathbb{R}^n$ may be replaced by the weaker hypothesis that ϕ is continuous at the lattice points $\mathbb{Z}^n$. Indeed, the continuity hypothesis may be replaced by the requirement that for each $m \in \mathbb{Z}^n$ there is a number ϕ_m satisfying $\lim_{\delta \to 0+} \delta^{-n} \int_{|x|<\delta} [\phi(m+x) - \phi_m]\, dx = 0$, in which case $\phi|_{\mathbb{Z}^n}$ at m is taken to mean ϕ_m. See [12, Lemma 3.16, p. 263].

It is possible to obtain analogues of Theorem 1.2 in dimension one for values of $\gamma \geq p-1$. However, in order to allow for the possibility of non-constant multipliers (see [11]), in this case the Schwartz class $\mathbb{S}(\mathbb{R})$ must be replaced by a smaller class such as $\mathbb{S}_{00}(\mathbb{R})$, the class of Schwartz functions on $\mathbb{R}$ with Fourier transform compactly supported away from the origin. We then have the following theorem.

Theorem 1.4. *Let $U(x) = |x|^2$ for $x \in \mathbb{R}$ and $u(x) = |x|^2$ for $x \in \mathbb{T}$. If $\phi \in M_{2,U}(\mathbb{R})$ is continuous on $\mathbb{R}$, then $\phi|_{\mathbb{Z}} \in M_{2,u}(\mathbb{T})$.*

Concerning extension of weighted multipliers on $\mathbb{T}$ to weighted multipliers on $\mathbb{R}$, we follow the method of Jodiet [9] in the unweighted case. There $\phi \in M_p(\mathbb{T})$ are extended to $\Phi \in M_p(\mathbb{R})$ by convolving ϕ with a suitable kernel Ψ. More precisely, $\Psi \in L^\infty(\mathbb{R})$ is chosen so that $\Phi(x) = \sum\limits_{m\in\mathbb{Z}} \phi(m)\Psi(x-m)$ exists a.e. and $\Phi \in M_p(\mathbb{R})$ whenever $\phi \in M_p(\mathbb{T})$. Two such choices of Ψ given in [9] are $\Psi(x) = \chi_{[0,1)}(x)$ and $\Psi(x) = \max(1-|x|, 0)$; other examples of suitable Ψ have been given in [8], [2], [4]. Characterizations of all such kernels Ψ in the cases $p = 1$ and $p = 2$ were given in [10].

We have the following theorem, which is in the nature of a converse to Theorem 1.4.

Theorem 1.5. *Let $U(x) = |x|^2$ for $x \in \mathbb{R}$ and $u(x) = |x|^2$ for $x \in \mathbb{T}$. Suppose $\Psi(x) = \Lambda(x+1) - \Lambda(x) + \chi_{[0,1)}(x)$ where $\Lambda : \mathbb{R} \to \mathbb{R}$ is such that*

(i) *$\Lambda(x) = 0$ if $x \in (-\infty, 0] \cup [1, \infty)$,*
(ii) *$\lim_{x\to 1-} \Lambda(x) = 1$,*
(iii) *Λ is absolutely continuous on $[0, a)$ for all $0 < a < 1$ and $\int_0^a |\Lambda'(x)|^2 dx \le C$.*

If $\phi \in M_{2,u}(\mathbb{T})$, then $\Phi(x) = \sum_{m\in\mathbb{Z}} \phi(m)\Psi(x-m)$ belongs to $M_{2,U}(\mathbb{R})$.

The proofs are given in the following sections. The proofs of the first two theorems are refinements of the proof of de Leeuw's Theorem in the unweighted case as presented in [12]. Thus, we begin with generalizations of certain lemmas presented there. For these, it is convenient to set $\omega_\delta(x) = e^{-\pi\delta|x|^2}$ for $\delta > 0$ and $x \in \mathbb{R}^n$.

2. Lemmas

Lemma 2.1. *For $\delta > 0$, ω_δ satisfies $||\omega_\delta||_{L^1(\mathbb{R}^n)} = \delta^{-n/2}$ and $\widehat{\omega_\delta}(x) = \delta^{-n/2}\omega_{1/\delta}(x)$; and if $\gamma \ge 0$, there is a constant $C_{n,\gamma}$ depending only on n and γ such that for $x \in \mathbb{T}^n$,*

$$\delta^{(\gamma+n)/2} \sum_{k\in\mathbb{Z}^n} \omega_\delta(x+k)|x+k|^\gamma \le C_{n,\gamma}(1+\delta)^{(\gamma+n)/2}. \tag{2.1}$$

Proof. It suffices to prove (2.1), as the other assertions are both well known and easily verified.

