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ON THE TOPOLOGY OF MANIFOLDS WITH POSITIVE ISOTROPIC CURVATURE

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ABSTRACT. We show that a closed orientable Riemannian n-manifold, $n \geq 5$, with positive isotropic curvature and free fundamental group is homeomorphic to the connected sum of copies of $S^{n-1} \times S^1$.

1. Introduction

Let (M, g) be a closed, orientable, Riemannian manifold with positive isotropic curvature. By [9], if M is simply connected, then M is homeomorphic to a sphere of the same dimension. We shall generalise this to the case when the fundamental group of M is a free group.

Theorem 1.1. *Let M be a closed, orientable Riemannian n-manifold with positive isotropic curvature. Suppose that $\pi_1(M)$ is a free group on k generators. Then, if $n \neq 4$ or $k = 1$ (i.e. $\pi_1(M) = \mathbb{Z}$), M is homeomorphic to the connected sum of k copies of $S^{n-1} \times S^1$.*

We note that a conjecture of M. Gromov ([4], Section 3 (b)) and A. Fraser [2], based on the work of Micallef-Wang [8], states that any compact manifold with positive isotropic curvature has a finite cover satisfying our hypothesis.

Conjecture 1 (M. Gromov, A. Fraser)**.** *$\pi_1(M)$ is virtually free; i.e., it is a finite extension of a free group.*

It is known by the work of A. Fraser [2] and A. Fraser and J. Wolfson [3] that $\pi_1(M)$ does not contain any subgroup isomorphic to the fundamental group of a closed surface of genus at least one.

Our starting point is the following fundamental result of M. Micallef and J. Moore [9].

Theorem 1.2 (M. Micallef, J. Moore)**.** *Suppose M is a closed manifold with positive isotropic curvature. Then $\pi_i(M) = 0$ for $2 \leq i \leq \frac{n}{2}$.*

It is clear that the following purely topological result, together with the Micallef-Moore theorem, implies Theorem 1.1.

Theorem 1.3. *Let M be a smooth, orientable, closed n-manifold such that $\pi_1(M)$ is a free group on k generators and $\pi_i(M) = 0$ for $2 \leq i \leq \frac{n}{2}$. If $n \neq 4$ or $k = 1$, then M is homeomorphic to the connected sum of k copies of $S^{n-1} \times S^1$.*

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Henceforth let M be a smooth, orientable, closed n-manifold such that $\pi_1(M)$ is a free group on k generators and $\pi_i(M) = 0$ for $2 \leq i \leq \frac{n}{2}$. We assume throughout that all manifolds we consider are orientable.

Let $\widetilde{M}$ be the universal cover of M. Hence $\pi_1(\widetilde{M})$ is trivial and so is $\pi_i(\widetilde{M}) = \pi_i(M)$ for $2 \leq i \leq \frac{n}{2}$. We shall show that the homology of $\widetilde{M}$ is isomorphic as $\pi_1(M)$-modules to that of the connected sum of k copies of $S^{n-1} \times S^1$. We then show that M is homotopy equivalent to the connected sum of k copies of $S^{n-1} \times S^1$ using theorems of Whitehead. Finally, recent results of Kreck and Lück allow us to conclude the result.

2. The homology of $\widetilde{M}$

Let X denote the wedge $\bigvee_{j=1}^{k} S^1$ of k circles and let x denote the common point on the circles. Choose and fix an isomorphism φ from $\pi_1(M,p)$ to $\pi_1(X,x)$ for some basepoint $p \in M$. We shall use this identification throughout. Denote $\pi_1(M,p) = \pi_1(X,x)$ by π.

As X is an Eilenberg-Mac Lane space, there is a map $f : (M,p) \to (X,z)$ inducing φ on fundamental groups and a map $s : (X,z) \to (M,p)$ so that $f \circ s : X \to X$ is homotopic to the identity.

We deduce the homology of $\widetilde{M}$ using the Hurewicz Theorem and Poincaré duality.

