ASYMPTOTIC DEPTH
OF TWISTED HIGHER DIRECT IMAGE SHEAVES

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Abstract. Let $\pi : X \to X_0$ be a projective morphism of schemes, such that $X_0$ is Noetherian and essentially of finite type over a field $K$. Let $i \in \mathbb{N}_0$, let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $\mathcal{L}$ be an ample invertible sheaf over $X$. Let $Z_0 \subseteq X_0$ be a closed set. We show that the depth of the higher direct image sheaf $R^i\pi_* (\mathcal{L}^n \otimes \mathcal{O}_X \mathcal{F})$ along $Z_0$ ultimately becomes constant as $n$ tends to $-\infty$, provided $X_0$ has dimension $\leq 2$. There are various examples which show that the mentioned asymptotic stability may fail if dim($X_0$) $\geq 3$.

To prove our stability result, we show that for a finitely generated graded module $M$ over a homogeneous Noetherian ring $R = \bigoplus_{n \geq 0} R_n$ for which $R_0$ is essentially of finite type over a field and an ideal $a_0 \subseteq R_0$, the $a_0$-depth of the $n$-th graded component $H^i_{R_+}(M)_n$ of the $i$-th local cohomology module of $M$ with respect to $R_+ := \bigoplus_{k > 0} R_k$ ultimately becomes constant in codimension $\leq 2$ as $n$ tends to $-\infty$.

1. Introduction

Let $\pi : X \to X_0$ be a projective morphism of schemes, such that $X_0$ is Noetherian and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules. Let $i \in \mathbb{N}_0$. In [2, Theorem 5.5] we did show:

(1.1) As $n \to -\infty$, the set

$$\text{Ass}_{X_0}(R^i\pi_* (\mathcal{L}^n \otimes \mathcal{O}_X \mathcal{F})) \leq 2 := \{x_0 \in \text{Ass}_{X_0}(R^i\pi_* (\mathcal{L}^n \otimes \mathcal{O}_X \mathcal{F})) \mid \dim(\mathcal{O}_{X_0,x_0}) \leq 2\}$$

ultimately becomes constant.

Our aim is to prove a corresponding but stronger stability result for the depths in codimension $\leq 2$ of the sheaves $R^i\pi_* (\mathcal{L}^n \otimes \mathcal{O}_X \mathcal{F})$ along a closed subset $Z_0 \subseteq X_0$.

To make this precise we introduce the following notion: If $Z_0 \subseteq X_0$ is a closed set, $\mathcal{G}$ is a coherent sheaf of $\mathcal{O}_{X_0}$-modules and $t \in \mathbb{N}_0$, we define the depth of $\mathcal{G}$ along $Z_0$ and the depth in codimension $\leq t$ of $\mathcal{G}$ along $Z_0$ respectively by

(1.2) $\text{depth}(Z_0, \mathcal{G}) := \inf \{\text{depth}(\mathcal{G}_{x_0}) \mid x_0 \in Z_0\};$

(1.3) $\text{depth}(Z_0, \mathcal{G}) \leq t := \inf \{\text{depth}(\mathcal{G}_{x_0}) \mid x_0 \in Z_0, \dim(\mathcal{O}_{X_0,x_0}) \leq t\}.$

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We use the convention that $\inf(\emptyset) := \infty$, so that
\[ \text{depth}(Z_0, \mathcal{G}) \leq t = \infty \text{ if codim}(Z_0, X_0) > t. \]

Our main result now says (cf. Theorem 3.5)
\[ \text{(1.4)} \]
As $n \to -\infty$, then number
\[ \text{depth}(Z_0, R^i\pi_*(L^n \otimes \mathcal{O}_X F)) \leq 2 \]
ultimately becomes constant.

As an immediate consequence we obtain (cf. Corollary 3.6)
\[ \text{(1.5)} \]
Let \( \dim(X_0) \leq 2 \). Then, as $n \to -\infty$, then number
\[ \text{depth}(Z_0, R^i\pi_*(L^n \otimes \mathcal{O}_X F)) \]
ultimately becomes constant.