We first prove (2.1) for $n = 1$. For fixed $\delta > 0$ and $\gamma \ge 0$, set $x_0 = \big(\gamma/(2\pi\delta)\big)^{1/2}$. Then $W(x) = \omega_\delta(x)|x|^\gamma$ is nondecreasing on $[0, x_0)$ and nonincreasing on (x_0, ∞). Thus,

$$\begin{aligned}
\delta^{(\gamma+1)/2} \sum_{k=0}^{\infty} W(k) &\le \delta^{(\gamma+1)/2}\Bigg(\sum_{0\le k\le \lfloor x_0\rfloor -1} W(k) + 2W(x_0) + \sum_{k\ge \lfloor x_0\rfloor+2} W(k)\Bigg) \\
&\le \delta^{(\gamma+1)/2}\left(2W(x_0) + \int_0^\infty W(x)\,dx\right) \qquad (2.2)\\
&= 2\left(\frac{\gamma}{2\pi}\right)^{\gamma/2} e^{-\gamma/2}\delta^{1/2} + \frac{1}{2}\frac{\Gamma(\frac{\gamma+1}{2})}{\pi^{\frac{\gamma+1}{2}}}.
\end{aligned}$$

Now, for $x \in \mathbb{T}$,

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \omega_\delta(x+k)|x+k|^\gamma &\leq \sum_{k=0}^{\infty} \omega_\delta(k)|k+1|^\gamma + \sum_{k=1}^{\infty} \omega_\delta(k-1)|k|^\gamma \\
&= 2\sum_{k=0}^{\infty} \omega_\delta(k)|k+1|^\gamma \\
&\leq 2\Big(1+2^\gamma \sum_{k=1}^{\infty} W(k)\Big).
\end{aligned} \tag{2.3}
$$

Combining (2.2) and (2.3) we obtain (2.1) for $n=1$ with

$$C_{1,\gamma} = 2 + 2^{\gamma+2}\big(\frac{\gamma}{2\pi}\big)^{\gamma/2} e^{-\gamma/2} + 2^\gamma \frac{\Gamma\big(\frac{\gamma+1}{2}\big)}{\pi^{\frac{\gamma+1}{2}}}.$$

Now, for $n>1$ and $x \in \mathbb{R}^n$, write $x=(y,z)$ with $y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}$. Then for $k=(l,m) \in \mathbb{Z}^{n-1}\times\mathbb{Z}$,

$$|x+k|^\gamma = \big(|y+l|^2 + |z+m|^2\big)^{\gamma/2} \leq 2^{\gamma/2}\big(|y+l|^\gamma + |z+m|^\gamma\big)$$

and therefore

$$\delta^{(\gamma+n)/2} \sum_{k\in\mathbb{Z}^n} \omega_\delta(x+k)|x+k|^\gamma \tag{2.4}$$

does not exceed $2^{\gamma/2}$ times the sum of

$$\Bigg[\delta^{(\gamma+n-1)/2} \sum_{l\in\mathbb{Z}^{n-1}} e^{-\pi\delta|y+l|^2}|y+l|^\gamma\Bigg]\Bigg[\delta^{1/2}\sum_{m\in\mathbb{Z}} e^{-\pi\delta|z+m|^2}\Bigg] \tag{2.5}$$

and

$$\Bigg[\delta^{(n-1)/2} \sum_{l\in\mathbb{Z}^{n-1}} e^{-\pi\delta|y+l|^2}\Bigg]\Bigg[\delta^{(\gamma+1)/2}\sum_{m\in\mathbb{Z}} e^{-\pi\delta|z+m|^2}|z+m|^\gamma\Bigg]. \tag{2.6}$$

An induction argument now shows that (2.1) holds for n, for if it holds for $n-1$, then (2.4), (2.5) and (2.6) show that it holds for n with

$$C_{n,\gamma} = 2^{\gamma/2}\big(C_{n-1,\gamma}C_{1,0} + C_{n-1,0}C_{1,\gamma}\big).$$

