PALINDROMIC RANDOM TRIGONOMETRIC POLYNOMIALS

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ABSTRACT. We show that if a real trigonometric polynomial has few real roots, then the trigonometric polynomial obtained by writing the coefficients in reverse order must have many real roots. This is used to show that a class of random trigonometric polynomials has, on average, many real roots. In the case that the coefficients of a real trigonometric polynomial are independently and identically distributed, but with no other assumptions on the distribution, the expected fraction of real zeros is at least one-half. This result is best possible.

1. INTRODUCTION

A random trigonometric polynomial is a function of the form

$$(1.1) \qquad F(x) = \sum_{k=0}^{K} a_k \cos(kx),$$

where the a_k are random variables.

We give a simple proof of the following very general result:

Theorem 1.1. *Suppose $C(x)$ is a real trigonometric polynomial,*

$$(1.2) \qquad C(x) = \sum_{k=L}^{M} a_k \cos(kx),$$

where the a_k are independently and identically distributed real random variables. Then, for those $C(x)$ which are not identically zero, the expected number of real zeros in each period is at least $L + M$.

In particular, on average at least half the zeros of $C(x)$ are real.

The surprising thing about the theorem is that we obtain a fairly strong conclusion with absolutely no assumption on the coefficients beyond the fact that they are iid. We now contrast this with previous results on this topic.

In the case that the a_k are normally distributed, the Kac-Rice formula [3] provides a quick way to determine the expected number of real zeros of F. Here "number of real zeros" refers to the number of real zeros in one period. In the case

that the normal distributions also are independent and have equal variance, Dunnage [2] showed that for large K the expected number of real zeros is asymptotically $\frac{2K}{\sqrt{3}}$. That is, about 57.7% of the zeros are real.

There are some exact results for the expected number of real zeros of (1.1) when the coefficients are not Gaussian. For example, [7] considered the case that the distribution of coefficients is in the domain of attraction of a stable law with exponent $0 < \alpha \le 2$. More generally, [9] established a result assuming that the expected value of $|a_k|^3$ is finite. Both of those results have an additional condition that the probability $a_k = 0$ is zero.

There are related results for random algebraic polynomials. The papers [4, 5] consider the case where a_k is in the domain of attraction of the normal law, and the probability $a_k = 0$ is zero. In [6] the requirement is that a_k have bounded density (which implies that the probability $a_k = 0$ is zero) and the expected value of $|a_k|^7$ is finite.

The need to assume that the probability $a_k = 0$ is zero arises because if there is a positive probability of the polynomial being identically zero, then the expected number of real zeros is meaningless. Thus, the identically zero polynomials must be eliminated before computing the expected value, and available methods cannot do that.

Restrictions on the moments of the a_k arise for several reasons. Typically upper bounds on the moments are needed for technical reasons, and it is possible that existing results can be improved. Lower bounds on the variance, as in [7, 8, 9], are unavoidable. This is required to rule out, for example, the case that every a_k is identically equal to 1, for which all the roots of (1.1) are real. This shows that the condition of a lower bound on the variance, as in [7, 8, 9], cannot be eliminated.

Despite the need for conditions on a_k in order to determine the exact expected number of real zeros of (1.1), we give an easy proof of Theorem 1.1, which has no conditions on the distribution. We avoid the technical issue of dealing with the possibility that the polynomial is identically zero because our method allows us to ignore that case. The cost is that we obtain a lower bound instead of an exact expected value. Nevertheless, our result is best possible because if a_k is very likely to be zero with a small chance of equaling 1, for example, then the nonzero $C(x)$ usually have only one term and our lower bound is obtained.

To state a more general version of the above theorem, we will refer to a joint probability distribution on a_k, $0 \le k \le K$, as *palindromic* if the a_k are independent and $(a_0, \ldots, a_K)$ has the same distribution as $(a_K, \ldots, a_0)$. Note that if $a_0, \ldots, a_K$ are iid, then the joint distribution is palindromic.

Theorem 1.2. *If the joint probability distribution of $a_0, \ldots, a_K$ is palindromic, then the expected number of real zeros of*

$$(1.3) \qquad\qquad C(x) = \sum_{k=0}^{K} a_k \cos(kx),$$

for those $C(x)$ which are not identically zero, is at least K.

To obtain Theorem 1.1 set $a_0, \ldots, a_{L-1}$ and $a_{K-L+1}, \ldots, a_K$ to always be zero, and let the other coefficients be independently and identically distributed.

Theorem 1.2 follows from a simple observation about trigonometric polynomials which doesn't actually have anything to do with randomness. We introduce a

small amount of notation and give the main proof in the next section. In Section 3 we discuss other cases in which these methods apply, such as odd trigonometric polynomials.