The examples constructed by Chardin-Cutkosky-Herzog-Srinivasan [5] illustrate that the conclusion of (1.5) need not hold if \( \dim(X_0) \geq 3 \).

The basic tool to prove our main result is a corresponding stability result for the depths of graded components of certain local cohomology modules. We shall establish this result in the next section.

2. Depth and local cohomology

By \( \mathbb{N}_0 \) we denote the set of non-negative integers, and by \( \mathbb{N} \) we denote the set of positive integers.

**Notation and Conventions 2.1.** (A) Throughout this paper let \( R = R_0 \oplus R_1 \oplus \cdots \) be a standard graded ring. So, \( R \) is \( \mathbb{N}_0 \)-graded, \( R_0 \) is Noetherian and there are finitely many elements \( a_1, \ldots, a_k \in R_1 \) such that \( R = R_0[a_1, \ldots, a_k] \). By \( R_+ \) we denote the irrelevant ideal of \( R \); thus \( R_+ = R_1 \oplus R_2 \oplus \cdots \).

(B) If \( i \in \mathbb{N}_0 \) and \( M \) is a graded \( R \)-module, we write \( H^i_{R_+}(M) \) for the \( i \)-th local cohomology module of \( M \) with respect to \( R_+ \), and we always furnish this module with its natural grading.

For \( n \in \mathbb{Z} \) we denote by \( H^i_{R_+}(M)_n \) the \( n \)-th graded component of \( H^i_{R_+}(M) \).

Keep in mind that \( H^i_{R_+}(M)_n \) is a finitely generated \( R_0 \)-module for all \( n \in \mathbb{Z} \) and vanishes for all \( n \gg 0 \), provided that the graded \( R \)-module \( M \) is finitely generated.

(C) Now, fix an ideal \( a_0 \subseteq R_0 \). We write \( \text{Var}(a_0) \) for the variety \( \{ p_0 \in \text{Spec}(R_0) \mid a_0 \subseteq p_0 \} \) of \( a_0 \). Keep in mind that for a finitely generated \( R_0 \)-module \( T \) we always have
\[ \text{depth}(a_0, T) = \inf \{ \text{depth}(T_{p_0}) \mid p_0 \in \text{Var}(a_0) \} . \]
Now, for any finitely generated \( R_0 \)-module \( T \) and any \( t \in \mathbb{N}_0 \) we define the depth in codimension \( \leq t \) of \( T \) with respect to \( a_0 \) by
\[ \text{depth}(a_0, T) \leq t = \inf \{ \text{depth}(T_{p_0}) \mid p_0 \in \text{Var}(a_0), \text{height}(p_0) \leq t \} . \]
Again we use the convention that \( \inf(\emptyset) = \infty \), so that
\[ \text{depth}(a_0, T) \leq t \in \{ 0, 1, \ldots, t, \infty \} \]
with
\[ \text{depth}(a_0, T) \leq t = \infty \iff \forall p_0 \in \text{Var}(a_0) \cap \text{Supp}(T) : \text{height}(p_0) > t. \]
(D) We say that a graded $R$-module $U = \bigoplus_{n \in \mathbb{Z}} U_n$ is *tame* if $$U_n = 0 \text{ for all } n \ll 0 \text{ or } U_n \neq 0 \text{ for all } n \ll 0.$$ 

(E) Let $(S_n)_{n \in \mathbb{Z}}$ be a family of numbers or sets. We say that $S_n$ is *asymptotically stable for $n \to -\infty$* if there is some $n_0 \in \mathbb{Z}$ such that $S_n = S_{n_0}$ for all $n \leq n_0$.

**Lemma 2.2.** Assume that $(R_0, m_0)$ is local and of dimension $\leq 2$. Let $i \in \mathbb{N}_0$ and let $M$ be a finitely generated graded $R$-module such that $\text{Ass}_{R_0}(H^i_{R^+_n}(M)_n)$ is asymptotically stable for $n \to -\infty$.