□

Lemma 2.2. *Let $\gamma > -n$ and set $c_{n,\gamma} = \Gamma\big((\gamma+n)/2\big)/[\pi^{\gamma/2}\Gamma(n/2)]$. Suppose f is periodic on $\mathbb{R}^n$ with period one. If, in addition, either*

(i) *f is continuous on $\mathbb{R}^n$ or*
(ii) *$\gamma \geq 0$ and $f \in L^1(\mathbb{T}^n)$,*

then

$$\lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2} \int_{\mathbb{R}^n} f(x)\omega_\epsilon(x)|x|^\gamma\,dx = c_{n,\gamma}\int_{\mathbb{T}^n} f(x)\,dx. \tag{2.7}$$

Proof. We first show that (2.7) holds for polynomials f. Set $g(x) = \omega_1(x)|x|^\gamma$ for $x \in \mathbb{R}^n$. Then $g \in L^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} g(x)\,dx = c_{n,\gamma}$ and therefore the Riemann-Lebesgue lemma shows that for $m \in \mathbb{Z}^n$,

$$\lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2}\int_{\mathbb{R}} e^{-2\pi i m\cdot x}\omega_\epsilon(x)|x|^\gamma\,dx = \lim_{\epsilon\to 0+}\hat{g}(m/\sqrt{\epsilon}) = \begin{cases} 0, & m\neq 0 \\ c_{n,\gamma}, & m=0. \end{cases}$$

Thus (2.7) holds for $f(x) = e^{-2\pi i m\cdot x}$ and hence also for all $f \in \mathbb{P}(\mathbb{T}^n)$.

Now, if f satisfies (i), there are $f_k \in \mathbb{P}(\mathbb{T}^n)$ with $||f - f_k||_{L^\infty(\mathbb{R}^n)} = ||f - f_k||_{L^\infty(\mathbb{T}^n)} \to 0$; so

$$\left| \lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2} \int_{\mathbb{R}^n} [f(x) - f_k(x)]\omega_\epsilon(x)|x|^\gamma\, dx \right| \le c_{n,\gamma}||f - f_k||_{L^\infty(\mathbb{R}^n)}$$

shows that

$$\lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2} \int_{\mathbb{R}^n} f(x)\omega_\epsilon(x)|x|^\gamma\, dx = \lim_{k\to\infty} \lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2} \int_{\mathbb{R}^n} f_k(x)\omega_\epsilon(x)|x|^\gamma\, dx.$$

Since $\int_{\mathbb{T}^n} f_k \to \int_{\mathbb{T}^n} f$, (2.7) for f follows from that for f_k already shown.

Similarly, if (ii) holds, there are $f_k \in \mathbb{P}(\mathbb{T}^n)$ with $||f - f_k||_{L^1(\mathbb{T})} \to 0$ and Lemma 2.1 shows there is a constant $C_{n,\gamma}$ such that

$$\begin{aligned} & \left| \lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2} \int_{\mathbb{R}^n} [f(x) - f_k(x)]\omega_\epsilon(x)|x|^\gamma\, dx \right| \\ = & \left| \lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2} \int_{\mathbb{T}^n} [f(x) - f_k(x)] \sum_{l\in\mathbb{Z}^n} \omega_\epsilon(x+l)|x+l|^\gamma\, dx \right| \\ \le & \quad C_{n,\gamma}||f - f_k||_{L^1(\mathbb{T}^n)}. \end{aligned}$$

Hence

$$\lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2} \int_{\mathbb{R}^n} f(x)\omega_\epsilon(x)|x|^\gamma\, dx = \lim_{k\to\infty} \lim_{\epsilon\to 0+} \epsilon^{(\gamma+n)/2} \int_{\mathbb{R}^n} f_k(x)\omega_\epsilon(x)|x|^\gamma\, dx,$$

and since $\int_{\mathbb{T}^n} f_k \to \int_{\mathbb{T}^n} f$, (2.7) for f again follows from that for f_k already shown. □