Lemma 2.1. *For $1 \leq i \leq n/2$, $H_i(\widetilde{M}, \mathbb{Z}) = 0$.*

Proof. As $\widetilde{M}$ is simply connected and $\pi_i(\widetilde{M}) = \pi_i(M) = 0$ for $1 < i \leq n/2$ (by hypothesis), by the Hurewicz theorem, $H_i(\widetilde{M}, \mathbb{Z}) = 0$ for $1 \leq i \leq n/2$. □

We deduce the homology in dimensions above $n/2$ using Poincaré duality for M with coefficients in the module $\mathbb{Z}[\pi]$, namely

$$H_{n-i}(M, \mathbb{Z}[\pi]) = H^i(M, \mathbb{Z}[\pi]).$$

Recall that $H_k(M, \mathbb{Z}[\pi]) = H_k(\widetilde{M}, \mathbb{Z})$ and the group $H^i(M, \mathbb{Z}[\pi])$ is the cohomology with compact support $H^i_c(\widetilde{M}, \mathbb{Z})$. Hence Poincaré duality with coefficients in $\mathbb{Z}[\pi]$ is the same as Poincaré duality for a non-compact manifold relating homology to cohomology with compact support.

To apply Poincaré duality, we need the following lemma.

Lemma 2.2. *For $1 \leq i \leq n/2$, the map $s : (X,z) \to (M,p)$ induces isomorphisms of modules with $s_* : H^i(M; \mathbb{Z}[\pi]) \to H^i(X; \mathbb{Z}[\pi])$.*

Proof. As the map s induces an isomorphism on homotopy groups in dimensions at most $n/2$, it induces isomorphisms on the cohomology groups with twisted coefficients. Specifically, we can add cells of dimensions $k \geq n/2 + 2$ to M to obtain an Eilenberg-MacLane space $\bar{M}$ for the group π, which is thus homotopy equivalent to X. For $i \leq n/2$ and any $\mathbb{Z}[\pi]$-module A, it follows that

$$H_i(M, A) = H_i(\bar{M}, A) = H_i(X, A),$$

where the first equality follows as the cells added to M to obtain $\bar{M}$ are of dimension at least $n/2 + 2$ and the second as the spaces are homotopy equivalent. □

By applying Poincaré duality, we obtain the following result.

Lemma 2.3. *Let M be a smooth, orientable, closed n-manifold such that $\pi_1(M)$ is a free group on k generators and $\pi_i(M) = 0$ for $2 \leq i \leq \frac{n}{2}$. Then, for the universal cover $\widetilde{M}$ of M,*

1. *$H_i(\widetilde{M}, \mathbb{Z}) = 0$ for $1 \leq i < n-1$.*
2. *We have an isomorphism $H_{n-1}(\widetilde{M}, \mathbb{Z}) = H^1_c(\widetilde{X}, \mathbb{Z})$, where $\widetilde{X}$ is the universal cover of X, determined by the isomorphisms $s_* : \pi_1(X, z) \to \pi_1(M, p)$ on fundamental groups.*

Proof. The statements follow from Lemmas 2.1 and 2.2 by using $H_*(\widetilde{M}, \mathbb{Z}) = H_*(M, \mathbb{Z}[\pi])$. □

3. Homotopy type

We now show that M is homotopy equivalent to the connected sum Y of k copies of $S^{n-1} \times S^1$. Our first step is to construct a map $g : Y \to M$. We shall then show that it is a homotopy equivalence.

Note that Y has the structure of a CW-complex obtained as follows. The 1-skeleton of Y is the wedge X of k circles. Let α_i denote the ith circle with a fixed orientation.

We attach k $(n-1)$-cells D_j, with the jth attaching map mapping ∂D^{n-1} to the midpoint x_j of the jth circle. Finally, we attach a single n-cell Δ.

We associate to D_j an element $A_j \in \pi_{n-1}(Y, x)$. Namely, as the attaching map is constant, the jth $(n-1)$-cell gives an element $B_j \in \pi_{n-1}(Y, x_j)$. We consider the subarc β_j of α_j joining z_j to x in the negative direction and let A_j be obtained from B_j by the change of basepoint isomorphism using β_j.