Then $\text{depth}_{R_0}(H^i_{R^+_n}(M)_n)$ is asymptotically stable for $n \to -\infty$.

**Proof.** By the asymptotic stability of $\text{Ass}_{R_0}(H^i_{R^+_n}(M)_n)$ for $n \to -\infty$ we have

$$\text{depth}_{R_0}(H^i_{R^+_n}(M)_n) = 0 \text{ for all } n \ll 0, \text{ or else}$$

$$\text{depth}_{R_0}(H^i_{R^+_n}(M)_n) > 0 \text{ for all } n \ll 0.$$ 

In the first case we are done. So, assume that we are in the second case.

According to [3] Proposition 5.10] the graded $R$-module $H^1_{m_0 R}(H^i_{R^+_n}(M))$ is Artinian and hence tame. Therefore either

$$H^1_{m_0}(H^i_{R^+_n}(M)_n) = 0 \text{ for all } n \ll 0 \text{ or else}$$

$$H^1_{m_0}(H^i_{R^+_n}(M)_n) \neq 0 \text{ for all } n \ll 0.$$ 

As $\dim(R_0) \leq 2$ we thus respectively have either

$$\text{depth}_{R_0}(H^i_{R^+_n}(M)_n) = 2 \text{ for all } n \ll 0 \text{ or else}$$

$$\text{depth}_{R_0}(H^i_{R^+_n}(M)_n) = 1 \text{ for all } n \ll 0. \quad \Box$$

**Proposition 2.3.** Let $a_0 \subseteq R_0$ be an ideal, let $i \in \mathbb{N}_0$ and let $M$ be a finitely generated graded $R$-module such that $\text{Ass}_{R_0}(H^i_{R^+_n}(M)_n)$$ \leq 2$ is asymptotically stable for $n \to -\infty$.

Then $\text{depth}(a_0, H^i_{R^+_n}(M)_n) \leq 2$ is asymptotically stable for $n \to -\infty$.

**Proof.** Let $t := \liminf_{n \to -\infty} \text{depth}(a_0, H^i_{R^+_n}(M)_n) \leq 2$. Observe that $t \in \{0, 1, 2, \infty\}$ (cf. 2.1 (C)). We have to show that $t(n) := \text{depth}(a_0, H^i_{R^+_n}(M)_n) \leq 2 = t$ for all $n \ll 0$.

If $t = \infty$, this is clear. So, let $t \in \{0, 1, 2\}$. We set

$$V := \{p_0 \in \text{Var}(a_0) \mid \text{height}(p_0) \leq 2\}. $$

By our hypothesis the set $V \cap \text{Ass}_{R_0}(H^i_{R^+_n}(M)_n)$ takes a constant value $U$ for all $n \ll 0$. If $U \neq \emptyset$, we have $\text{depth}(a_0, H^i_{R^+_n}(M)_n) \leq 2 = 0$ for all $n \ll 0$. If $U = \emptyset$, we have $\text{depth}(a_0, H^i_{R^+_n}(M)_n) \leq 2 > 0$ for all $n \ll 0$. This gives our claim if $t = 0$.

So, assume that $t \in \{1, 2\}$. Then $\text{depth}(a_0, H^i_{R^+_n}(M)_n) \leq 2 \geq 1$ for all $n \ll 0$. By our hypothesis, the set $V \cap \text{Supp}_{R_0}(H^i_{R^+_n}(M)_n) = V \cap \text{Ass}_{R_0}(H^i_{R^+_n}(M)_n)$ takes a constant value $W$ for all $n \ll 0$. If $\text{height}(p_0) \leq 1$ for some $p_0 \in W$, we have $\text{depth}(a_0, H^i_{R^+_n}(M)_n) \leq \text{height}(p_0) \leq 1$ for all $n \ll 0$, so that $\text{depth}(a_0, H^i_{R^+_n}(M)_n) = 1$ for all $n \ll 0$. Thus, our claim follows in this case.