Lemma 2.3. *Let ϕ be bounded and continuous on $\mathbb{R}^n$ and let T_ϕ and $T_{\phi|_{\mathbb{Z}^n}}$ be defined on $\mathbb{S}(\mathbb{R}^n)$ and $\mathbb{P}(\mathbb{T}^n)$ respectively by (1.1). Suppose U is a non-negative periodic weight function on $\mathbb{R}^n$ of period one and let u denote its restriction to $\mathbb{T}^n$. If $u \in L^1(\mathbb{T}^n)$, then for all $P, Q \in \mathbb{P}(\mathbb{T}^n)$,*

$$\lim_{\epsilon\to 0+} \epsilon^{n/2} \int_{\mathbb{R}^n} (T_\phi P\omega_{\alpha\epsilon})(x)\overline{Q(x)}\omega_{\beta\epsilon}(x)U(x)\, dx = \int_{\mathbb{T}^n} (T_{\phi|_{\mathbb{Z}^n}} P)(x)\overline{Q(x)}u(x)\, dx$$

whenever $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta = 1$.

Proof. Since $u \in L^1(\mathbb{T}^n)$, there are $u_k \in \mathbb{P}(\mathbb{T}^n)$ with $||u - u_k||_{L^1(\mathbb{T}^n)} \to 0$, and since $Q\overline{u_k} \in \mathbb{P}(\mathbb{T}^n)$, Lemma 3.11 of [12, p.261] shows that

$$\begin{aligned} &\lim_{\epsilon\to 0+} \epsilon^{n/2} \int_{\mathbb{R}^n} (T_\phi P\omega_{\alpha\epsilon})(x)\overline{Q(x)}u_k(x)\omega_{\beta\epsilon}(x)\, dx \\ &= \int_{\mathbb{T}^n} (T_{\phi|_{\mathbb{Z}^n}} P)(x)\overline{Q(x)}u_k(x)\, dx. \end{aligned} \tag{2.8}$$

Now since $(T_{\phi|_{\mathbb{Z}^n}} P)(x)\overline{Q(x)} \in \mathbb{P}(\mathbb{T}^n)$ we have

$$\left| \int_{\mathbb{T}^n} \big(T_{\phi|_{\mathbb{Z}^n}} P\big)(x)\overline{Q(x)}[u_k(x) - u(x)]\, dx \right| \le ||[T_{\phi|_{\mathbb{Z}^n}} P]\bar{Q}||_{L^\infty(\mathbb{T}^n)}||u_k - u||_{L^1(\mathbb{T}^n)},$$

which shows that

$$\lim_{k\to\infty} \int_{\mathbb{T}^n} \big(T_{\phi|_{\mathbb{Z}^n}} P\big)(x)\overline{Q(x)}u_k(x)\, dx = \int_{\mathbb{T}^n} \big(T_{\phi|_{\mathbb{Z}^n}} P\big)(x)\overline{Q(x)}u(x)\, dx. \tag{2.9}$$

On the other hand, for $\delta > 0$ and $m \in \mathbb{Z}^n$,

$$\begin{aligned} ||T_\phi\big(e^{2\pi i m\cdot(\cdot)}\omega_\delta(\cdot)\big)||_{L^\infty(\mathbb{R}^n)} &\leq ||\phi(\cdot)[\widehat{e^{2\pi i m\cdot(\cdot)}\omega_\delta}(\cdot)]||_{L^1(\mathbb{R}^n)} \\ &\leq ||\phi||_{L^\infty(\mathbb{R}^n)}||\delta^{-n/2}\omega_{1/\delta}(\cdot - m)||_{L^1(\mathbb{R}^n)} \\ &= ||\phi||_{L^\infty(\mathbb{R}^n)}, \end{aligned}$$

so that if $P(x) = \sum_m a_m e^{2\pi i m\cdot x}$, then we have

$$||T_\phi(P\omega_\delta)||_{L^\infty(\mathbb{R}^n)} \leq \sum_m |a_m|\,||T_\phi(e^{2\pi i m\cdot(\cdot)}\omega_\delta(\cdot)||_{L^\infty(\mathbb{R}^n)} \leq C_P||\phi||_{L^\infty(\mathbb{R}^n)}.$$