Note that if we instead chose the arc joining z_j to x in the positive direction, then the resulting element is $-\alpha_j \cdot A_j$. By the construction of Y, it follows that the attaching map of the $(n-1)$-cell represents the element

$$\partial\Delta = \Sigma_j (A_j - \alpha_j \cdot A_j)$$

in $\pi_{n-1}(Y)$ regarded as a module over $\pi_1(Y)$. This can be seen for instance by using Poincaré duality.

We now construct the map $g : Y \to M$. Recall that we have a map $s : (X, z) \to (M, p)$ inducing the isomorphism φ^{-1} on fundamental groups. We define g on the 1-skeleton X of Y by $g|_X = s$. We henceforth identify the fundamental groups of Y and M using the isomorphism φ, i.e., $\pi_1(Y, z)$ is identified with π.

We next extend g to the n-cell of Y as follows. By the Hurewicz theorem and Lemma 2.3, we have isomorphisms of π-modules $\pi_{n-1}(M, p) = H_{n-1}(\widetilde{M}, \mathbb{Z})$ and $\pi_{n-1}(Y, z) = H_{n-1}(\widetilde{M}, \mathbb{Z})$. By Lemma 2.3, each of these modules is isomorphic to $H^1_c(\widetilde{X}, \mathbb{Z})$ with the isomorphisms determined by the identifications of the fundamental groups.

Under the above isomorphisms the elements A_j correspond to elements A'_j in $\pi_{n-1}(M, p)$. Consider the element B'_j of $\pi_{n-1}(M, g(z_j))$ obtained from A'_j by the basechange map using the arc $f(\beta_j)$. We define the map g on D_j by extending the constant map on its boundary to be a representative of B_j.

As the π-modules $\pi_{n-1}(M, p)$ and $\pi_{n-1}(Y, z)$ are isomorphic, the image g of $\partial\Delta$ is homotopically trivial. Hence we can extend the map g across the cell Δ.

Lemma 3.1. *The map $g : Y \to M$ is a homotopy equivalence.*

Proof. Let $G : \widetilde{Y} \to \widetilde{M}$ be the induced map on the universal covers. By Lemma 2.3 applied to M and Y, we see that $H_p(\widetilde{Y}) = H_p(\widetilde{M}) = 0$ for $0 < p \neq n-1$ and G induces an isomorphism on H_{n-1}. Thus the map G is a homology equivalence. By a theorem of Whitehead [10], a homology equivalence between simply connected CW-complexes is a homotopy equivalence.

It follows that G induces isomorphisms $G_* : \pi_k(\widetilde{Y}) \to \pi_k(\widetilde{M})$ for $k > 1$. As covering maps induce isomorphims on higher homotopy groups, and g induces an isomorphism on π_1, it follows that g is a weak homotopy equivalence, hence a homotopy equivalence (see [5]). □

4. Proof of Theorem 1.3

The rest of the proof of Theorem 1.3 is based on results of Kreck-Lück [7]. In [7], the authors define a manifold N to be a *Borel manifold* if any manifold homotopy equivalent to N is homeomorphic to N. We have shown that a manifold M satisfying the hypothesis of Theorem 1.3 is homotopy equivalent to the connected sum Y of k copies of $S^{n-1} \times S^1$. Hence it suffices to observe that Y is Borel.

By Theorem 0.13(b) of [7], the manifold $S^{n-1} \times S^1$ is Borel for $n \geq 4$. This completes the proof in the case when $\pi_1(M) = \mathbb{Z}$. Further, if $n \geq 5$, then Theorem 0.9 of [7] says that the connected sum of Borel manifolds is Borel; hence Y is Borel. This concludes the proof for $\pi_1(M)$ a free group and $n \geq 5$. □

Finally, in the case when $n = 3$ by the Knesser conjecture (proved by Stallings) the manifold M is a connected sum of manifolds whose fundamental group is $\mathbb{Z}$. As M is orientable, it follows that if M is expressed as a connected sum of prime manifolds (such a decomposition exists and is unique by the Knesser-Milnor theorem), then each prime component is either $S^2 \times S^1$ or a homotopy sphere. By the Poincaré conjecture (Perelman's theorem), every homotopy 3-sphere is homeomorphic to a sphere. It follows that M is the connected sum of k copies of $S^2 \times S^1$. □

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