Therefore, we may assume that $\text{height}(p_0) = 2$ for all $p_0 \in W$. As $W$ is closed in $V$ and $V$ is closed under generalization, the set $W$ must be finite. As $t < \infty$ we must have $\text{depth}(a_0, H^i_{R^+_n}(M)_n) \leq 2 < \infty$ for infinitely many $n \ll 0$. Therefore $W \neq \emptyset$ (cf. 2.1 (C)).
Now by Lemma 2.4 and the Flat Base-Change Property of local cohomology (cf. [4 Theorem 13.1.8]) we get that depth_{(R_0)_{p_0}}((H^i_{R_+}(M)_n)_{p_0}) is asymptotically stable for \( n \to -\infty \) for all \( p_0 \in W \). As \( W \) is finite it follows that depth(\( a_0, H^i_{R_+}(M)_n \)) = \( \min\{\text{depth}_{(R_0)_{p_0}}((H^i_{R_+}(M)_n)_{p_0}) | p_0 \in W \} \) is asymptotically stable for \( n \to -\infty \).

**Corollary 2.4.** Assume that \( R_0 \) is essentially of finite type over a field. Let \( i \in \mathbb{N}_0 \), let \( a_0 \subseteq R_0 \) be an ideal and let \( M \) be a finitely generated graded \( R \)-module.

Then depth(\( a_0, H^i_{R_+}(M)_n \)) is asymptotically stable for \( n \to -\infty \).

**Proof.** According to [2 Proposition 3.5] the set Ass_{\( R_0 \)}(\( H^i_{R_+}(M)_n \)) is asymptotically stable for \( n \to -\infty \). Now, we may conclude by Proposition 2.3.

**Corollary 2.5.** Assume that \( \dim(R_0) \leq 2 \) and \( R_0 \) is essentially of finite type over a field. Let \( i \in \mathbb{N}_0 \), let \( a_0 \subseteq R_0 \) be an ideal and let \( M \) be a finitely generated graded \( R \)-module.

Then depth(\( a_0, H^i_{R_+}(M)_n \)) is asymptotically stable for \( n \to -\infty \).

This result actually is shown in [1].

**Remark 2.6.** (A) According to [3], there is a normal homogeneous domain \( R = R_0 \oplus R_1 \oplus \cdots \), such that \( (R_0, m_0) \) is local of dimension 3, essentially of finite type over \( \mathbb{C} \) and such that \( H^2_{R_0}(R) \) is not tame. In particular depth(\( m_0, H^2_{R_+}(M)_n \)) is not asymptotically stable for \( n \to -\infty \). So in codimensions \( \geq 3 \) the depth of \( H^i_{R_+}(M)_n \) need not be asymptotically stable.

(B) Let \( R = R_0 \oplus R_1 \oplus \cdots \) be as in 2.1 (A), let \( a_0 \subseteq R_0 \) be an ideal and let \( M \) be a finitely generated graded \( R \)-module. Let

\[
c := \sup\{i \in \mathbb{N}_0 | H^i_{R_+}(M) \neq 0\}
\]

be the **cohomological dimension of \( M \) with respect to \( R_+ \)** and let

\[
f := \inf\{i \in \mathbb{N} | H^i_{R_+}(M) \text{ not finitely generated}\}
\]

be the **cohomological finiteness dimension of \( M \) with respect to \( R_+ \)**.

If \( f < \infty \), then clearly \( f \leq c \). Moreover, it is well known that Ass_{\( R_0 \)}(\( H^f_{R_+}(M)_n \)) need not be asymptotically stable for \( n \to -\infty \) (cf. [5] for example) and that Ass_{\( R_0 \)}(\( H^f_{R_+}(M)_n \)) is asymptotically stable for \( n \to -\infty \). In [7] it is shown:

If \( f = c \), then depth(\( a_0, H^f_{R_+}(M)_n \)) is asymptotically stable for \( n \to -\infty \).

### 3. Depth and higher direct images

For the unexplained terminology of this section we refer to [6].