Therefore

$$\begin{aligned} &\left|\epsilon^{n/2}\int_{\mathbb{R}^n}\big(T_\phi(P\omega_{\alpha\epsilon})\big)(x)\overline{Q(x)}\omega_{\beta\epsilon}(x)[u_k(x)-U(x)]\,dx\right| \\ &\leq C_P||\phi||_{L^\infty(\mathbb{R}^n)}||Q||_{L^\infty(\mathbb{R}^n)}\left[\epsilon^{n/2}\int_{\mathbb{R}^n}|u_k(x)-U(x)|\omega_{\beta\epsilon}(x)\,dx\right] \end{aligned} \tag{2.10}$$

and Lemma 2.2 shows that the last term in (2.10) tends to $\beta^{-n/2}||u_k - u||_{L^1(\mathbb{T}^n)}$ as $\epsilon \to 0+$. Thus,

$$\begin{aligned} &\lim_{k\to\infty}\lim_{\epsilon\to 0+}\epsilon^{n/2}\int_{\mathbb{R}^n}\big(T_\phi(P\omega_{\alpha\epsilon})\big)(x)\overline{Q(x)}\omega_{\beta\epsilon}(x)u_k(x)\,dx \\ &\quad = \lim_{\epsilon\to 0+}\epsilon^{n/2}\int_{\mathbb{R}^n}\big(T_\phi(P\omega_{\alpha\epsilon})\big)(x)\overline{Q(x)}\omega_{\beta\epsilon}(x)U(x)\,dx. \end{aligned}$$

Combining this with (2.8) and (2.9) completes the proof. □

3. Proof of Theorem 1.1

Proof. Let $P, Q \in \mathbb{P}(\mathbb{T}^n)$. Then Lemma 2.3 with $\alpha = 1/p$ and $\beta = 1/p'$ together with Hölder's inequality yields

$$\begin{aligned} &\left|\int_{\mathbb{T}^n}\big(T_{\phi|_{\mathbb{Z}^n}}P\big)(x)\overline{Q(x)}u(x)\,dx\right| \\ &\leq \lim_{\epsilon\to 0+}\left(\epsilon^{n/2}\int_{\mathbb{R}^n}\left|\big(T_\phi(P\omega_{\epsilon/p})\big)(x)\right|^p U(x)\,dx\right)^{1/p} \\ &\qquad\times\left(\epsilon^{n/2}\int_{\mathbb{R}^n}|Q(x)|^{p'}\omega_\epsilon(x)U(x)\,dx\right)^{1/p'} \end{aligned}$$

and $T_\phi \in M_{p,U}(\mathbb{R}^n)$ shows that this does not exceed

$$\begin{aligned} &||T_\phi||_{p,U}\lim_{\epsilon\to 0+}\left(\epsilon^{n/2}\int_{\mathbb{R}^n}|P(x)|^p\omega_\epsilon(x)U(x)\,dx\right)^{1/p} \\ &\qquad\times\left(\epsilon^{n/2}\int_{\mathbb{R}^n}|Q(x)|^{p'}\omega_\epsilon(x)U(x)\,dx\right)^{1/p'}. \end{aligned} \tag{3.1}$$

Since both $|P(x)|^p U(x)$ and $|Q(x)|^{p'}U(x)$ are periodic on $\mathbb{R}^n$ with period one and are integrable on $\mathbb{T}^n$, Lemma 2.2 shows that (3.1) equals

$$||T_\phi||_{p,U}||P||_{L^p_u(\mathbb{T}^n)}||Q||_{L^{p'}_u(\mathbb{T}^n)}.$$

Thus we have shown that

$$\left|\int_{\mathbb{T}^n}\big(T_{\phi|_{\mathbb{Z}^n}}P\big)(x)\overline{Q(x)}u(x)\,dx\right| \leq ||T_\phi||_{p,U}||P||_{L^p_u(\mathbb{T}^n)}||Q||_{L^{p'}_u(\mathbb{T}^n)} \tag{3.2}$$

for all $P, Q \in \mathbb{P}(\mathbb{T}^n)$. Since u is integrable, $\mathbb{P}(\mathbb{T}^n)$ is dense in both $L^p_u(\mathbb{T}^n)$ and $L^{p'}_u(\mathbb{T}^n)$; therefore (3.2) shows that $T_{\phi|_{\mathbb{Z}^n}} \in M_{p,u}(\mathbb{T}^n)$ with $||T_{\phi|_{\mathbb{Z}^n}}||_{p,u} \leq ||T_\phi||_{p,U}$. □