2. Cosine polynomials

Throughout this section a_k is either a real number or a real random variable. If $a = (a_0, \dots, a_K)$ define

$$(2.1) \qquad C_a(x) = \sum_{k=0}^{K} a_k \cos(kx)$$

and let $\overline{a} = (a_K, \dots, a_0)$. Finally, if f is a trigonometric polynomial let $Z(f)$ denote the number of real zeros of f in one period, counted with multiplicity.

All our results on random polynomials follow from the following completely deterministic statement.

Proposition 2.1. *If* $a = (a_0, a_1, \dots, a_K) \neq (0, 0, \dots, 0)$*, then*

$$(2.2) \qquad Z(C_a) + Z(C_{\overline{a}}) \geq 2K.$$

That is, an even trigonometric polynomial and its "reverse" have on average at least half of their zeros real.

The above proposition is a slight generalization and strengthening of Theorem 2 of [1]. Our result does not assume that any particular a_k is nonzero (except that $Z(0)$ is not defined), and a slightly larger lower bound than stated in [1] follows by setting $a_0 = a_K = 0$. For the case of $\{0, 1\}$ polynomials in Theorem 3 of [1], we obtain a lower bound of $n + 1$, which improves their lower bound of $n/4$.

Proof of Theorem 1.2. Take the expected value of (2.2). By assumption $Z(C_a)$ and $Z(C_{\overline{a}})$ have the same distribution and therefore the same expected value. $\square$

Note that the expected value of $Z(C_a)$ exists because it is the average of a bounded measurable function.

As another application of Proposition 2.1 we have the following, which appears to be nontrivial if $L \geq 2$.

Corollary 2.2. *If* $a = (a_0, \dots, a_K)$ *with* $a_k = 0$ *for* $k \leq L$*, then* $Z(C_a) \geq 2L$*.*

Proof. If $C_a \equiv 0$, then the conclusion is true. Otherwise, $C_{\overline{a}}$ has (trigonometric) degree at most $K - L$, and so can have at most $2(K - L)$ zeros. In other words, $Z(C_{\overline{a}}) \leq 2K - 2L$. $\square$

2.1. **Proof of Proposition 2.1.** Real trigonometric polynomials are just another view of self-reciprocal polynomials, and our proof involves counting and estimating, in two different ways, the number of zeros on the unit circle of a self-reciprocal polynomial.

Let

$$(2.3) \qquad f(z) = \sum_{k=0}^{K} a_k z^k$$

and set

$$(2.4) \qquad h(z) = z^K f(1/z) + z^{-K} f(z).$$

Note that

$$(2.5) \qquad h(e^{i\theta}) = 2\sum_{k=0}^{K} a_k \cos((K-k)\theta),$$

so real zeros of $C_{\overline{a}}$ correspond to zeros of $h(z)$ on $|z| = 1$.

Since $h(z)$ is real on $|z| = 1$, the number of zeros on $|z| = 1$ is at least as large as the number of sign changes of $h(e^{2\pi ij/2K})$ for $j = 0, \ldots, 2K$. Here a "sign change" of $\lambda_j = h(e^{2\pi ij/2K})$ means that $\lambda_j \lambda_{j+1} \leq 0$.

We have

$$h(e^{2\pi ij/2K}) = (-1)^j (f(e^{-2\pi ij/2K}) + f(e^{2\pi ij/2K}))$$

$$= 2(-1)^j \sum_{k=0}^{K} a_k \cos(2\pi j/2K)$$

$$(2.6) \qquad = 2(-1)^j C_a(2\pi j/2K).$$

Putting everything together:

$$Z(C_{\overline{a}}) \geq \#\{\text{sign changes of } h(e^{2\pi ij/2K})\}$$

$$= \#\{\text{sign changes of } (-1)^j C_a(2\pi j/2K)\}$$

$$\geq 2K - \#\{\text{sign changes of } C_a(2\pi j/2K)\}$$

$$(2.7) \qquad \geq 2K - Z(C_a),$$

as claimed.

3. Sine polynomials and other cases

The proof of Proposition 2.1 is easily modified to handle other cases, such as odd trigonometric polynomials: $\sum b_k \sin(kx)$. For that case merely replace $h(z)$ by

$$(3.1) \qquad h^-(z) = z^K f(1/z) - z^{-K} f(z).$$

More generally, fix φ and define

$$(3.2) \qquad h^\varphi(z) = e^{i\varphi} z^K f(1/z) - e^{-i\varphi} z^{-K} f(z).$$

Following the proof of Proposition 2.1 we find that if $X = \tan(\varphi)$ is any real number (or ∞, suitably interpreted), then

$$(3.3) \qquad Z(F_{X,a}) + Z(F_{X,\overline{a}}) \geq 2K,$$

where

$$(3.4) \qquad F_{X,a}(x) = \sum_{k=0}^{K} a_k \cos(kx) + X \sum_{k=0}^{K} a_k \sin(kx).$$

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