**Notation and Conventions 3.1.** (A) For the rest of this paper let \( X_0 \) denote a Noetherian scheme, let \( \pi : X \to X_0 \) denote a projective scheme over \( X_0 \) with very ample sheaf \( \mathcal{O}_X(1) \) and let \( Z_0 \subseteq X_0 \) be a closed set.

(B) If \( \mathcal{G} \) is a coherent sheaf of \( \mathcal{O}_{X_0} \)-modules and \( t \in \mathbb{N}_0 \) we always use the notation introduced in (1.2) and (1.3).
Proposition 3.2. Assume that $X_0$ is affine and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$.

Then $\text{depth}(Z_0, H^i(X, \mathcal{F}(n))) \leq 2$ is asymptotically stable for $n \to -\infty$.

Proof. Let $R_0 := \mathcal{O}(X_0)$, $a_0 := I(Z_0) \subseteq R_0$. Then, there is a homogeneous Noetherian $R_0$-algebra $R = R_0 \oplus R_1 \oplus \cdots$ with $X = \text{Proj}(R)$ and $\mathcal{O}_X(1) = R(1)^\sim$.

Moreover there is a finitely generated graded $R$-module $M$ such that $\mathcal{F} = \tilde{M}$. Now, for each $n \in \mathbb{Z}$ the Serre-Grothendieck Correspondence gives rise to a short exact sequence of $R_0$-modules

$$0 \to H^0_{R_0}(M)_n \to M_n \to H^0(X, \mathcal{F}(n)) \to H^1_{R_0}(M)_n \to 0$$

and to isomorphisms of $R_0$-modules

$$H^i(X, \mathcal{F}(n)) \cong H^{i+1}_{R_0}(M)_n \text{ for all } j > 0.$$ 

Therefore, our claim follows by Corollary 2.4.

Proposition 3.3. Assume that $X_0$ is affine and essentially of finite type over a field. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules, let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules and let $i \in \mathbb{N}_0$.

Then, $\text{depth}(Z_0, H^i(X, \mathcal{L}^n \otimes \mathcal{O}_X \cdot \mathcal{F})) \leq 2$ is asymptotically stable for $n \to -\infty$.

Proof. This follows from Proposition 3.2 by essentially the same arguments as used in the proof of [2, Theorem 5.3].

Corollary 3.4. Let $X_0$, $\mathcal{L}$, $\mathcal{F}$ and $i$ be as in Proposition 3.3. Assume in addition that $\dim(X_0) \leq 2$.

Then, $\text{depth}(Z_0, H^i(X, \mathcal{L}^n \otimes \mathcal{O}_X \cdot \mathcal{F}))$ is asymptotically stable for $n \to -\infty$.

Theorem 3.5. Let $X_0$ be essentially of finite type over a field. Let $\mathcal{L}$ be an ample invertible sheaf of $\mathcal{O}_X$-modules and let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $i \in \mathbb{N}_0$.

Then $\text{depth}(Z_0, \mathcal{R}^1\pi_*(\mathcal{L}^n \otimes \mathcal{O}_X \cdot \mathcal{F})) \leq 2$ is asymptotically stable for $n \to -\infty$.

Proof. We may assume that $X_0$ is affine. Now, we can conclude by Proposition 3.3 as $\mathcal{R}^1\pi_*(\mathcal{L}^n \otimes \mathcal{O}_X \cdot \mathcal{F}) \cong H^1(X, \mathcal{L}^n \otimes \mathcal{O}_X \cdot \mathcal{F})^\sim$.

Corollary 3.6. Let $X_0$, $\mathcal{L}$, $\mathcal{F}$ and $i$ be as in Theorem 3.5. Assume in addition that $\dim(X_0) \leq 2$.

Then $\text{depth}(Z_0, \mathcal{R}^1\pi_*(\mathcal{L}^n \otimes \mathcal{O}_X \cdot \mathcal{F}))$ is asymptotically stable for $n \to -\infty$.

Remark 3.7. The observations made in Remark 2.4 show that the conclusions of Corollaries 3.4 and 3.6 need not hold if $\dim(X_0) \geq 3$. 
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