4. Proof of Theorem 1.2

Proof. For $P, Q \in \mathbb{P}(\mathbb{T}^n)$ and $U(x) = |x|^\gamma$, Hölder's inequality shows that

$$\epsilon^{n/2}\left|\int_{\mathbb{R}^n}\big(T_\phi(P\omega_{\epsilon/p})\big)(x)\overline{Q(x)}\omega_{\epsilon/p'}(x)\,dx\right| \tag{4.1}$$

does not exceed the product of

$$\left(\epsilon^{(\gamma+n)/2}\int_{\mathbb{R}^n}|\big(T_\phi(P\omega_{\epsilon/p})\big)(x)|^p|x|^\gamma\,dx\right)^{1/p} \tag{4.2}$$

and

$$\left(\epsilon^{(-p'\gamma/p+n)/2}\int_{\mathbb{R}^n}|Q(x)|^{p'}\omega_\epsilon(x)|x|^{-p'\gamma/p}\,dx\right)^{1/p'}.$$

Since $T_\phi \in M_{p,U}(\mathbb{R}^n)$, (4.2) does not exceed

$$||T_\phi||_{p,U}\left(\epsilon^{(\gamma+n)/2}\int_{\mathbb{R}^n}|P(x)|^p\omega_\epsilon(x)|x|^\gamma\,dx\right)^{1/p}.$$

Thus, in view of Lemmas 2.3 and 2.2, taking the limit in (4.1) as $\epsilon \to 0+$ yields

$$\left|\int_{\mathbb{T}^n}\big(T_{\phi|_{\mathbb{Z}^n}}P\big)(x)\overline{Q(x)}\,dx\right| \leq c_{n,\gamma}^{1/p}\,c_{n,-p'\gamma/p}^{1/p'}||T_\phi||_{p,U}||P||_{L^p(\mathbb{T}^n)}||Q||_{L^{p'}(\mathbb{T}^n)}.$$

Since $\mathbb{P}(\mathbb{T}^n)$ is dense in both $L^p(\mathbb{T}^n)$ and $L^{p'}(\mathbb{T}^n)$, the proof is complete. □

5. Proof of Corollary 1.3

Proof. Let $\phi_\epsilon(x) = \phi(\epsilon x)$ and $f_\epsilon(x) = f(\epsilon x)$ for $\epsilon > 0$ and $x \in \mathbb{R}^n$. It is easy to verify from (1.1) that $(T_{\phi_\epsilon}f)(x) = (T_\phi f_\epsilon)(x/\epsilon)$ and hence also, with $U(x) = |x|^\gamma$, that

$$||T_{\phi_\epsilon}||_{p,U} \leq ||T_\phi||_{p,U}.$$

Theorem 1.2 then shows that there is a constant $C_{n,p,\gamma}$ independent of ϵ such that

$$||T_{\phi_\epsilon|_{\mathbb{Z}^n}}||_p \leq C_{n,p,\gamma}||T_\phi||_{p,U},$$

and the corollary now follows from Theorem 3.18 of [12, p. 264]. □

6. Proof of Theorem 1.4

Proof. Let $U(x) = |x|^2$ for $x \in \mathbb{R}$ and let $u(x) = |x|^2$ for $x \in \mathbb{T}$. Since $\phi \in M_{2,U}(\mathbb{R})$, Theorem 2.2 of [11] shows that ϕ has a weak derivative ϕ' on $\mathbb{R} \setminus \{0\}$ satisfying

$$||\phi||_{L^\infty(\mathbb{R})} + \sup_{r>0}\left[r\int_{r\leq|x|\leq 2r}|\phi'(x)|^2\,dx\right]^{1/2} \leq C||T_\phi||_{2,U} \tag{6.1}$$

for some constant C. Fix an integer $m \neq -1, 0$, and for $0 < \delta < 1/2$ let g_δ be infinitely differentiable on $\mathbb{R}$, satisfying $g_\delta(x) = 1$ for $x \in [m+\delta, m+1-\delta]$ and

$g_\delta(x) = 0$ for $x \in \mathbb{R}\backslash[m, m+1]$, with g_δ monotone in $[m, m+\delta]$ and $[m+1-\delta, m+1]$. Since ϕ is continuous and $\phi'(x)$ is locally integrable on $[m, m+1]$, we have

$$\begin{aligned}\phi(m+1) - \phi(m) &= -\lim_{\delta\to 0+}\int \phi(x) g_\delta'(x)\,dx \\ &= \lim_{\delta\to 0+}\int \phi'(x) g_\delta(x)\,dx = \int_m^{m+1}\phi'(x)\,dx.\end{aligned} \tag{6.2}$$

Thus, (6.2) and Jensen's inequality show that for each integer $j \geq 2$ we have

$$\begin{aligned}&\sup_{\mathbb{Z}}\big|\phi|_{\mathbb{Z}}\big| + \Big[j\sum_{j\leq|m|\leq 2j}|\phi(m+1)-\phi(m)|^2\Big]^{1/2} \\ &\leq ||\phi||_{L^\infty(\mathbb{R})} + \Big[j\sum_{j\leq|m|\leq 2j}\int_m^{m+1}|\phi'(x)|^2\,dx\Big]^{1/2},\end{aligned} \tag{6.3}$$

and (6.1) shows this does not exceed $3C||T_\phi||_{2,U}$. A similar estimate, noting that $|\phi(-1)-\phi(0)| \leq 2||\phi||_{L^\infty(\mathbb{R})}$, shows that this bound of the left-hand side of (6.3) also holds for $j = 1$. Hence, Theorem 10.1 of [11] shows that $\phi|_{\mathbb{Z}} \in M_{2,u}(\mathbb{T})$ with $||T_{\phi|_{\mathbb{Z}}}||_{p,u} \leq c||T_\phi||_{p,U}$ for some constant c. □

7. Proof of Theorem 1.5

Proof. From the definition,

$$\Phi(x) = \sum_{m\in\mathbb{Z}}\phi(m)[\Lambda(x+1-m) - \Lambda(x-m)] + \sum_{m\in\mathbb{Z}}\phi(m)\chi_{[0,1)}(x-m).$$

A summation by parts in the first sum shows that $\Phi(x) = \sum_{m\in\mathbb{Z}}\Psi_{m,\phi}(x)$ where

$$\Psi_{m,\phi}(x) = [\phi(m+1)-\phi(m)]\Lambda(x-m) + \phi(m)\chi_{[0,1)}(x-m).$$

Since $\Psi_{m,\phi}(x)$ is zero on $\mathbb{R}\backslash[m, m+1)$ and continuous on $(m, m+1)$ with $\Psi_{m,\phi}(m) = \phi(m)$ and $\lim_{x\to(m+1)-}\Psi_{m,\phi}(x) = \phi(m+1)$, this shows that Φ is continuous on $\mathbb{R}$, with $\Phi(m) = \phi(m)$. Moreover,

$$|\Phi(x)| \leq \big(2\sup_{x\in[0,1]}|\Lambda(x)| + 1\big)\sup_{m\in\mathbb{Z}}|\phi(m)|, \tag{7.1}$$

and Φ is absolutely continuous on every interval $[m, m+1-a]$ for $0 < a < 1$ and $m \in \mathbb{Z}$ with

$$\Phi'(x) = \sum_{m\in\mathbb{Z}}[\phi(m+1)-\phi(m)]\Lambda'(x-m)$$

for almost all $x \in \mathbb{R}$. Thus, for $r \geq 1$,

$$\begin{aligned}&r\int_{r\leq|x|\leq 2r}|\Phi'(x)|^2\,dx \leq r\int_{\lfloor r\rfloor\leq|x|\leq\lfloor 2r\rfloor+1}|\Phi'(x)|^2\,dx \\ &\leq r\sum_{\lfloor r\rfloor\leq|m|\leq\lfloor 2r\rfloor+1}\int_m^{m+1}|\Lambda'(x-m)|^2|\phi(m+1)-\phi(m)|^2\,dx \\ &\leq ||\Lambda'||^2_{L^2(\mathbb{T})}(\lfloor r\rfloor+1)\Bigg(\sum_{\lfloor r\rfloor\leq|m|\leq 2\lfloor r\rfloor} + \sum_{\lfloor r\rfloor+1\leq|m|\leq 2\lfloor r\rfloor+2}\Bigg)|\phi(m+1)-\phi(m)|^2,\end{aligned}$$

and Theorem 10.2 of [11] shows that this does not exceed a constant times

$$||\Lambda'||^2_{L^2(\mathbb{T})}||T_\phi||^2_{2,u}. \tag{7.2}$$

Combining (7.1) and (7.2), Theorem 2.1 of [11] shows that $T_\Phi \in M_{p,U}(\mathbb{R})$. □

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